FRAMES AND TIME-FREQUENCY ANALYSIS

LECTURE 5: MODULATION SPACES AND APPLICATIONS

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READING

For background on Banach spaces, Hilbert spaces, and operator theory:

Chapters 1–2 in C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.

For background on the Fourier transform:

Chapter 9 in C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.

Today's lecture is based upon:

K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

C. Heil, *Integral operators, pseudodifferential operators, and Gabor frames*, **in: "Advances in Gabor Analysis," Birkhäuser, Boston, 2003, pp. 153–169.**

For further reading:

C. Heil, *History and evolution of the Density Theorem for Gabor frames*, **J. Fourier Anal. Appl., 13 (2007), pp. 113–166.**

C. Heil, *An introduction to weighted Wiener amalgams*, **in: "Wavelets and their Applications," Allied Publishers, New Delhi (2003), pp. 183–216.**

Translation: $(T_a f)(t) \; = \; f(t-a), \qquad a \in \mathbb{R}.$

Modulation: $(M_b f)(t) \; = \; e^{2\pi i b t} f(t), \qquad b \in \mathbb{R}.$

(Regular or Lattice) Gabor (Gah-bor) System:

$$\mathcal{G}(g,a,b) \; = \; \{M_{bn}T_{ak}g\}_{k,n\in\mathbb{Z}} \; = \; \{e^{2\pi i bnt} g(t-ak)\}_{k,n\in\mathbb{Z}}.$$

Analysis map: $C_g f \; = \; \Big\{\langle f, M_{bn}T_{ak}g\rangle\Big\}_{k,n\in\mathbb{Z}}.$

STFT: $V_g f(x,\xi) \; = \; \langle f, M_\xi T_x g\rangle \; = \; \int f(t)\, \overline{g(t-x)}\, e^{-2\pi i \xi t}\, dt, \qquad (x,\xi) \in \mathbb{R}^2.$

Orthogonality relations: $\Big\langle V_{g_1}f_1,\, V_{g_2}f_2\Big\rangle \; = \; \langle f_1, f_2\rangle\, \overline{\langle g_1, g_2\rangle}.$

Inversion: $f \; = \; \dfrac{1}{\langle \gamma, g\rangle} \displaystyle\iint V_g f(x,\xi)\, M_\xi T_x \gamma \, d\xi\, dx$ **(weakly).**

Modulation space $M^p(\mathbb{R})$**:** $\|f\|_{M^p} \; = \; \|V_\phi f\|_p \; = \; \left(\displaystyle\iint |V_\phi f|^p\right)^{1/p}.$

ADJOINT OF THE STFT

Given $1 \le p \le \infty$, **let** p' **be its dual index, defined by** $\dfrac{1}{p} + \dfrac{1}{p'} = 1.$

The STFT is (by definition) an isometric map of $M^{p'}(\mathbb{R})$ **into** $L^{p'}(\mathbb{R}^2)$**:**

$$V_\phi \colon M^{p'}(\mathbb{R}) \to L^{p'}(\mathbb{R}), \qquad \|f\|_{M^{p'}} = \|V_\phi f\|_{p'}.$$

Therefore it has an adjoint that is a bounded linear map

$$V_\phi^* \colon L^p(\mathbb{R}^2) \to M^p(\mathbb{R}).$$

If $F \in L^p(\mathbb{R}^2)$ **is given, then** $V_\phi^* F$ **is a uniquely defined element of** $M^p(\mathbb{R})$. **Given** $F \in L^p(\mathbb{R}^2)$ **and** $f \in M^{p'}(\mathbb{R})$, **we have**

$$\begin{aligned}
\langle V_\phi^* F, f\rangle &= \langle F, V_\phi f\rangle \qquad \textbf{(definition of the adjoint)} \\
&= \iint F(x,\xi)\, \overline{V_\phi f(x,\xi)}\, dx\, d\xi \\
&= \iint F(x,\xi)\, \langle M_\xi T_x \phi, f\rangle\, dx\, d\xi = \left\langle \iint F(x,\xi)\, M_\xi T_x \phi,\ f \right\rangle.
\end{aligned}$$

Therefore

$$V_\phi^* F = \iint F(x,\xi)\, M_\xi T_x \phi,$$

where we interpret the integral weakly.

$$V_\phi^* F \;=\; \iint F(x,\xi)\, M_\xi T_x \phi.$$

Corollary 1. If $f \in M^p(\mathbb{R})$ then $V_\phi^* V_\phi f = f$.

Proof.

$$\begin{aligned} V_\phi^* V_\phi f \;&=\; \iint V_\phi f(x,\xi)\, M_\xi T_x \phi \\ &=\; \frac{1}{\langle \phi, \phi \rangle} \iint V_\phi f(x,\xi)\, M_\xi T_x \phi \;=\; f \qquad \text{(by Inversion).} \end{aligned}$$

□

Corollary 1 and most other results, can be formulated in terms of other windows. A first step is to extend to windows in $\mathcal{S}(\mathbb{R})$, but then once all the tools are in place, we can extend to any window in $M^1(\mathbb{R})$. In this sense the Feichtinger algebra $M_1(\mathbb{R})$ is the "correct" space of windows for time-frequency analysis. We will concentrate on the Gaussian window for convenience.

Corollary 1 does not imply that V_ϕ is invertible. It is injective, but it does not map onto $L^p(\mathbb{R}^2)$. Even so, from $f = V_\phi^* V_\phi f$ we obtain the reproducing formula

$$V_\phi f \;=\; V_\phi V_\phi^* V_\phi f.$$

This will be very useful when we combine it with the next (seemingly technical but very important) result.

Theorem 2. If $F \in L^p(\mathbb{R}^2)$, then $|V_\phi V_\phi^* F| \leq |F| * |V_\phi \phi|$.

Proof.

$$\begin{aligned}
|V_\phi V_\phi^* F(u,\eta)| &= \big|\big\langle V_\phi^* F,\, M_\eta T_u \phi \big\rangle\big| \\
&= \big|\big\langle F,\, V_\phi(M_\eta T_u \phi)\big\rangle\big| \\
&= \left| \iint F(x,\xi)\, \overline{V_\phi(M_\eta T_u \phi)(x,\xi)} \right| dx\, d\xi \\
&\leq \iint |F(x,\xi)|\, |\langle M_\eta T_u \phi, M_\xi T_x \phi\rangle|\, dx\, d\xi \\
&= \iint |F(x,\xi)|\, |\langle M_{\xi-\eta} T_{x-u} \phi, \phi\rangle|\, dx\, d\xi \\
&= \iint |F(x,\xi)|\, |V_\phi \phi(u-x, \eta-\xi)|\, dx\, d\xi \\
&= \Big(|F| * |V_\phi \phi|\Big)(u,\eta).
\end{aligned}$$

□

Remark: Since $\phi \in \mathcal{S}(\mathbb{R})$, we know that $V_\phi \phi \in \mathcal{S}(\mathbb{R}^2)$. In fact, $V_\phi \phi$ is a two-dimensional Gaussian. When going to general windows, the important fact is that if $g,\, \gamma \in M^1(\mathbb{R})$, then $V_\gamma g \in W(C, \ell^1)$, so we have both local and global control of $V_\gamma g$.

$$|V_\phi V_\phi^* F| \;\leq\; |F| * |V_\phi \phi|.$$

Corollary 3. If $f \in M^p(\mathbb{R})$, then $F = V_\phi f \in L^p(\mathbb{R}^2)$ satisfies

$$|V_\phi f| \;=\; |V_\phi V_\phi^* V_\phi f| \;\leq\; |V_\phi f| * |V_\phi \phi|. \qquad \diamondsuit$$

Theorem 4 (Inclusions for Amalgams)**.** If $p_1 \geq p_2$ and $q_1 \leq q_2$, then

$$W(L^{p_1}, \ell^{q_1}) \subseteq W(L^{p_2}, \ell^{q_2}). \qquad \diamondsuit$$

Theorem 5 (Convolution for Amalgams)**.** For appropriate indices,

$$W(L^{p_1}, \ell^{q_1}) * W(L^{p_2}, \ell^{q_2}) \subseteq W(L^{p_1} * L^{p_2}, \ell^{q_1} * \ell^{q_2}). \qquad \diamondsuit$$

Consequently,

$$\begin{aligned} L^p * W(L^\infty, \ell^1) &= W(L^p, \ell^p) * W(L^\infty, \ell^1) \\ &\subseteq W(L^1, \ell^p) * W(L^\infty, \ell^1) \\ &\subseteq W(L^1 * L^\infty, \ell^p * \ell^1) \\ &\subseteq W(L^\infty, \ell^p). \end{aligned}$$

$$|V_\phi V_\phi^* F| \;\le\; |F| * |V_\phi \phi|$$

$$L^p \,*\, W(L^\infty, \ell^1) \;\subseteq\; W(L^\infty, \ell^p)$$

Corollary 6. If $f \in M^p(\mathbb{R})$, then $V_\phi f \in W(L^\infty, \ell^p)$.

Proof. We have

$$|V_\phi f| \;=\; |V_\phi V_\phi^* V_\phi f| \;\le\; |V_\phi f| * |V_\phi \phi| \;\in\; L^p \,*\, W(L^\infty, \ell^1) \;\subseteq\; W(L^\infty, \ell^p).$$

Amalgam spaces are *solid*, i.e.,

$$g \in W(L^p, \ell^q) \text{ and } |h| \le |g| \qquad \Longrightarrow \qquad h \in W(L^p, \ell^q).$$

Therefore $V_\phi f \in W(L^\infty, \ell^p)$. □

The mapping in Corollary 6 is bounded; a slightly more careful proof shows that

$$\|V_\phi f\|_{W(L^\infty,\ell^p)} \;\le\; C\, \|f\|_{M^p}\, \|V_\phi \phi\|_{W(L^\infty,\ell^1)}.$$

Not every space is solid! Spaces defined in terms of the Fourier transform, like the Wiener algebra $A(\mathbb{R})$ or the Sobolev spaces $H^s(\mathbb{R})$, are not solid. The modulation spaces $M^p(\mathbb{R})$ are not solid either.

$$f \in M^p(\mathbb{R}) \quad \Longrightarrow \quad V_\phi f \in W(L^\infty, \ell^p)$$

The *analysis map* is a sampling of the STFT:

$$C_g f \;=\; \Bigl\{\langle f, M_{bn}T_{ak}g\rangle\Bigr\}_{k,n\in\mathbb{Z}} \;=\; \Bigl\{V_g f(ak, bn)\Bigr\}_{k,n\in\mathbb{Z}}.$$

Samples are a *local property* of the STFT, while what ℓ^p space they belong to is a *global property*. We have a relation between the samples and the STFT because the STFT belongs to an amalgam space!

Theorem 7. Fix a, $b > 0$. Analysis (using the Gaussian window) is a bounded map of $M^p(\mathbb{R})$ into $\ell^p(\mathbb{Z}^2)$:

$$C_\phi \colon M^p(\mathbb{R}) \to \ell^p(\mathbb{Z}^2) \quad \text{is bounded.}$$

Proof. Let $Q_{kn} = [ak, a(k+1)] \times [bn, b(n+1)]$. Then

$$\begin{aligned} \|C_\phi f\|_{\ell^p}^p \;&=\; \Bigl\|\bigl\{\langle f, M_{bn}T_{ak}\phi\rangle\bigr\}_{k,n\in\mathbb{Z}}\Bigr\|_p^p \;=\; \sum_{k,n\in\mathbb{Z}} |V_\phi f(ak, bn)|^p \;\le\; \sum_{k,n\in\mathbb{Z}} \sup_{(x,\xi)\in Q_{kn}} |V_\phi f(x,\xi)|^p \\ &=\; \|V_\phi f\|_{W(L^\infty,\ell^p)}^p \;\le\; C\,\|f\|_{M^p}^p\,\|V_\phi \phi\|_{W(L^\infty,\ell^1)}^p. \end{aligned}$$

□

Restated explicitly, there exists a constant $B > 0$ $\big(B = C\,\|V_\phi\phi\|^p_{W(L^\infty,\ell^1)}\big)$ **such that**

$$\sum_{k,n\in\mathbb{Z}} |\langle f, M_{bn}T_{ak}\phi\rangle|^p \;\le\; B\,\|f\|^p_{M^p}, \qquad \textbf{all } f \in M^p(\mathbb{R}).$$

In essence, $\mathcal{G}(\phi, a, b)$ **is a "*Bessel sequence*" for** $M^p(\mathbb{R})$. **The same is true if we replace** ϕ **by any other window** $g \in M^1(\mathbb{R})$.

Since *synthesis* is the adjoint of *analysis*, we obtain the following.

Corollary 8. Let $R_\phi\colon \ell^p(\mathbb{Z}^2) \to M^p(\mathbb{R})$ be the adjoint of $C_\phi\colon M^{p'}(\mathbb{R}) \to \ell^{p'}(\mathbb{Z}^2)$. Then R_ϕ is bounded. $\diamondsuit$

Assignment 1. Prove that if $c = \{c_{kn}\}_{k,n\in\mathbb{Z}}$ has only finitely many nonzero components, then

$$R_\phi c \;=\; \sum_{k,n\in\mathbb{Z}} c_{kn}\, M_{bn}T_{ak}\phi. \qquad \diamondsuit$$

Here (and elsewhere) we are sweeping under the rug some technical issues concerning $p = \infty$. **Sometimes this involves interpreting series or integrals in a weak or weak* sense.**

Theorem 9. If $1 \le p < \infty$ and $c \in \ell^p(\mathbb{Z}^2)$, then

$$R_\phi c \;=\; \sum_{k,n\in\mathbb{Z}} c_{kn}\, M_{bn} T_{ak}\phi, \tag{1}$$

with (unconditional) convergence in M^p-norm.

Proof. Fix $c \in \ell^p(\mathbb{Z}^2)$, and let

$$G \;=\; V_\phi \phi \qquad \text{and} \qquad \chi \;=\; \chi_{[0,a]\times[0,b]}.$$

Since $G \in W(L^\infty, \ell^1)$, we have

$$0 \;\le\; G \;\le\; \sum_{j,\ell\in\mathbb{Z}} a_{j\ell}\, T_{(aj,b\ell)}\chi, \qquad \text{where } a = \{a_{j\ell}\} \text{ is summable.}$$

Aside. To show that a series $\sum_{n=1}^\infty$ converges, we must show that the partial sums are Cauchy. A difference of two partial sums has the form

$$\sum_{n=1}^{N} - \sum_{n=1}^{M} \;=\; \sum_{n=M+1}^{N}.$$

We want this $< \varepsilon$ when M, N are large enough. For unconditional convergence, we want

$$\sum_{n\in F} \;<\; \varepsilon$$

whenever F is a finite subset of the index set and $\min(F)$ is large enough.

Let F be a finite subset of $\mathbb{Z}^2$, and let c_F be c restricted to F. By Assignment 1, $R_\phi c_F = \sum_{k,n\in\mathbb{Z}} c_{kn}\, M_{bn} T_{ak}\phi$, so to estimate $\|R_\phi c_F\|_{M^p} = \|V_\phi(R_\phi c_F)\|_{L^p}$ we first compute that

$$
\begin{aligned}
|V_\phi(R_\phi c_F)| \;=\; \Bigl|\sum_{(k,n)\in F} c_{kn}\, V_\phi(M_{bn}T_{ak}\phi)\Bigr| &\le \sum_{(k,n)\in F} |c_{kn}|\; |V_\phi(\cdot - ak, \cdot - bn)| \\
&\le \sum_{(k,n)\in F} |c_{kn}|\; T_{(ak,bn)} G \\
&\le \sum_{(k,n)\in F} |c_{kn}|\; T_{(ak,bn)} \Bigl(\sum_{j,\ell\in\mathbb{Z}} a_{j\ell}\, T_{(aj,b\ell)}\chi\Bigr) \\
&\le \sum_{(k,n)\in F} |c_{kn}| \sum_{j,\ell\in\mathbb{Z}} a_{j\ell}\, T_{(aj+ak,b\ell+bn)}\chi \\
&\le \sum_{(k,n)\in F} |c_{kn}| \sum_{j,\ell\in\mathbb{Z}} a_{j-k,\ell-n}\, T_{(aj,b\ell)}\chi \\
&= \sum_{j,\ell\in\mathbb{Z}} \Bigl(\sum_{(k,n)\in F} |c_{kn}|\, a_{j-k,\ell-n}\Bigr) T_{(aj,b\ell)}\chi \\
&= \sum_{j,\ell\in\mathbb{Z}} \bigl(|c_F| * a\bigr)_{j\ell}\, T_{(aj,b\ell)}\chi.
\end{aligned}
$$

We've shown:

$$\left| \sum_{(k,n)\in F} c_{kn}\, V_\phi(M_{bn}T_{ak}\phi) \right| \;\le\; \sum_{j,\ell\in\mathbb{Z}} \big(|c_F| * a\big)_{j\ell}\; T_{(aj,b\ell)}\chi.$$

Therefore

$$\begin{aligned}
\|R_\phi c_F\|_{M^p} \;=\; \|V_\phi(R_\phi c_F)\|_{L^p} &= \left\| \sum_{(k,n)\in F} c_{kn}\, M_{bn}T_{ak}\phi \right\|_{M^p} \\
&= \left\| \sum_{(k,n)\in F} c_{kn}\, V_\phi(M_{bn}T_{ak}\phi) \right\|_{L^p} \\
&\le \left\| \sum_{j,\ell\in\mathbb{Z}} \big(|c_F| * a\big)_{j\ell}\; T_{(aj,b\ell)}\chi \right\|_{L^p} \\
&= C \,\Big\|\, |c_F| * a \,\Big\|_{\ell^p} \qquad \text{(it's a step function)} \\
&\le C\, \|c_F\|_{\ell^p}\, \|a\|_{\ell^1} \;<\; \varepsilon \qquad \text{for all large enough tails } F.
\end{aligned}$$

This shows that the series

$$R_\phi c \;=\; \sum_{k,n\in\mathbb{Z}} c_{kn}\, M_{bn}T_{ak}\phi \tag{2}$$

converges unconditionally in M^p-norm, and repeating the above work with $F = \mathbb{Z}^2$ shows that

$$\|R_\phi c\|_{M^p} \;\le\; B\, \|c\|_{\ell^p}, \qquad \text{where } B = C\, \|a\|_{\ell^1}.$$

Finally, repeating Assignment 1 with $F = \mathbb{Z}^2$ shows that (2) really is the adjoint of the analysis map C_ϕ. $\square$

Remark: For general windows, the only important point is to have $V_\gamma g \in W(L^\infty, \ell^1)$, **which is ensured if** $g,\ \gamma \in M^1(\mathbb{R})$.

Summary: If $g,\ \gamma$ **belong to** $M^1(\mathbb{R})$, **then**

$$C_g\colon M^p(\mathbb{R}) \to \ell^p(\mathbb{Z}^2) \qquad \textbf{and} \qquad R_\gamma\colon \ell^p(\mathbb{Z}^2) \to M^p(\mathbb{R})$$

are bounded, and therefore

$$S_{g,\gamma} f \;=\; R_\gamma C_g f \;=\; \sum_{k,n\in\mathbb{Z}} \langle f, M_{bn} T_{ak} g\rangle\, M_{bn} T_{ak} \gamma, \qquad f \in M^p(\mathbb{R}),$$

is a bounded mapping of $M^p(\mathbb{R})$ **into itself.**

What is $S_{g,\gamma}$**? If** $\mathcal{G}(g,a,b)$ **is a Gabor frame for** $L^2(\mathbb{R})$ **and** $\gamma = \widetilde{g}$ **is the dual window, then** $S_{g,\gamma} = I$ ***on the space*** $L^2(\mathbb{R})$.

What all this work shows is that if we start with a Gabor frame for $L^2(\mathbb{R})$ whose window g belongs to the "correct" window class $M^1(\mathbb{R})$ ($\subsetneq W(C,\ell^1)$), then *it is a Gabor frame for every modulation space* $M^p(\mathbb{R})$*, not just* $L^2(\mathbb{R})$.

Corollary 10. Assume $g \in M^1(\mathbb{R})$ is such that $\mathcal{G}(g,a,b)$ is a Gabor frame for $L^2(\mathbb{R})$, and let $\widetilde{g}$ be the dual window. Assume $\widetilde{g} \in M^1(\mathbb{R})$ (this turns out to be automatic). Then there exist constants A, $B > 0$ such that for each $1 \le p < \infty$ we have

$$A\,\|f\|_{M^p}^p \;\le\; \sum_{k,n\in\mathbb{Z}} |\langle f, M_{bn}T_{ak}g\rangle|^p \;\le\; B\,\|f\|_{M^p}^p, \qquad f \in M^p(\mathbb{R}),$$

and

$$f \;=\; \sum_{k,n\in\mathbb{Z}} \langle f, M_{bn}T_{ak}\widetilde{g}\,\rangle\, M_{bn}T_{ak}g \;=\; \sum_{k,n\in\mathbb{Z}} \langle f, M_{bn}T_{ak}g\rangle\, M_{bn}T_{ak}\widetilde{g}, \qquad f \in M^p(\mathbb{R}). \tag{3}$$

Proof. The upper inequality in (3) is a restatement of the boundedness of $C_g\colon M^p(\mathbb{R}) \to \ell^p(\mathbb{Z}^2)$. A close examination of the preceding results shows that the bound (which is the operator norm of C_g) does not depend on p.

Fix $f \in \mathcal{S}(\mathbb{R})$, which is dense in $M^p(\mathbb{R})$. Then

$$R_{\widetilde{g}}\,C_g f \;=\; \sum_{k,n\in\mathbb{Z}} \langle f, M_{bn}T_{ak}\widetilde{g}\,\rangle\, M_{bn}T_{ak}g \qquad \text{(convergence in } M^p\text{-norm).}$$

$$R_{\widetilde{g}}\, C_g f \;=\; \sum_{k,n\in\mathbb{Z}} \langle f, M_{bn}T_{ak}\widetilde{g}\,\rangle\, M_{bn}T_{ak}g \qquad \text{(convergence in } M^p\text{-norm).}$$

Because $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbb{R})$, we have

$$f \;=\; \sum_{k,n\in\mathbb{Z}} \langle f, M_{bn}T_{ak}\widetilde{g}\,\rangle\, M_{bn}T_{ak}g \qquad \text{(convergence in } L^2\text{-norm).}$$

This implies (after some work) that $R_{\widetilde{g}}\, C_g f = f$. Using the boundedness of $R_{\widetilde{g}}$ and C_g and the density of the Schwartz space, this extends to all $f \in M^p(\mathbb{R})$. Finally,

$$\|f\|_{M^p} \;=\; \|R_{\widetilde{g}}\, C_g f\|_{M^p} \;\leq\; \|R_{\widetilde{g}}\|\, \|_g C f\|_{M^p},$$

which gives the lower inequality in (3). □

Remark 1: For $p = \infty$, we still obtain the norm equivalence, but the series converge weak* instead of in norm.

Remark 2: These results extend to general mixed-norm, weighted modulation spaces $M^{p,q}_w(\mathbb{R})$.

Remark 3: A deep result of Gröchenig and Leinert shows that if $g \in M^1(\mathbb{R})$ and $\mathcal{G}(g,a,b)$ is frame for $L^2(\mathbb{R})$, then $\widetilde{g}$ belongs to $M^1(\mathbb{R})$.

Corollary 11. Assume $g \in M^1(\mathbb{R})$ is such that $\mathcal{G}(g,a,b)$ is a tight Gabor frame for $L^2(\mathbb{R})$. Then $f \in M^p(\mathbb{R})$ if and only if $C_g f \in \ell^p(\mathbb{Z}^2)$, and

$$f \;=\; \frac{1}{A} \sum_{k,n\in\mathbb{Z}} \langle f, M_{bn}T_{ak}g\rangle\, M_{bn}T_{ak}g, \qquad f \in M^p(\mathbb{R}),$$

with unconditional convergence of this series in M^p-norm.

Further results: Although Gabor frames do not yield unique representations and hence are not Schauder bases for the modulation spaces, there is a remarkable related construction of *Wilson bases* that share the same norm equivalence and series representation theorems as Gabor frames *with unique representations.* Consequently Wilson bases are unconditional Schauder bases for the modulation spaces. As a consequence, the norm equivalence implies an actual *isomorphism* between M^p and ℓ^p:

$$M^p(\mathbb{R}) \cong \ell^p(\mathbb{Z}^2) \qquad \text{(in the sense of topological isomorphism)}.$$

This implies additional results. In particular, the modulation spaces are ordered identically to the ℓ^p spaces:

$$1 \le p \le q \le \infty \implies M^p(\mathbb{R}) \subseteq M^q(\mathbb{R}).$$

Definition 12. (a) Given $k \in L^2(\mathbb{R}^2)$, the *integral operator* $A_k \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is

$$A_k f(x) \;=\; \int k(x,y)\, f(y)\, dy, \qquad f \in L^2(\mathbb{R}).$$

(Analogous to matrix-vector multiplication: $(Au)_j \;=\; \sum_j a_{ij} u_i$.)

(b) Given a *symbol function* $\sigma \in L^2(\mathbb{R}^2)$, the *Weyl transform* of σ is $L_\sigma \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$\begin{aligned} L_\sigma f(x) \;&=\; \iint \sigma\Bigl(\frac{x+y}{2}, \xi\Bigr)\, e^{2\pi i (x-y)\xi}\, f(y)\, dy\, d\xi \\ &=\; \int \biggl(\int \sigma\Bigl(\frac{x+y}{2}, \xi\Bigr)\, e^{2\pi i (x-y)\xi}\, d\xi \biggr)\, f(y)\, dy. \end{aligned}$$

(c) Given a *symbol function* $\tau \in L^2(\mathbb{R}^2)$, the Kohn-Nirenberg transform of τ is $K_\tau \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$\begin{aligned} K_\tau f(x) \;&=\; \int \tau(x,\xi)\, \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi \\ &=\; \iint \tau(x,\xi)\, e^{2\pi i (x-y)\xi}\, f(y)\, dy\, d\xi. \qquad \diamondsuit \end{aligned}$$

L_σ **and** K_τ **are** ***pseudodifferential operators.***

Contrast the following formulas.

Identity:

$$f(x) \;=\; \big(\widehat{f}\,\big)^{\vee}(x) \;=\; \int \widehat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi$$

Filtering (convolution):

$$(f * g)(x) \;=\; (f * g)^{\wedge\vee}(x) \;=\; (\widehat{f}\,\widehat{g}\,)^{\vee}(x) \;=\; \int \widehat{f}(\xi)\,\widehat{g}(\xi)\, e^{2\pi i \xi x}\, d\xi$$

(f with frequency amplitudes adjusted, like a signal through an equalizer.)

Time-varying filtering (Kohn-Nirenberg transform):

$$K_\tau f(x) \;=\; \int \hat{f}(\xi)\, \tau(x,\xi)\, e^{2\pi i x \xi}\, d\xi$$

(f with frequency amplitudes adjusted continuously with time.)

(f after transmission through a wireless channel; superposition of time delays and doppler shifts.)

To convert from Weyl to integral operator:

$$k(x,y) \;=\; \int \sigma\Big(\frac{x+y}{2}, \xi\Big)\, e^{2\pi i (x-y)\xi}\, d\xi$$

This is a composition of a change of variable and a partial Fourier transform, and M^p **is invariant under both of these. Therefore**

$$\sigma \in M^p(\mathbb{R}^2) \quad\Longleftrightarrow\quad k \in M^p(\mathbb{R}^2) \quad\Longleftrightarrow\quad \tau \in M^p(\mathbb{R}^2).$$

As far as modulation properties go, we can work either with integral operators or pseudodifferential operators. We'll focus on integral operators, but the application (and generalization) we have in mind is based on pseudodifferential operators.

If $k \in L^2(\mathbb{R}^2)$, then the corresponding integral operator A_k is bounded on $L^2(\mathbb{R})$, because

$$\begin{aligned}
\|A_k f\|_2^2 \;=\; \int |A_k f(x)|^2\,dx \;&=\; \int \left| \int k(x,y)\,f(y)\,dy \right|^2 dx \\
&\le\; \int \left(\int |k(x,y)|^2\,dy \right) \left(\int |f(y)|^2\,dy \right) dx \\
&=\; \|f\|_2^2 \iint |k(x,y)|^2\,dy\,dx \\
&=\; \|f\|_2^2\,\|k\|_2^2.
\end{aligned}$$

Thus,

$$\|A_k f\|_2 \;\le\; \|k\|_2\,\|f\|_2.$$

Taking the supremum over all unit vectors f, we see that A_k is bounded and its operator norm satisfies

$$\|A_k\| \;=\; \sup_{\|f\|_2=1} \|A_k f\|_2 \;\le\; \|k\|_2.$$

More is true: A_k is a *compact operator* when k is square-integrable. To prove this, let $\{e_n\}_{n\in\mathbb{N}}$ be an ONB for $L^2(\mathbb{R})$. Set

$$e_{mn}(x,y) \;=\; (e_m \otimes \bar{e}_n)(x,y) \;=\; e_m(x)\,\overline{e_n(y)}.$$

Then $\{e_{mn}\}_{m,n\in\mathbb{N}}$ is an ONB for $L^2(\mathbb{R}^2)$. Since $k \in L^2(\mathbb{R}^2)$, we have

$$k \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \langle k, e_{mn}\rangle e_{mn},$$

with convergence in L^2-norm. Make an approximation to k:

$$k_N \;=\; \sum_{m=1}^{N}\sum_{n=1}^{N} \langle k, e_{mn}\rangle e_{mn}.$$

Let $A_N = A_{k_N}$ be the corresponding integral operator:

$$\begin{aligned} A_N f(x) \;&=\; \int k_N(x,y)\, f(y)\, dy \;=\; \sum_{m=1}^{N}\sum_{n=1}^{N} \langle k, e_{mn}\rangle \int e_{mn}(x,y)\, f(y)\, dy \\ &=\; \sum_{m=1}^{N}\sum_{n=1}^{N} \langle k, e_{mn}\rangle \int e_m(x)\,\overline{e_n(y)}\, f(y)\, dy \\ &=\; \sum_{m=1}^{N}\sum_{n=1}^{N} \langle k, e_{mn}\rangle\, \langle f, e_n\rangle\, e_m(x). \end{aligned}$$

$$
\begin{aligned}
A_N f &= \sum_{m=1}^{N} \sum_{n=1}^{N} \langle k, e_{mn} \rangle \langle f, e_n \rangle \, e_m \\
&\in \operatorname{span}\{e_1, \ldots, e_N\}.
\end{aligned}
$$

Hence A_N has finite-dimensional range, it is a *linear, continuous, finite-rank operator.* All such operators are *compact* (the image of the unit ball has compact closure). The integral operator A does not have finite-dimensional range, but

$$
\|A_k - A_N\| = \|A_{k-k_N}\| \le \|k - k_N\|_2 \to 0.
$$

It is the limit (in operator norm) of compact operators, and this implies that it is compact.

In general A_k is not self-adjoint:

$$
A_k \textbf{ is self-adjoint} \quad \Longleftrightarrow \quad k(x,y) = \overline{k(y,x)},
$$

but on the other hand the composition $A_k^* A_k$ is both compact and self-adjoint.

The (most basic) version of the *Spectral Theorem* says that $A_k^* A_k$ **has an *ONB of eigenvectors* and corresponding nonnegative eigenvalues** $\mu_n \to 0$. **The *singular values* of** A_k **are**

$$s_n \;=\; \sqrt{\mu_n}.$$

If A_k **happens to be self-adjoint, then**

$$s_n \;=\; |\lambda_n|,$$

where λ_n **are the eigenvalues of** A_k.

Theorem 13. If $k \in L^2(\mathbb{R}^2)$, then the singular numbers of A_k satisfy

$$\sum_{n=1}^{\infty} s_n^2 \;=\; \|k\|_2^2 \;<\; \infty.$$

(Therefore we say that A_k is a *Hilbert–Schmidt operator*.) $\diamondsuit$

We say that A_k **is *trace-class* if the singular values are *summable*, i.e., if**

$$\sum_{n=1}^{\infty} s_n \;<\; \infty.$$

We can give a very simple sufficient condition to be trace-class.

Theorem 14. If $k \in M^1(\mathbb{R}^2)$, then A_k is trace-class.

Proof. Choose $g \in M^1(\mathbb{R})$ so that $\mathcal{G}(g,a,b)$ is a Parseval frame for $L^2(\mathbb{R})$. Enumerate this Gabor system as $\mathcal{G}(g,a,b) = \{e_n\}_{n\in\mathbb{N}}$, and let $e_{mn} = e_n \otimes \bar{e}_m$. Then $\{e_{mn}\}_{m,n\in\mathbb{N}}$ is a Parseval frame for every modulation space M^p! In particular,

$$k \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \langle k, e_{mn}\rangle\, e_{mn}$$

with convergence in M^1-norm, and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\langle k, e_{mn}\rangle| \;<\; \infty.$$

Via linearity and absolute convergence,

$$A_k \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \langle k, e_{mn}\rangle\, A_{e_{mn}}. \tag{4}$$

As before, $A_{e_{mn}}$ has *one-dimensional range*. It has a single nonzero singular value, and that singular value is $\|g\|_2^2$. As a consequence $A_{e_{mn}}$ is a trace-class operator. The space of trace-class operators is a Banach space, so an absolutely convergence series of trace-class operators is trace-class. Therefore equation (4) converges absolutely in trace-class norm, so A_k is trace-class. □

More refined analysis using weighted modulation spaces gives, with a simple proof, an improvement to known results: If $k \in M_s^2$ (= intersection of weighted L^2 space and a Sobolev space) then A_k is trace-class. *Further*, the invariance of the modulation spaces gives us equivalent theorems for pseudodifferential operators instead of integral operators.

Thank You!