

FRAMES AND TIME-FREQUENCY ANALYSIS

LECTURE 3: GABOR FRAMES AT THE CRITICAL DENSITY (and beyond)

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READING

For background on Hilbert spaces and operator theory:

Chapters 1–2 in C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.

For background on the Fourier transform:

Chapter 9 in C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.

Today's lecture is based upon:

Chapter 11 (Sections 11.6–11.8 and 11.3) in C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.

Also see:

C. E. Heil and D. F. Walnut, *Continuous and discrete wavelet transforms*, SIAM Review, 31 (1989), pp. 628–666.

For further reading:

K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

C. Heil, *History and evolution of the Density Theorem for Gabor frames*, J. Fourier Anal. Appl., 13 (2007), pp. 113–166.

REVIEW

Translation: $(T_a f)(x) = f(x - a), \quad a \in \mathbb{R}.$

Modulation: $(M_b f)(x) = e^{2\pi i b x} f(x), \quad b \in \mathbb{R}.$

Time-frequency shifts are $M_b T_a$ and $T_a M_b$ (note $T_a M_b = e^{-2\pi i a b} M_b T_a$)

(Regular or Lattice) Gabor (Gah-bor) System:

$$\mathcal{G}(g, a, b) = \{M_{bn} T_{ak} g\}_{k,n \in \mathbb{Z}} = \{e^{2\pi i b n x} g(x - ak)\}_{k,n \in \mathbb{Z}}.$$

Terminology for a sequence $\{x_n\}_{n \in \mathbb{N}}$ in a Hilbert space:

- It is a *frame* if there exist $A, B > 0$ such that

$$A \|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq B \|x\|^2.$$

- It is *complete* if its finite span is dense.
- It is *minimal* if and only if it has a biorthogonal sequence, $\langle x_m, y_n \rangle = \delta_{mn}$.
- It is *exact* if it is both minimal and complete.
- It is a *Riesz basis* if it is the image of an ONB under a continuous bijection.
Equivalently, it is a *bounded unconditional basis*, and an *exact frame*.

Density trichotomy for Gabor systems:

- If $ab > 1$ then $\mathcal{G}(g, a, b)$ is not a frame.
- $\mathcal{G}(g, a, b)$ is a Riesz basis if and only if it is a frame and $ab = 1$.
- If $\mathcal{G}(g, a, b)$ is a frame and $0 < ab < 1$ then it is a redundant frame.

The value $1/(ab)$ is called the *density* of the Gabor system, and density $1/(ab) = 1$ is the *critical density*. Because

$$\mathcal{G}(g, a, b) \text{ is a frame} \iff \mathcal{G}(D_r g, a/r, br) \text{ is a frame.}$$

when working at the critical density ($ab = 1$) it suffices to consider $a = b = 1$.

THE ZAK TRANSFORM

Set

$$Q = [0, 1]^2 \quad \text{and} \quad E_{nk}(x, \xi) = e^{2\pi i n x} e^{-2\pi i k \xi}.$$

Then $\{E_{nk}\}_{k,n \in \mathbb{Z}}$ is an ONB for $L^2(Q)$.

Definition 1. The *Zak transform* is the unique unitary map $Z: L^2(\mathbb{R}) \rightarrow L^2(Q)$ that satisfies

$$Z(M_n T_k \chi_{[0,1]}) = E_{nk}, \quad k, n \in \mathbb{Z}. \quad \diamond$$

This definition easily gives us the *existence* of Z , but we will see another “more generalizable” definition shortly.

Discovered by:

- Gel’fand, 1950 (hence *Gel’fand mapping*)
- Weil, 1964 (on LCA groups)
- Brezin, 1970 (hence *Weil–Brezin mapping*)
- Zak, 1967 (who called it the *k-q mapping*)

In applied math and signal processing, “Zak transform” has become customary. Influential survey paper by Janssen in 1988.

Theorem 2. If $f \in L^2(\mathbb{R})$, then

$$Zf(x, \xi) = \sum_{j \in \mathbb{Z}} f(x - j) e^{2\pi i j \xi}, \quad (x, \xi) \in Q = [0, 1]^2, \quad (1)$$

where this series converges unconditionally in the norm of $L^2(Q)$.

Proof. Direct calculation: $\{f(x - j) e^{2\pi i j \xi}\}_{j \in \mathbb{Z}}$ is an orthogonal sequence in $L^2(Q)$. Hence

$$\begin{aligned} \left\| \sum_{j \in \mathbb{Z}} f(x - j) e^{2\pi i j \xi} \right\|_{L^2(Q)}^2 &= \sum_{j \in \mathbb{Z}} \|f(x - j) e^{2\pi i j \xi}\|_{L^2(Q)}^2 \quad (\text{Pythagoras}) \\ &= \sum_{j \in \mathbb{Z}} \int_0^1 \int_0^1 |f(x - j) e^{2\pi i j \xi}|^2 dx d\xi \\ &= \sum_{j \in \mathbb{Z}} \int_0^1 |f(x - j)|^2 dx = \|f\|_2^2. \end{aligned}$$

Hence the series in (1) converges (more precisely, a similar argument shows the partial sums are Cauchy, regardless of ordering). Therefore Z defined by (1) is an isometry on $L^2(Q)$. Also,

$$\begin{aligned} Z(M_n T_k \chi_{[0,1]})(x, \xi) &= \sum_{j \in \mathbb{Z}} M_n T_k \chi_{[0,1]}(x - j) e^{2\pi i j \xi} \\ &= \sum_{j \in \mathbb{Z}} e^{2\pi i n(x-j)} \chi_{[0,1]}(x - j - k) e^{2\pi i j \xi} \\ &= e^{2\pi i n(x+k)} e^{-2\pi i k \xi} = e^{2\pi i n x} e^{-2\pi i k \xi} = E_{nk}(x, \xi). \end{aligned}$$

Hence the isometry given by (1) equals the Zak transform. □

Theorem 3. Let $\chi = \chi_{[0,1]}$. Let $C: L^2(\mathbb{R}) \rightarrow \ell^2(\mathbb{Z}^2)$ be the coefficient mapping with respect to the ONB

$$\mathcal{G}(\chi, 1, 1) = \{M_n T_k \chi\}_{k,n \in \mathbb{Z}},$$

i.e.,

$$Cf = \{\langle f, M_n T_k \chi \rangle\}_{k,n \in \mathbb{Z}}, \quad f \in L^2(\mathbb{R}).$$

Let $\mathcal{F}: \ell^2(\mathbb{Z}^2) \rightarrow L^2(Q)$ be the Fourier transform on the group \mathbb{Z}^2 :

$$(\mathcal{F}c)(x, \xi) = \sum_{k,n \in \mathbb{Z}} c_{kn} E_{nk}(x, \xi) = \sum_{k,n \in \mathbb{Z}} c_{kn} e^{2\pi i n x} e^{-2\pi i k \xi}, \quad c = (c_{kn})_{k,n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}^2).$$

Then $Z: L^2(\mathbb{R}) \rightarrow L^2(Q)$ is

$$Z = \mathcal{F}C.$$

Proof.

$$\mathcal{F}Cf = \mathcal{F}(\{\langle f, M_n T_k \chi \rangle\}_{k,n \in \mathbb{Z}}) = \sum_{k,n \in \mathbb{Z}} \langle f, M_n T_k \chi \rangle E_{nk}.$$

This is precisely the unitary map that sends $M_n T_k \chi$ to E_{nk} , so it is the Zak transform. \square

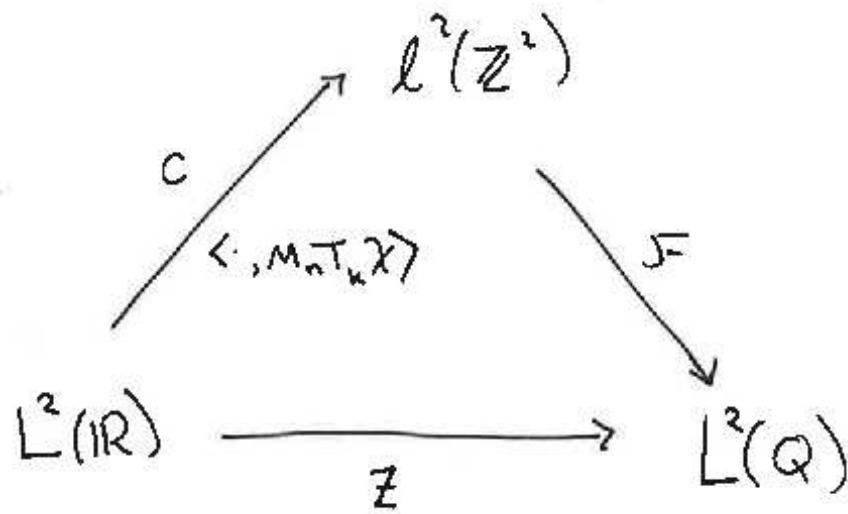


FIGURE 1. Factorization of the Zak transform ($Q = [0, 1]^2$).

Theorem 4. (a) If $1 \leq p \leq \infty$ then for each $f \in W(L^p, \ell^1)$ the series

$$Zf(x, \xi) = \sum_{j \in \mathbb{Z}} f(x - j) e^{2\pi i j \xi}, \quad (x, \xi) \in Q, \quad (2)$$

converges absolutely in $L^p(Q)$, and Z is a bounded mapping of $W(L^p, \ell^1)$ into $L^p(Q)$.

(b) For each $f \in W(C, \ell^1)$ the series in (2) converges absolutely in $C(Q)$ with respect to the uniform norm, and Z is a bounded mapping of $W(C, \ell^1)$ into $C(Q)$.

Proof. (a) If $f \in W(L^p, \ell^1)$ with p finite then

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \|f(x - j) e^{2\pi i j \xi}\|_{L^p(Q)} &= \sum_{j \in \mathbb{Z}} \left(\int_0^1 \int_0^1 |f(x - j) e^{2\pi i j \xi}|^p dx d\xi \right)^{1/p} \\ &= \sum_{j \in \mathbb{Z}} \|f \cdot \chi_{[j, j+1]}\|_p \\ &= \|f\|_{W(L^p, \ell^1)} < \infty, \end{aligned}$$

so the series defining Zf converges absolutely in $L^p(Q)$. A similar calculation holds if $p = \infty$, and these calculations show that $\|Zf\|_{L^p(Q)} \leq \|f\|_{W(L^p, \ell^1)}$.

(b) If $f \in W(C, \ell^1) \subseteq W(L^\infty, \ell^1)$ then $Zf \in L^\infty(Q)$ by part (a), and the series defining Zf converges absolutely in the uniform norm. As each term $f(x - j) e^{2\pi i j \xi}$ is continuous on Q and the uniform limit of continuous functions is continuous, Zf is continuous on Q . \square

Summary: Using the definition

$$Zf(x, \xi) = \sum_{j \in \mathbb{Z}} f(x - j) e^{2\pi i j \xi}, \quad (x, \xi) \in Q, \quad (3)$$

we have

- $Z: L^2(\mathbb{R}) \rightarrow L^2(Q)$ is unitary,
- $Z: W(L^p, \ell^1) \rightarrow L^p(Q)$ is bounded for $1 \leq p \leq \infty$, and
- $Z: W(C, \ell^1) \rightarrow C(Q)$ is bounded.

Assignment 1 (Quasiperiodicity). Given $f \in L^2(\mathbb{R})$ or $f \in W(L^p, \ell^1)$, use equation (3) to extend the definition of $Zf(x, \xi)$ to $(x, \xi) \in \mathbb{R}^2$. Then

$$Zf(x + m, \xi + n) = e^{2\pi i m \xi} Zf(x, \xi), \quad m, n \in \mathbb{Z},$$

where the equality holds pointwise everywhere on \mathbb{R}^2 if Zf is continuous, and almost everywhere otherwise. \diamond

Consequently, if $f \in W(C, \ell^1)$, then Zf is *continuous* and “*quasiperiodic*.”

It is easy to create *periodic* functions that are continuous. For example, a constant function is both periodic and continuous on \mathbb{R}^2 . But what kind of functions are both *quasiperiodic* and continuous? Is a constant function quasiperiodic on \mathbb{R}^2 ?

The constant function is continuous on $[0, 1)^2$, but cannot be extended to a quasiperiodic function that is continuous on $Q = [0, 1]^2$.

A quasiperiodic function is continuous on Q (with boundary) if and only if it is continuous on \mathbb{R}^2 . Such functions do exist, but they all have an interesting property.

Theorem 5. A quasiperiodic function F that is continuous on Q must vanish at some point of Q .

Proof. We give (with some handwaving) an argument due to Janssen. Suppose F is continuous, quasiperiodic, and everywhere nonzero on \mathbb{R}^2 . Then for each fixed $x \in \mathbb{R}$, the function

$$F_x(\xi) = F(x, \xi), \quad \xi \in \mathbb{R},$$

is continuous, 1-periodic, and nonzero on \mathbb{R} . As ξ varies from 0 to 1, the values $F_x(\xi)$ trace out a closed curve J_x in the complex plane that never intersects the origin. Such a curve has a well-defined *winding number* N_x that is an integer representing the total number of times the curve J_x travels counterclockwise around the origin.

Since F is continuous, the curves J_x deform continuously as we vary x . Further, since

$$F_1(\xi) = F(1, \xi) = e^{2\pi i \xi} F(0, \xi) = e^{2\pi i \xi} F_0(\xi),$$

the curve J_1 winds one more time around the origin than does J_0 .

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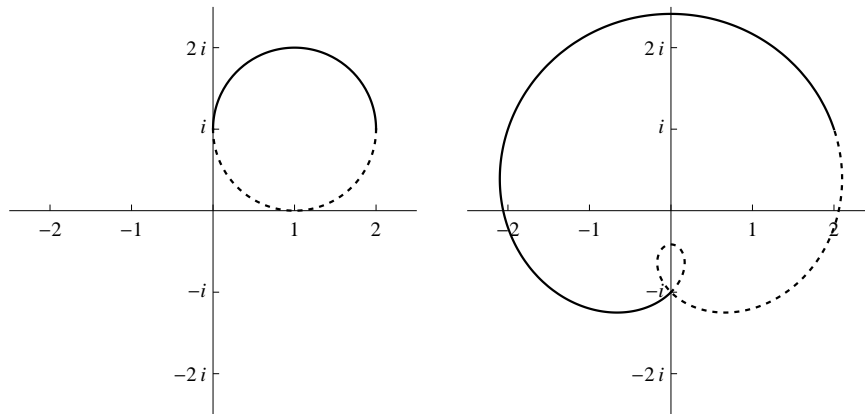


FIGURE 2. Plots of the complex-valued functions $F_0(\xi) = 1+i+e^{2\pi i \xi}$ and $F_1(\xi) = e^{2\pi i \xi} F_0(\xi)$ for $0 \leq \xi \leq 1$. The graph is shown as a solid line for $0 \leq \xi \leq 1/2$, and as a dashed line for $1/2 \leq \xi \leq 1$. The winding number of the left-hand graph is zero, while it is one for the right-hand graph.

However, there is no way to continuously deform a curve that winds N_0 times around the origin into one that winds $N_1 = N_0 + 1$ times around the origin without having the curve pass through the origin at some time. Hence there must be at least one value of x such that the curve J_x passes through the origin, which says that $F(x, \xi) = 0$ for some ξ . \square

GABOR SYSTEMS AT THE CRITICAL DENSITY

Gabor system at the critical density: $\mathcal{G}(g, 1, 1) = \{M_n T_k g\}_{k, n \in \mathbb{Z}}$. (Typo in printouts.)

Theorem 6. If $g \in L^2(\mathbb{R})$, then

$$Z(M_n T_k g) = E_{nk} \cdot Zg \text{ a.e.}, \quad k, n \in \mathbb{Z},$$

where $E_{nk}(x, \xi) = e^{2\pi i n x} e^{-2\pi i k \xi}$.

Proof. For integer k and n ,

$$\begin{aligned} Z(M_n T_k g)(x, \xi) &= \sum_{j \in \mathbb{Z}} (M_n T_k g)(x - j) e^{2\pi i j \xi} \\ &= \sum_{j \in \mathbb{Z}} e^{2\pi i n(x-j)} g(x - k - j) e^{2\pi i j \xi} \\ &= \sum_{j \in \mathbb{Z}} e^{2\pi i n(x-j+k)} g(x - j) e^{2\pi i(j-k)\xi} \\ &= e^{2\pi i n x} e^{-2\pi i k \xi} \sum_{j \in \mathbb{Z}} g(x - j) e^{2\pi i j \xi} \quad (e^{-2\pi i n j} = e^{2\pi i n k} = 1) \\ &= E_{nk}(x, \xi) Zg(x, \xi). \end{aligned}$$

The series above converge in $L^2(Q)$, not pointwise, but this does not affect the calculation. \square

Consequently,

$$Z(\mathcal{G}(g, 1, 1)) = \{Z(M_n T_k g)\}_{k,n \in \mathbb{Z}} = \{E_{nk} \cdot Zg\}_{k,n \in \mathbb{Z}}.$$

Note that $\{E_{nk}\}_{k,n \in \mathbb{Z}}$ is an ONB for $L^2(Q)$! What happens when you take an ONB and multiply each element by a single function?

Since Z is unitary,

$\mathcal{G}(g, 1, 1)$ is complete in $L^2(\mathbb{R}) \iff Z(\mathcal{G}(g, 1, 1)) = \{E_{nk} \cdot Zg\}_{k,n \in \mathbb{Z}}$ is complete in $L^2(Q)$,

and similarly for the properties

- minimal,
- exact,
- frame,
- Riesz basis,
- ONB.

When does $\{E_{nk} \cdot Zg\}_{k,n \in \mathbb{Z}}$ have these properties?

$$Z(\mathcal{G}(g, 1, 1)) = \{E_{nk} \cdot Zg\}_{k,n \in \mathbb{Z}}.$$

Theorem 7. Let $g \in L^2(\mathbb{R})$ be fixed.

- (a) $\mathcal{G}(g, 1, 1)$ is **complete** in $L^2(\mathbb{R})$ if and only if $Zg \neq 0$ a.e.
- (b) $\mathcal{G}(g, 1, 1)$ is **minimal** in $L^2(\mathbb{R})$ if and only if $1/Zg \in L^2(Q)$. In this case, $\mathcal{G}(g, 1, 1)$ is exact and its biorthogonal system is $\mathcal{G}(\tilde{g}, 1, 1)$ where $\tilde{g} \in L^2(\mathbb{R})$ satisfies $Z\tilde{g} = 1/\overline{Zg}$.
- (c) $\mathcal{G}(g, 1, 1)$ is a **Bessel sequence** in $L^2(\mathbb{R})$ if and only if $Zg \in L^\infty(\mathbb{R})$, and in this case $B = \|Zg\|_\infty^2$ is a Bessel bound.
- (d) $\mathcal{G}(g, 1, 1)$ is a **frame** for $L^2(\mathbb{R})$ if and only if $0 < A \leq |Zg(x, \xi)|^2 \leq B < \infty$ a.e. In this case $\mathcal{G}(g, 1, 1)$ is a Riesz basis and A, B are frame bounds.
- (e) $\mathcal{G}(g, 1, 1)$ is an **orthonormal basis** for $L^2(\mathbb{R})$ if and only if $|Zg(x, \xi)| = 1$ a.e. \diamond

Proof. (e) The claim is that

$$\mathcal{G}(g, 1, 1) \text{ is an ONB} \iff |Zg| = 1 \text{ a.e.}$$

Recall that $Z(\mathcal{G}(g, 1, 1)) = \{E_{nk} \cdot Zg\}_{k,n \in \mathbb{Z}}$.

\Leftarrow . Assume $|Zg| = 1$ a.e. Then

$$\langle E_{nk} \cdot Zg, E_{n'k'} \cdot Zg \rangle = \int_0^1 \int_0^1 E_{nk} Zg \overline{E_{n'k'} Zg} = \int_0^1 \int_0^1 E_{nk} \overline{E_{n'k'}} = \langle E_{nk}, E_{n'k'} \rangle,$$

so $\{E_{nk} \cdot Zg\}_{k,n \in \mathbb{Z}}$ is ON. If $F \in L^2(Q)$, then

$$\begin{aligned} \sum_{k,n \in \mathbb{Z}} |\langle F, E_{nk} \cdot Zg \rangle|^2 &= \sum_{k,n \in \mathbb{Z}} \left| \int_0^1 \int_0^1 F(x, \xi) \overline{E_{nk}(x, \xi) Zg(x, \xi)} dx d\xi \right|^2 \\ &= \sum_{k,n \in \mathbb{Z}} |\langle F \cdot \overline{Zg}, E_{nk} \rangle|^2 \\ &= \|F \cdot \overline{Zg}\|_2^2 \\ &= \|F\|_2^2. \end{aligned}$$

The Plancherel Equality holds, so $\{E_{nk} \cdot Zg\}_{k,n \in \mathbb{Z}}$ is an ON tight frame, and hence is an ONB.

(c) The claim is that

$$\mathcal{G}(g, 1, 1) \text{ is Bessel} \iff |Zg|^2 \leq B \text{ a.e.}$$

Recall that $Z(\mathcal{G}(g, 1, 1)) = \{E_{nk} \cdot Zg\}_{k,n \in \mathbb{Z}}$.

\Rightarrow . Assume Bessel with Bessel bound B , and choose $F \in L^2(Q)$. Then

$$\|F \cdot \overline{Zg}\|_2^2 = \sum_{k,n \in \mathbb{Z}} |\langle F \cdot \overline{Zg}, E_{nk} \rangle|^2 = \sum_{k,n \in \mathbb{Z}} |\langle F, E_{nk} \cdot Zg \rangle|^2 \leq B \|F\|_2^2.$$

Therefore

$$\begin{aligned} \int_0^1 \int_0^1 (B - |Zg|^2) |F|^2 &= \int_0^1 \int_0^1 B |F|^2 - \int_0^1 \int_0^1 |Zg|^2 |F|^2 \\ &= B \|F\|_2^2 - \|F \cdot \overline{Zg}\|_2^2 \geq 0. \end{aligned}$$

Since this is true for every F , we must have $B - |Zg|^2 \geq 0$ a.e.

(a) The claim is that

$$\mathcal{G}(g, 1, 1) \text{ is complete} \iff Zg \neq 0 \text{ a.e.}$$

Complete means the closed span is dense; equivalently, only the zero function is orthogonal to every element. Recall that $Z(\mathcal{G}(g, 1, 1)) = \{E_{nk} \cdot Zg\}_{k,n \in \mathbb{Z}}$.

\Leftarrow . Suppose $Zg \neq 0$ a.e. and $F \in L^2(Q)$ satisfies $F \perp E_{nk} Zg$ for all k, n . Let $G = F \cdot \overline{Zg}$. Then $G \in L^1(Q)$, and its Fourier coefficients are

$$\begin{aligned} \widehat{G}(n, k) &= \langle G, E_{nk} \rangle_{L^2(Q)} = \langle F \cdot \overline{Zg}, E_{nk} \rangle_{L^2(Q)} \\ &= \int_0^1 \int_0^1 F(x, \xi) \overline{Zg(x, \xi)} \overline{E_{nk}(x, \xi)} dx d\xi \\ &= \langle F, E_{nk} \cdot Zg \rangle_{L^2(Q)} = 0. \end{aligned}$$

But *functions in $L^1(Q)$ are uniquely determined by their Fourier coefficients*. Therefore

$$G = F \cdot \overline{Zg} = 0 \text{ a.e.}$$

As $Zg \neq 0$ a.e., this implies $F = 0$ a.e. □

Assignment 2. Prove the remaining implications. ◇

THE BLT (sorry, did we have lunch yet?)

By earlier results, if $\mathcal{G}(g, a, b)$ is a Riesz basis for $L^2(\mathbb{R})$, then $ab = 1$ (critical density), and by dilating g we can take $a = b = 1$. Moreover, here at the critical density we also know that

$$\mathcal{G}(g, 1, 1) \text{ is a frame} \iff \mathcal{G}(g, 1, 1) \text{ is a Riesz basis.}$$

Theorem 8 (Amalgam BLT; H, 1990). If $\mathcal{G}(g, 1, 1)$ is a Riesz basis for $L^2(\mathbb{R})$ then $g \notin W(C, \ell^1)$. Specifically, either

$$g \text{ is not continuous} \quad \text{or} \quad \sum_{k \in \mathbb{Z}} \|g \cdot \chi_{[k, k+1]}\|_{\infty} = \infty.$$

Likewise, $\widehat{g} \notin W(C, \ell^1)$.

Proof. We proved earlier that if $g \in W(C, \ell^1)$, then $Zg \in C(Q)$. Therefore Zg is continuous on Q , so it must have a zero. Consequently we do not have $|Zg| \geq A > 0$ a.e., so $\mathcal{G}(g, 1, 1)$ does not have a positive lower frame bound. The same reasoning transfers to \widehat{g} by using the fact that

$$(M_n T_k g)^\wedge = T_n M_{-k} \widehat{g} = e^{2\pi i k n} M_{-k} T_n \widehat{g} = M_{-k} T_n \widehat{g},$$

and therefore

$$(\mathcal{G}(g, 1, 1))^\wedge = \mathcal{G}(\widehat{g}, 1, 1). \quad \square$$

Theorem 9 (Classical BLT). If $\mathcal{G}(g, 1, 1)$ is a Riesz basis for $L^2(\mathbb{R})$ then

$$\left(\int_{-\infty}^{\infty} |xg(x)|^2 dx \right) \left(\int_{-\infty}^{\infty} |\xi \widehat{g}(\xi)|^2 d\xi \right) = \infty. \quad \diamond$$

Compare the classical BLT to the following classical result.

Theorem 10 (Classical Uncertainty Principle). If $g \in L^2(\mathbb{R})$, then

$$\left(\int_{-\infty}^{\infty} |xg(x)|^2 dx \right) \left(\int_{-\infty}^{\infty} |\xi \widehat{g}(\xi)|^2 d\xi \right) \geq \frac{1}{4\pi} \int_{-\infty}^{\infty} |g(x)|^2 dx. \quad \diamond$$

Hence the generator of a Gabor Riesz basis “maximizes uncertainty.”

The proof of the Classical BLT has a long and involved history. The proofs given (independently) by Battle and Low each contained a “gap”, which was filled by Coifman, Semmes, and Daubechies. Battle gave an elegant proof for Gabor ONBs; Daubechies and Janssen extended this to a proof for Gabor Riesz bases. Benedetto/H/Walnut gave modified proofs that avoid the use of distributional calculations (such as distributional differentiation).

We give Battle's elegant proof for Gabor ONBs.

Proof. In mathematical terms, the quantum mechanics operators of position and momentum are

$$Pf(x) = xf(x) \quad \text{and} \quad Mf = \frac{1}{2\pi i} f' = (P\widehat{f})^\vee = (\xi\widehat{f}(\xi))^\vee.$$

These are unbounded operators on $L^2(\mathbb{R})$, defined not on the entire space but only on dense subspaces. Suppose $g \in L^2(\mathbb{R})$ and $\mathcal{G}(g, 1, 1)$ is an ONB, and both $\int |xg(x)|^2 dx$ and $\int |\xi\widehat{g}(\xi)|^2 d\xi$ are finite, i.e., $Pg \in L^2(\mathbb{R})$ and $P\widehat{g} \in L^2(\mathbb{R})$. Then

$$\begin{aligned} \langle Pg, M_n T_k g \rangle &= \int_{-\infty}^{\infty} xg(x) e^{-2\pi i n x} \overline{g(x-k)} dx = \langle g, P M_n T_k g \rangle \\ &= \int_{-\infty}^{\infty} g(x) \overline{e^{2\pi i n x} (x-k) g(x-k)} dx + k \int_{-\infty}^{\infty} g(x) \overline{e^{2\pi i n x} g(x-k)} dx \\ &= \langle g, M_n T_k P g \rangle + k \langle g, M_n T_k g \rangle \\ &= \langle g, M_n T_k P g \rangle + 0 \quad (\text{since } \langle g, M_n T_k g \rangle = \delta_{0k} \delta_{0n}) \\ &= \langle T_{-k} M_{-n} g, P g \rangle = \langle M_{-n} T_{-k} g, P g \rangle \quad (M_n \text{ and } T_k \text{ commute because } ab = 1!). \end{aligned}$$

A similar calculation (switch to the Fourier side) shows that

$$\langle M g, M_n T_k g \rangle = \langle M_{-n} T_{-k} g, M g \rangle.$$

Expand Pg and Mg in the ONB:

$$\begin{aligned}
\langle Mg, Pg \rangle &= \left\langle \sum_{k,n \in \mathbb{Z}} \langle Mg, M_n T_k g \rangle M_n T_k g, Pg \right\rangle \\
&= \sum_{k,n \in \mathbb{Z}} \langle Mg, M_n T_k g \rangle \langle M_n T_k g, Pg \rangle \\
&= \sum_{k,n \in \mathbb{Z}} \langle M_{-n} T_{-k} g, Mg \rangle \langle Pg, M_{-n} T_{-k} g \rangle \\
&= \sum_{k,n \in \mathbb{Z}} \langle Pg, M_n T_k g \rangle \langle M_n T_k g, Mg \rangle = \langle Pg, Mg \rangle.
\end{aligned}$$

Remark: Because P and M are self-adjoint, if we restrict to nice enough g then this implies that the commutator $[P, M] = PM - MP$ satisfies

$$\langle [P, M]g, g \rangle = \langle (PM - MP)g, g \rangle = \langle PMg, g \rangle - \langle MPg, g \rangle = \langle Mg, Pg \rangle - \langle Pg, Mg \rangle = 0.$$

But this cannot be right—we'll show that P and M do not commute.

By integrating by parts (valid for absolutely continuous functions),

$$\begin{aligned}
\langle Mg, Pg \rangle &= \frac{1}{2\pi i} \lim_{\substack{a \rightarrow -\infty \\ b \rightarrow \infty}} \int_a^b g'(x) \overline{xg(x)} dx \\
&= \frac{1}{2\pi i} \lim_{\substack{a \rightarrow -\infty \\ b \rightarrow \infty}} \int_a^b (xg'(x) + g(x) - g(x)) \overline{g(x)} dx \\
&= \frac{1}{2\pi i} \lim_{\substack{a \rightarrow -\infty \\ b \rightarrow \infty}} \left(b|g(b)|^2 - a|g(a)|^2 - \int_a^b xg(x) \overline{g'(x)} dx - \int_a^b |g(x)|^2 dx \right) \\
&= \int_{-\infty}^{\infty} Pg(x) \overline{Mg(x)} dx - \frac{1}{2\pi i} \|g\|_{L^2}^2 \\
&= \langle Pg, Mg \rangle - \frac{1}{2\pi i}.
\end{aligned}$$

This is a contradiction. □

Remark: The calculation above shows that, at least when we restrict the domain to a sufficiently nice subset of $L^2(\mathbb{R})$,

$$[P, M] = PM - MP = -\frac{1}{2\pi i}I.$$

Position and momentum do not commute.

BEYOND THE CRITICAL DENSITY

Suppose $g \in W(L^\infty, \ell^1)$ and $\mathcal{G}(g, a, b)$ is a Parseval frame ($A = B = 1$). Then:

- $ab \leq 1$,
- the frame operator is $S = I$,
- the Walnut representation of the frame operator is

$$Sf = b^{-1} \sum_{n \in \mathbb{Z}} G_n \cdot T_{\frac{n}{b}} f, \quad f \in L^2(\mathbb{R}),$$

where this series converges absolutely in L^2 -norm,

- the autocorrelation functions G_n are bounded and a -periodic (and can be expressed explicitly).

What else can we determine about g ?

Suppose f and h are bounded and supported in $[0, b^{-1}]$. Then

$$\langle T_{\frac{\ell}{b}} f, T_{\frac{m}{b}} h \rangle = \delta_{\ell m} \langle f, h \rangle.$$

But $S = I$, so

using $f = Sf = b^{-1} \sum_n G_n \cdot T_{\frac{n}{b}} f$,

$$\begin{aligned}
\delta_{\ell m} \langle f, h \rangle &= \langle T_{\frac{\ell}{b}} f, T_{\frac{m}{b}} h \rangle = \langle ST_{\frac{\ell}{b}} f, T_{\frac{m}{b}} h \rangle \\
&= b^{-1} \sum_{n \in \mathbb{Z}} \langle G_n \cdot T_{\frac{n}{b}} (T_{\frac{\ell}{b}} f), T_{\frac{m}{b}} h \rangle \\
&= b^{-1} \sum_{n \in \mathbb{Z}} \langle G_n \cdot T_{\frac{n+\ell}{b}} f, T_{\frac{m}{b}} h \rangle \\
&= b^{-1} \langle G_{m-\ell} \cdot T_{\frac{m}{b}} f, T_{\frac{m}{b}} h \rangle \quad (\text{only nonzero term is } n + \ell = m) \\
&= b^{-1} \langle T_{-\frac{m}{b}} (G_{m-\ell} \cdot T_{\frac{m}{b}} f), h \rangle \\
&= b^{-1} \langle T_{-\frac{m}{b}} G_{m-\ell} \cdot f, h \rangle.
\end{aligned}$$

This holds for bounded $f, h \in L^2[0, b^{-1}]$, but extends by density to all $f, h \in L^2[0, b^{-1}]$.

Consequently

$$\delta_{\ell m} f = b^{-1} T_{-\frac{m}{b}} G_{m-\ell} \cdot f, \quad \text{all } f \in L^2[0, b^{-1}],$$

and hence

$$\delta_{\ell m} = b^{-1} T_{-\frac{m}{b}} G_{m-\ell} \quad \text{on } [0, b^{-1}].$$

$$\delta_{\ell m} = b^{-1} T_{-\frac{m}{b}} G_{m-\ell} \quad \text{on } [0, b^{-1}].$$

Varying m and ℓ , we obtain

$$G_0 = b \quad \text{and} \quad G_n = 0 \text{ for } n \neq 0.$$

Now remember that G_n is the bounded, a -periodic function

$$G_n = \sum_{k \in \mathbb{Z}} T_{ak} g \cdot T_{ak + \frac{n}{b}} \bar{g} = \sum_{k \in \mathbb{Z}} T_{ak} (g \cdot T_{\frac{n}{b}} \bar{g}).$$

Since $\{a^{-1} e^{2\pi i k x / a}\}_{k \in \mathbb{Z}}$ is an ONB for $L^2[0, a]$ and $G_n \in L^\infty[0, a] \subseteq L^2[0, a]$, we can also write G_n as a Fourier series:

$$G_n(x) = a^{-1} \sum_{k \in \mathbb{Z}} \widehat{G}_n(k) e^{2\pi i k x / a},$$

where

$$\widehat{G}_n(k) = a^{-1} \int_0^a G_n(x) e^{-2\pi i k x / a} dx.$$

But $G_0 = b$ and $G_n = 0$ for $n \neq 0$, so

$$\widehat{G}_n(k) = b \delta_{k0} \delta_{n0}.$$

Therefore,

$$\begin{aligned}
b \delta_{k0} \delta_{n0} &= \widehat{G}_n(k) = a^{-1} \int_0^a G_n(x) e^{-2\pi i k x/a} dx \\
&= a^{-1} \sum_{k \in \mathbb{Z}} \int_0^a T_{ak}(g \cdot T_{\frac{n}{b}} \bar{g})(x) e^{-2\pi i k x/a} dx \\
&= a^{-1} \sum_{k \in \mathbb{Z}} \int_0^a (g \cdot T_{\frac{n}{b}} \bar{g})(x - ak) e^{-2\pi i k(x-ak)/a} dx \\
&= a^{-1} \int_{-\infty}^{\infty} (g \cdot T_{\frac{n}{b}} \bar{g})(x) e^{-2\pi i k x/a} dx \\
&= a^{-1} \int_{-\infty}^{\infty} g \cdot \overline{M_{\frac{k}{a}} T_{\frac{n}{b}} g} \\
&= a^{-1} \langle g, M_{\frac{k}{a}} T_{\frac{n}{b}} g \rangle.
\end{aligned}$$

Thus, if $g \in W(L^\infty, \ell^1)$ and $\mathcal{G}(g, a, b)$ is a Parseval frame (redundant if $ab < 1!$), then

$$\langle g, M_{\frac{k}{a}} T_{\frac{n}{b}} g \rangle = (ab) \delta_{k0} \delta_{n0},$$

which says that

$$(ab)^{-1} \mathcal{G}(g, \frac{1}{b}, \frac{1}{a}) = \{(ab)^{-1} M_{\frac{k}{a}} T_{\frac{n}{b}} g\}_{k,n \in \mathbb{Z}} \text{ is ORTHONORMAL.}$$

More generally, we have the following result.

Theorem 11 (Wexler–Raz Biorthogonality Relations). Given $g, \gamma \in W(L^\infty, \ell^1)$, the following two statements are equivalent.

- $S_{g,\gamma} = S_{\gamma,g} = I$ on $L^2(\mathbb{R})$, where $S_{g,\gamma}f = \sum_{k,n \in \mathbb{Z}} \langle f, M_{bn}T_{ak}g \rangle M_{bn}T_{ak}g$.
- $\langle \gamma, M_{\frac{k}{a}}T_{\frac{n}{b}}g \rangle = ab \delta_{k0} \delta_{n0}$. \diamond

In particular, if $\mathcal{G}(g, a, b)$ is a frame, $\gamma = \tilde{g} = S^{-1}$ is the dual window, and g, γ belong to $W(L^\infty, \ell^1)$, then (except for normalization),

$$\mathcal{G}(g, \frac{1}{b}, \frac{1}{a}) \text{ and } \mathcal{G}(\tilde{g}, \frac{1}{b}, \frac{1}{a}) \text{ are biorthogonal.}$$

We call

$$\Lambda^\circ = \frac{1}{b}\mathbb{Z} \times \frac{1}{a}\mathbb{Z}$$

the *adjoint lattice* of $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$.

The hypothesis that g and γ both belong to $W(L^\infty, \ell^1)$ can be weakened. Essentially, Wexler–Raz holds whenever $\mathcal{G}(g, a, b)$ and $\mathcal{G}(\gamma, a, b)$ are both Bessel.

Even more is true!

Theorem 12 (The Duality Principle). If $g \in L^2(\mathbb{R})$ and $a, b > 0$, then the following two statements are equivalent.

- $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$.
- $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is a *Riesz sequence* (a Riesz basis for its closed span). \diamond

Note that we must necessarily have $ab \leq 1$. If $ab < 1$, then the frame $\mathcal{G}(g, a, b)$ is redundant, and since $(ab)^{-1} > 1$ the Riesz sequence $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ *cannot* be complete.

Theorem 12 was proved independently and essentially simultaneously (circa 1995), with three very different proofs, by

- Janssen,
- Ron and Shen,
- Daubechies, Landau, and Landau.

PROBLEMS ON GABOR SYSTEMS

Numbers correspond to “A Basis Theory Primer.”

11.22. Given $f \in L^1(\mathbb{R}) = W(L^1, \ell^1)$, we know $Zf \in L^1(Q)$. Prove the following.

(a) $f(x) = \int_0^1 Zf(x, \xi) d\xi$ for almost every $x \in \mathbb{R}$.

(b) If Zf is continuous, then f is continuous.

(c) Z is an injective mapping of $L^1(\mathbb{R})$ into $L^1(Q)$, and the range of $Z: L^1(\mathbb{R}) \rightarrow L^1(Q)$ is a proper, dense subspace of $L^1(Q)$.

(d) $Z^{-1}: L^1(Q) \rightarrow L^1(\mathbb{R})$ is unbounded.

11.29. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space. Show that the position and momentum operators map $\mathcal{S}(\mathbb{R})$ into itself, and are self-adjoint when restricted to this domain, i.e., for $f, g \in \mathcal{S}(\mathbb{R})$ we have

$$\langle Pf, g \rangle = \langle f, Pg \rangle \quad \text{and} \quad \langle Mf, g \rangle = \langle f, Mg \rangle.$$

11.31. This exercise gives an operator-theoretic version of the Classical Uncertainty Principle. Let S be a subspace of a Hilbert space H , and let $A, B: S \rightarrow H$ be linear but possibly unbounded operators. By replacing S with the smaller space $\text{domain}(AB) \cap \text{domain}(BA)$ if necessary, we may assume that A, B, AB , and BA are all defined on S .

(a) Show that if A, B are self-adjoint in the sense that

$$\forall f, g \in S, \quad \langle Af, g \rangle = \langle f, Ag \rangle \quad \text{and} \quad \langle Bf, g \rangle = \langle f, Bg \rangle,$$

then

$$\forall f \in S, \quad \|Af\| \|Bf\| \geq \frac{1}{2} |\langle [A, B]f, f \rangle|,$$

where $[A, B] = AB - BA$ is the commutator of A and B .

(b) Show that equality holds in part (a) if and only if $Af = icBf$ for some $c \in \mathbb{R}$.

11.32. Apply Exercises 11.29–11.31 to the position and momentum operators P and M to derive the Classical Uncertainty Principle for functions $g \in \mathcal{S}(\mathbb{R})$.

Remark: Extension by density or integration by parts for absolutely continuous functions can be used to extend to all g such that $\|xg(x)\|_{L^2} \|\xi \widehat{g}(\xi)\|_{L^2}$ is finite.