# FRAMES AND TIME-FREQUENCY ANALYSIS

## LECTURE 1: FRAMES

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## READING

**For background on Hilbert spaces and operator theory:**

**Chapters 1–2 in C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.**

**For background on the Fourier transform:**

**Chapter 9 in C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.**

**A short general survey article on frames:**

**C. Heil,** *WHAT IS . . . a frame?*, **Notices Amer. Math. Soc., 60 (2013), pp. 748–750.**

**Today's lecture is based upon:**

**Chapter 7, Chapter 8 (Sections 8.1-8.5) in C. Heil, *A Basis Theory Primer*, Birkhäuser, Boston, 2011.**

**Also see:**

**C. E. Heil and D. F. Walnut,** *Continuous and discrete wavelet transforms*, **SIAM Review, 31 (1989), pp. 628–666.**

**For an introduction to frames in *finite dimensions*:**

**D. Han, K. Kornelson, D. Larson, and E. Weber, *Frames for Undergraduates*, AMS, Providence, 2007.**

**Theorem 1.** Let $H$ be a Hilbert space, let $\mathcal{E} = \{e_n\}_{n \in \mathbb{N}}$ be an orthonormal sequence in $H$, and set

$$M \;=\; \overline{\operatorname{span}}(\mathcal{E}) \;=\; \overline{\operatorname{span}}\{e_n\}_{n \in \mathbb{N}}.$$

Then the following statements hold.

(a) *Bessel's Inequality*: $$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \;\le\; \|x\|^2 \qquad \text{for all } x \in H.$$

(b) If the series $x = \sum_{n=1}^{\infty} c_n e_n$ converges, then $c_n = \langle x, e_n\rangle$ for each $n \in \mathbb{N}$.

(c) $\sum_{n=1}^{\infty} c_n e_n$ converges $\iff$ $\sum_{n=1}^{\infty} |c_n|^2 < \infty$.

(d) If $x = \sum_{n=1}^{\infty} c_n e_n$ converges, then it converges *unconditionally*, i.e., it converges regardless of the ordering of the index set.

(e) If $x \in H$, then: $$p \;=\; \sum_{n=1}^{\infty} \langle x, e_n\rangle\, e_n$$
is the orthogonal projection of $x$ onto $M$, and $\|p\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2$.

(f) If $x \in H$, then

$$x \in M \quad\iff\quad x = \sum_{n=1}^{\infty} \langle x, e_n\rangle\, e_n \quad\iff\quad \|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2. \qquad \diamondsuit$$

**Theorem 2.** If $H$ is a Hilbert space and $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence in $H$, then the following five statements are equivalent.

(a) $\{e_n\}_{n\in\mathbb{N}}$ is *complete*, i.e., $\overline{\text{span}}\{e_n\}_{n\in\mathbb{N}} = H$.

(b) $\{e_n\}_{n\in\mathbb{N}}$ is a *Schauder basis* for $H$, i.e., for each $x \in H$ there exists a *unique* sequence of scalars $(c_n)_{n\in\mathbb{N}}$ such that $x = \sum_{n=1}^{\infty} c_n e_n$.

(c) If $x \in H$, then

$$x \;=\; \sum_{n=1}^{\infty} \langle x, e_n\rangle\, e_n,$$

where this series converges in the norm of $H$ (in this case it converges unconditionally).

(d) *Plancherel's Equality*:

$$\|x\|^2 \;=\; \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \qquad \text{for all } x \in H.$$

(e) *Parseval's Equality*:

$$\langle x, y\rangle \;=\; \sum_{n=1}^{\infty} \langle x, e_n\rangle\, \langle e_n, y\rangle \qquad \text{for all } x, y \in H. \qquad \diamondsuit$$

**Benefits:** ***uniqueness, stability, unconditionality.***

**Let** $H = \mathbb{R}^2$**, and set**

$$x_1 = (1,0), \quad x_2 = (0,1), \quad x_3 = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \quad x_4 = \left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right).$$

**Each of** $\{x_1, x_2\}$ **and** $\{x_3, x_4\}$ **is an orthonormal basis for** $\mathbb{R}^2$**, so**

$$\sum_{n=1}^{4} |\langle x, x_n\rangle|^2 \;=\; 2\,\|x\|^2, \qquad x \in \mathbb{R}^2.$$

**Therefore**

$$\{2^{-1/2}x_1,\; 2^{-1/2}x_2,\; 2^{-1/2}x_3,\; 2^{-1/2}x_4\}$$

**satisfies the Plancherel Equality, but it is not orthogonal and is not a basis for** $\mathbb{R}^2$**.**

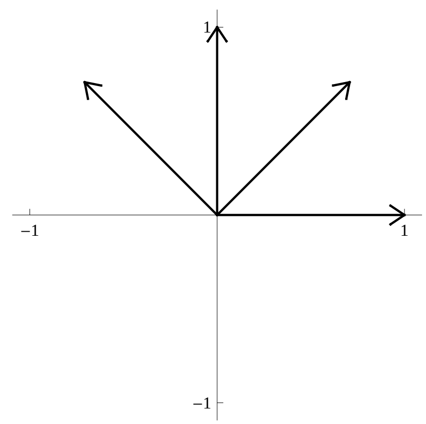


FIGURE 1. A union of two orthonormal bases.

**Let $H = \mathbb{R}^2$, and set**

$$x_1 = (0,1), \quad x_2 = \left(-\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\right), \quad x_3 = \left(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\right).$$

**Then $\sum_{n=1}^{3} |\langle x, x_n\rangle|^2 = \frac{3}{2}\,\|x\|^2$ for all $x \in \mathbb{R}^2$. Therefore, if we set $c = (2/3)^{1/2}$, then**

$$\{cx_1, cx_2, cx_3\}$$

**satisfies $\sum_{n=1}^{3} |\langle x, cx_n\rangle|^2 = \|x\|^2$, but it is not orthogonal, not a basis, and not a union of orthonormal bases.**

**(Can you see that the Mercedes frame is the orthogonal projection of an orthonormal basis for $\mathbb{R}^3$ onto a two-dimensional plane?)**

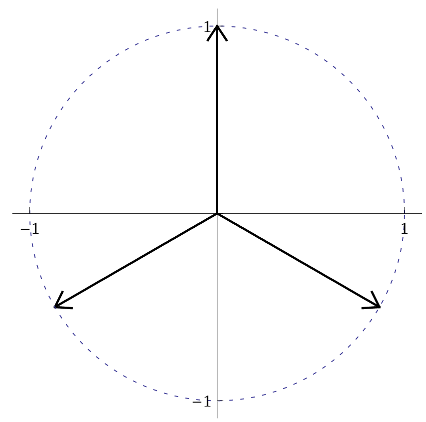


FIGURE 2. The three vectors of the Mercedes frame. A dashed unit circle is included for comparison, and also to motivate its alternative name (the *peace frame*).

**Plancherel Equality for an ONB:** $\displaystyle\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \;=\; \|x\|^2$, **all** $x \in H$.

**Definition 3** (Frame)**.** [Duffin and Schaeffer, 1952] A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space $H$ is a *frame* for $H$ if there exist constants $A$, $B > 0$ such that the following *pseudo-Plancherel formula* holds:

$$\forall\, x \in H, \quad A\,\|x\|^2 \;\le\; \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \;\le\; B\,\|x\|^2.$$

(a) $A$ and $B$ are called *frame bounds.* $A$ is a *lower frame bound*, and $B$ is an *upper frame bound.*

(b) The largest possible lower frame bound is the *optimal lower frame bound*, and the smallest possible upper frame bound is the *optimal upper frame bound.*

(c) The frame is *tight* if we can take $A = B$.

(d) It is a *Parseval frame* if the Plancherel Equality holds (i.e., we can take $A = B = 1$).

(e) It is an *exact frame* if it ceases to be a frame whenever any single element is deleted from the sequence. $\diamondsuit$

## EXAMPLES

**(a) ONB $\iff$ exact Parseval frame.**

**(b) Let $\{e_n\}_{n\in\mathbb{N}}$ be an ONB. Then $\{0, e_1, e_2, \dots\}$ is an inexact Parseval frame.**

**(c) $\{e_1, e_1, e_2, e_2, \dots\}$ is an inexact tight frame ($A = B = 2$).**

**(d) $\{e_1,\ e_2/\sqrt{2},\ e_2/\sqrt{2},\ e_3/\sqrt{3},\ e_3/\sqrt{3},\ e_3/\sqrt{3},\ \dots\}$ is an inexact Parseval frame, and no nonredundant subsequence is a frame.**

**Let** $L^2[0,1]$ **be the Lebesgue space of complex-valued, square-integrable functions on** $[0,1]$ **with inner product**

$$\langle f, g \rangle \;=\; \int_0^1 f(t)\,\overline{g(t)}\,dt.$$

**The *trigonomentric system*** 

$$\mathcal{E}_1 \;=\; \{e_n\}_{n\in\mathbb{Z}} \;=\; \{e^{2\pi int}\}_{n\in\mathbb{Z}}$$

**is an ONB for** $L^2[0,1]$. **Fix** $b > 0$ **and consider**

$$\mathcal{E}_b \;=\; \{e_{bn}\}_{n\in\mathbb{Z}} \;=\; \{e^{2\pi ibnt}\}_{n\in\mathbb{Z}},$$

**as a system in** $L^2[0,1]$ **(even though** $e_{bn}$ **is** $1/b$**-periodic, restrict the domain to** $[0,1]$**).**

***Case 1*:** $b > 1$. **Each element of** $\mathcal{E}_b$ **is** $1/b$**-periodic, and** $1/b < 1$. **We can construct a function** $f \in L^2[0,1]$ **such that**

$$f \perp e_{bn}, \qquad \text{all } n \in \mathbb{Z}.$$

**Hence** $\mathcal{E}_b$ **is *not complete*, i.e.,** $\overline{\mathrm{span}}(\mathcal{E}_b) \subsetneq L^2[0,1]$.

***Case 2*:** $b < 1$. **Each element of** $\mathcal{E}_b$ **is** $1/b$**-periodic, but** $1/b > 1$. **Think of** $L^2[0,1]$ **as a proper subspace of** $L^2[0,1/b]$ **(extend functions in** $L^2[0,1]$ **by zero). Let** $P$ **be the**

**orthogonal projection of $L^2[0,1/b]$ onto $L^2[0,1]$:**

$$Pf = f\,\chi_{[0,1]}, \qquad f \in L^2[0,1/b].$$

**Set**

$$\{f_{bn}\}_{n\in\mathbb{Z}} \;=\; \{b^{1/2}\,e_{bn}\}_{n\in\mathbb{Z}}, \qquad \textbf{domain} \;= [0,1/b].$$

**This is an ONB for $L^2[0,1/b]$ and**

$$b^{1/2}\mathcal{E}_b \;=\; \{b^{1/2}e_{bn}\}_{n\in\mathbb{Z}} \;=\; \{Pf_{bn}\}_{n\in\mathbb{Z}}, \qquad \textbf{domain} \;= [0,1],$$

**is the image of an ONB under an orthogonal projection. Given $f \in L^2[0,1]$, let $g \in L^2[0,1/b]$ be $f$ extended by zero to $[0,1/b]$. Then**

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |\langle f, e_{bn}\rangle_{L^2[0,1]}|^2 \;&=\; \sum_{n\in\mathbb{Z}} |\langle f, e_{bn}\rangle_{L^2[0,1]}|^2 \;=\; \sum_{n\in\mathbb{Z}} \left|\int_0^1 f(t)\,\overline{e_{bn}(t)}\,dt\right|^2 \\
&= \sum_{n\in\mathbb{Z}} |\langle Pg, b^{-1/2}Pf_{bn}\rangle_{L^2[0,1/b]}|^2 \qquad \textbf{(extend by zero to } [0,1/b]\textbf{)} \\
&= \frac{1}{b}\sum_{n\in\mathbb{Z}} |\langle P^2 g, f_{bn}\rangle_{L^2[0,1/b]}|^2 \qquad (P^* = P, P^2 = P, Pg = g) \\
&= \frac{1}{b}\,\|g\|^2_{L^2[0,1/b]} \;=\; \frac{1}{b}\int_0^{1/b} |g|^2 \qquad \textbf{(Plancherel: } \sum |\langle h, f_{bn}\rangle|^2 = \|h\|\textbf{)} \\
&= \frac{1}{b}\,\|f\|^2_{L^2[0,1]} \;=\; \frac{1}{b}\int_0^1 |f|^2.
\end{aligned}$$

**We have proved the following.**

**Theorem 4.** $\mathcal{E}_b = \{e_{bn}\}_{n\in\mathbb{Z}}$ is a *tight frame* for $L^2[0,1]$, with $A = B = 1/b$. Consequently, for all $f \in L^2[0,1]$ we have

$$\frac{1}{b}\, f \;=\; \sum_{n\in\mathbb{Z}} \langle f, e_{bn}\rangle\, e_{bn} \;=\; \sum_{n\in\mathbb{Z}} \langle f, e^{2\pi i bnt}\rangle\, e^{2\pi i bnt},$$

where this series converges (unconditionally) in $L^2$-norm. $\diamondsuit$

**We will show that the *Classical (Shannon) Sampling Theorem* is a corollary of Theorem 4!**

**Assignment 1.** (a) Show that $\mathcal{E}_b$ is not a Schauder basis for $L^2[0,1]$ by finding two two distinct ways to write the constant function as $1 = \sum_{n\in\mathbb{Z}} c_n e_{bn}$, where the series converges in $L^2$-norm.

(b) Prove that $\{e_{nb}\}_{n\neq 0}$ is complete in $L^2[0,1]$ (only the zero function is orthogonal to every $e_{nb}$ with $n \neq 0$).

(c) Is $\mathcal{E}_b$ is linearly dependent in the linear algebra sense (does not some nontrivial finite linear combination equal the zero function)?

(d) (Harder) Is $\mathcal{E}_b$ "infinitely overcomplete" or just "finitely overcomplete"? (How many elements can be removed an still leave a complete set?) $\diamondsuit$

**The *Fourier transform* of $f \in L^1(\mathbb{R})$ is**

$$\widehat{f}(\xi) \;=\; \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx, \qquad \xi \in \mathbb{R}.$$

**The *inverse Fourier transform* of $f \in L^1(\mathbb{R})$ is**

$$\check{f}(\xi) \;=\; \int_{-\infty}^{\infty} f(x)\, e^{2\pi i \xi x}\, dx, \qquad \xi \in \mathbb{R}.$$

**Theorem 5** (Inversion Formula)**.** If $f$, $\widehat{f} \in L^1(\mathbb{R})$, then $f$ and $\widehat{f}$ are continuous, and

$$f(x) \;=\; \big(\widehat{f}\big)^{\vee}(x) \;=\; \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi$$

with equality holding pointwise for each $x$. $\diamondsuit$

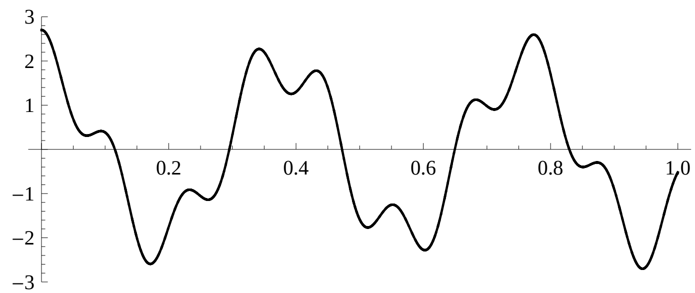


FIGURE 3. Graph of $\varphi(x) = 2\cos(2\pi\sqrt{7}x) + 0.7\cos(2\pi 9x)$ for $0 \le x \le 1$.

**Lemma 6.** If $f \in C_c^2(\mathbb{R})$ then $\widehat{f} \in L^2(\mathbb{R})$ and $\|\widehat{f}\|_2 = \|f\|_2$. $\diamondsuit$

**Hence** $\mathcal{F}\colon f \mapsto \widehat{f}$ **is an isometric map from a dense subset of** $L^2(\mathbb{R})$ **into** $L^2(\mathbb{R})$. **Consequently,** $\mathcal{F}$ **has an extension to a unitary map**

$$\begin{aligned} \mathcal{F}\colon L^2(\mathbb{R}) &\to L^2(\mathbb{R}) \\ f &\mapsto \widehat{f} \end{aligned} \qquad \textbf{(linear, bijective, isometric } \|\widehat{f}\|_2 = \|f\|_2\textbf{)}.$$

**Recall that** $\{e_{bn}\}_{n\in\mathbb{Z}}$ **is a tight frame for** $L^2[-\frac{1}{2}, \frac{1}{2}]$. **Extend** $e_{bn}$ **by zero to all of** $\mathbb{R}$**:**

$$e_{bn}(x) = e^{2\pi ibnt}\,\chi(t) = \begin{cases} e^{2\pi ibnt}, & |t| \le \frac{1}{2}, \\ 0, & |t| > \frac{1}{2}, \end{cases} \qquad \textbf{where } \chi = \chi_{[-\frac{1}{2},\frac{1}{2}]}.$$

**If we think of** $L^2[-\frac{1}{2}, \frac{1}{2}]$ **as being a subspace of** $L^2(\mathbb{R})$, **then** $\{e_{bn}\}_{n\in\mathbb{Z}}$ **is a tight frame** ***for that subspace.*** **As the Fourier transform is unitary,**

$$\{s_{bn}\}_{n\in\mathbb{Z}} = \{(e_{bn}\chi)^{\vee}\}_{n\in\mathbb{Z}}$$

**is a tight frame for the** ***Paley–Wiener space***

$$\mathrm{PW} = \bigl\{f \in L^2(\mathbb{R}) : \mathbf{supp}(\widehat{f}) \subseteq [-\tfrac{1}{2}, \tfrac{1}{2}]\bigr\}.$$

**(Functions in this space are said to be** ***bandlimited*** **to** $[-\frac{1}{2}, \frac{1}{2}]$**). Hence for all** $f \in \mathrm{PW}$,

$$f = b\sum_{n\in\mathbb{Z}} \langle f, s_{bn}\rangle\, s_{bn}.$$

**To compute** $s_{bn} = (e_{bn}\chi)^\vee$**, first verify that** $\overset{\vee}{\chi}(x) = \int_{-\infty}^{\infty} \chi_{[-\frac{1}{2},\frac{1}{2}]}(t)\, e^{2\pi i x t}\, dt = \dfrac{\sin \pi x}{\pi x}$.
**This is the** ***sinc function.*** **Then**

$$s_{bn}(x) = (e^{2\pi i b n t}\chi)^\vee(x) = \int_{-\infty}^{\infty} e^{2\pi i b n t}\, \chi_{[-\frac{1}{2},\frac{1}{2}]}(t)\, e^{2\pi i x t}\, dt = \frac{\sin \pi(x+bn)}{\pi(x+bn)}.$$

**Furthermore,**

$$\langle f, s_{bn}\rangle = \big\langle \widehat{f}, (s_{bn})^\wedge \big\rangle = \big\langle \widehat{f}, e_{bn} \big\rangle = \int_{-1/2}^{1/2} \widehat{f}(\xi)\, \overline{e_{bn}}(\xi)\, d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{-2\pi i b n \xi}\, d\xi$$
$$= \big(\widehat{f}\big)^\vee(-bn) = f(-bn).$$

**Corollary 7** (Classical Sampling Theorem)**.** Fix $f \in \mathrm{PW}$, i.e., $f \in L^2(\mathbb{R})$ is bandlimited to $[-\frac{1}{2}, \frac{1}{2}]$. Then for $0 < b \le 1$,

$$f(x) = b \sum_{n \in \mathbb{Z}} \langle f, s_{bn}\rangle\, s_{bn}(x) = b \sum_{n \in \mathbb{Z}} f(bn)\, \frac{\sin \pi(x-bn)}{\pi(x-bn)},$$

where this series converges (unconditionally) in $L^2$-norm. $\diamondsuit$

**Every bandlimited function is determined by its** ***samples*** $f(bn)$ **(if we sample quickly enough). This is the basis for digital encoding of analog signals. If** $b < 1$ **then the frame is not exact, which gives enhanced** ***robustness against noise.***

**Definition 8** (Bessel Sequence)**.** A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space $H$ is a *Bessel sequence* if

$$\forall\, x \in H, \quad \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \;<\; \infty.$$

**Theorem 9.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a Bessel sequence, and let $Cx = \big(\langle x, x_n\rangle\big)_{n\in\mathbb{N}}$ for $x \in H$ (the *analysis map* or *coefficient map*). Then the following statements hold.

(a) $C$ is a bounded mapping of $H$ into $\ell^2$, and there exists a constant $B > 0$ such that

$$\forall\, x \in H, \quad \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \;\le\; B\,\|x\|^2.$$

(b) If $(c_n) \in \ell^2$, then the series $\sum c_n x_n$ converges unconditionally in $H$, and $Rc = \sum c_n x_n$ defines a bounded map of $\ell^2$ into $H$ (the *synthesis map* or *reconstruction map*).

(c) $R = C^*$ and $\|R\| = \|C\| \le B^{1/2}$. Consequently,

$$\forall\,(c_n) \in \ell^2, \quad \left\|\sum_{n=1}^{\infty} c_n x_n\right\|^2 \;\le\; B \sum_{n=1}^{\infty} |c_n|^2.$$

(d) If $\{x_n\}_{n\in\mathbb{N}}$ is complete, then $C$ is injective and range$(R)$ is dense in $H$. $\quad\diamondsuit$

**Assignment 2.** Prove statement (a). Uniform Boundedness Principle!

*Proof.* (b) Choose $c = (c_n)_{n\in\mathbb{N}} \in \ell^2$. Then for $0 \le M < N$ we have

$$\begin{aligned}
\left\| \sum_{n=M+1}^{N} c_n x_n \right\| &= \sup_{\|y\|=1} \left| \sum_{n=M+1}^{N} \langle c_n x_n, y\rangle \right| \\
&\le \sup_{\|y\|=1} \sum_{n=M+1}^{N} |c_n|\, |\langle x_n, y\rangle| \\
&\le \sup_{\|y\|=1} \left( \sum_{n=M+1}^{N} |c_n|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} |\langle x_n, y\rangle|^2 \right)^{1/2} \\
&\le \sup_{\|y\|=1} B\, \|y\| \left( \sum_{n=M+1}^{N} |c_n|^2 \right)^{1/2} \\
&= B \left( \sum_{n=M+1}^{N} |c_n|^2 \right)^{1/2},
\end{aligned}$$

Therefore $\sum c_n x_n$ is a Cauchy series and hence converges. If $\sigma\colon \mathbb{N} \to \mathbb{N}$ is a bijection, then the same argument applies to $\sum c_{\sigma(n)} x_{\sigma(n)}$. Therefore the series converges unconditionally. Taking $M = 1$ and letting $N \to \infty$, we obtain

$$\|Rc\| = \left\| \sum_{n=1}^{\infty} c_n x_n \right\| \le B \left( \sum_{n=1}^{\infty} |c_n|^2 \right)^{1/2} = B\, \|c\|_2,$$

so $R$ is bounded and $\|R\| \le B$.

(c) Given $c = (c_n) \in \ell^2$ and $y \in H$, we have

$$Rc \;=\; \sum_{n=1}^{\infty} c_n x_n \;\in\; H \qquad \text{and} \qquad Cy \;=\; \big\{\langle x_n, y\rangle_{n\in\mathbb{N}}\big\} \;\in\; \ell^2.$$

Therefore

$$\begin{aligned}
\langle c, R^* y\rangle \;&=\; \langle Rc, y\rangle \\
&=\; \left\langle \sum_{n=1}^{\infty} c_n x_n,\; y \right\rangle \\
&=\; \sum_{n=1}^{\infty} c_n \,\langle x_n, y\rangle \\
&=\; \Big\langle (c_n),\; \big\{\langle y, x_n\rangle\big\} \Big\rangle \;=\; \langle c, Cy\rangle,
\end{aligned}$$

so $C = R^*$. □

**Benefits of a Bessel sequence:** $\sum c_n x_n$ ***converges unconditionally; bounded analysis and synthesis maps.***

**Drawbacks: What does** $\sum c_n x_n$ **converge to?** $C$ ***and*** $R$ ***need not be invertible (lack of stability).*** $\{x_n\}_{n\in\mathbb{N}}$ ***need not be complete (its span need not be dense).***

**Assignment 3.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a Hilbert space $H$, and let $\{\delta_n\}$ be the standard basis for $\ell^2$. Prove that the following statements are equivalent.

(a) $\{x_n\}_{n\in\mathbb{N}}$ is a Bessel sequence in $H$, i.e., $\sum |\langle x, x_n\rangle|^2 < \infty$ for every $x$.

(b) There exists a constant $B > 0$ and a dense set $E \subseteq H$ such that

$$\forall\, x \in E, \quad \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \;\le\; B\,\|x\|^2.$$

(c) There exists a constant $B > 0$ such that

$$\forall\, N \in \mathbb{N}, \quad \forall\, c_1, \dots, c_N \in \mathbb{F}, \quad \Big\|\sum_{n=1}^{N} c_n x_n\Big\|^2 \;\le\; B \sum_{n=1}^{N} |c_n|^2.$$

(d) The series $\sum c_n x_n$ converges for each sequence $(c_n) \in \ell^2$.

(e) There exists a bounded operator $R\colon \ell^2 \to H$ such that $R\delta_n = x_n$ for each $n \in \mathbb{N}$.

(f) There exists an orthonormal sequence $\{e_n\}_{n\in\mathbb{Z}}$ in $H$ and a bounded operator $T \in \mathcal{B}(H)$ such that $Te_n = x_n$ for each $n \in \mathbb{N}$.

Further, when these hold, the operator $R$ appearing in part (e) is the synthesis operator for $\{x_n\}_{n\in\mathbb{N}}$, and $\overline{\text{span}}\{x_n\}_{n\in\mathbb{N}} = \overline{\text{range}}(R)$.

***Bessel sequences are images of ONBs under bounded maps.*** **(This statement makes Bessel sequences seem much simpler than they are!)**

## THE FRAME OPERATOR

**The *frame operator* for a Bessel sequence is**

$$Sx \;=\; RCx \;=\; \sum_{n=1}^{\infty} \langle x, x_n\rangle\, x_n, \qquad x \in H.$$

**Note that**

$$\langle Sx, x\rangle \;=\; \sum_{n=1}^{\infty} \langle x, x_n\rangle\, \langle x_n, x\rangle \;=\; \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2.$$

**Example: If $\{x_n\}_{n\in\mathbb{N}}$ is an ONB, then $S = I$!** $\diamondsuit$

**Fact (good assignment, but we'll prove more): If $\{x_n\}_{n\in\mathbb{N}}$ is a tight frame ($A = B$), then $S = AI$! This can happen even when the frame is not exact ("includes redundancy" in some sense). Hence**

$$\{x_n\}_{n\in\mathbb{N}} \textbf{ tight frame} \quad\Longrightarrow\quad Ax \;=\; \sum_{n=1}^{\infty} \langle x, x_n\rangle\, x_n, \quad x \in H. \qquad \diamondsuit$$

**This is for a tight frame, but we will prove that if $\{x_n\}$ is *any* frame then we have *stable frame expansions.***

**Theorem 10.** If $\{x_n\}_{n\in\mathbb{N}}$ is a frame for $H$ with frame bounds $A$, $B$, then the following statements hold.

(a) The frame operator $S\colon H \to H$ is a *topological isomorphism* (linear, bijective, continuous, with continuous inverse). It need not be isometric, but it satisfies the operator inequalities $AI \le S \le BI$.

(b) $S^{-1}$ is a topological isomorphism, and $B^{-1}I \le S^{-1} \le A^{-1}I$.

(c) $\{S^{-1}x_n\}$ is a frame with frame bounds $B^{-1}$, $A^{-1}$.

(d) For each $x \in H$,

$$x \;=\; \sum_{n=1}^{\infty} \langle x, S^{-1}x_n\rangle\, x_n \;=\; \sum_{n=1}^{\infty} \langle x, x_n\rangle\, S^{-1}x_n,$$

and these series converge unconditionally in the norm of $H$.

(e) If the frame is $A$-tight, then $S = AI$, $S^{-1} = A^{-1}I$, and

$$\forall\, x \in H, \quad x \;=\; A^{-1}\sum_{n=1}^{\infty} \langle x, x_n\rangle\, x_n. \qquad \diamondsuit$$

**We write $\widetilde{x}_n = S^{-1}x_n$, and call $\{\widetilde{x}_n\}$ the *canonical dual frame.***

*Proof.* (a) $\{x_n\}_{n\in\mathbb{N}}$ is Bessel, so $S = RC = C^*C$ is bounded and positive. Also,

$$\langle Sx, x\rangle \;=\; \left\langle \sum_{n=1}^{\infty} \langle x, x_n\rangle\, x_n,\; x\right\rangle \;=\; \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \;\leq\; B\,\|x\|^2 \;=\; \langle BIx, x\rangle.$$

The definition of the operator inequality $S \leq BI$ is that

$$S \leq BI \qquad \text{means} \qquad \langle Sx, x\rangle \leq \langle BIx, x\rangle \text{ for every } x.$$

Hence $S \leq BI$, and similarly $S \geq AI$.

Reasonable, but needs justification:

$$AI \leq S \leq BI \implies S \text{ is bounded, linear, and } S^{-1} \text{ is invertible.}$$

Example argument:

$$A\,\|x\|^2 \;=\; \langle AIx, x\rangle \;\leq\; \langle Sx, x\rangle \;\leq\; \|Sx\|\,\|x\|,$$

so $\|Sx\| \geq A\,\|x\|$. This implies $S$ is injective and has closed range.

(b) Also reasonable, but needs justification:

$$AI \leq S \leq BI \qquad \implies \qquad S^{-1}AI \leq S^{-1}S \leq S^{-1}BI,$$

which yields $B^{-1}I \leq S^{-1} \leq A^{-1}I$. ($\leftarrow\ \exists$ typo in printouts here!)

(c) The image of a frame under a topological isomorphism is a frame, so $\{S^{-1}x_n\}$ is a frame.

(d) Recall

$$Sx \;=\; RCx \;=\; \sum_{n=1}^{\infty} \langle x, x_n\rangle\, x_n.$$

Since $S$ is self-adjoint, we therefore have

$$x \;=\; S(S^{-1}x) \;=\; \sum_{n=1}^{\infty} \langle S^{-1}x, x_n\rangle\, x_n \;=\; \sum_{n=1}^{\infty} \langle x, S^{-1}x_n\rangle\, x_n.$$

Also, since $S^{-1}$ is continuous

$$x \;=\; S^{-1}(Sx) \;=\; S^{-1}\biggl(\sum_{n=1}^{\infty} \langle x, x_n\rangle\, x_n\biggr) \;=\; \sum_{n=1}^{\infty} \langle x, x_n\rangle\, S^{-1}x_n. \qquad \square$$

**Hence, with** $\widetilde{x}_n = S^{-1}x_n$,

$$x \;=\; \sum_{n=1}^{\infty} \langle x, \widetilde{x}_n\rangle\, x_n \;=\; \sum_{n=1}^{\infty} \langle x, x_n\rangle\, \widetilde{x}_n, \qquad x \in H.$$

**An *alternative dual* is a sequence** $\{y_n\}$ **such that**

$$x \;=\; \sum_{n=1}^{\infty} \langle x, y_n\rangle\, x_n, \qquad x \in H.$$

**Since $S$ and $S^{-1}$ are both positive operators, they have square roots $S^{1/2}$ and $S^{-1/2}$.**

**Corollary 11.** (Every frame is the image of a tight frame under a topological isomorphism.) Let $\{x_n\}_{n\in\mathbb{N}}$ be a frame.

(a) $S^{-1/2}$ is a topological isomorphism, and $\{S^{-1/2}x_n\}$ is a Parseval frame.

(b) $\langle x_n, \widetilde{x}_n\rangle = \|S^{-1/2}x_n\|^2$, and $0 \le \langle x_n, \widetilde{x}_n\rangle \le 1$ for every $n$.

(c) $\{x_n\}_{n\in\mathbb{N}}$ is exact if and only if $\{S^{-1/2}x_n\}$ is an ONB.

*Proof.* (a) $S^{-1/2}$ is a topological isomorphism because $S$ is. Therefore $\{S^{-1/2}x_n\}$ is a frame, and

$$\sum_{n=1}^{\infty} \langle x, S^{-1/2}x_n\rangle\, S^{-1/2}x_n \;=\; S^{-1/2}SS^{-1/2}x \;=\; x \;=\; Ix.$$

It follows that $\{S^{-1/2}x_n\}$ is a Parseval frame.

(b) Since $S^{-1/2}$ is self-adjoint,

$$\langle x_n, \widetilde{x}_n\rangle \;=\; \big\langle x_n,\, S^{-1}x_n\big\rangle \;=\; \big\langle S^{-1/2}x_n,\, S^{-1/2}x_n\big\rangle \;=\; \|S^{-1/2}x_n\|^2.$$

Since $\{S^{-1/2}x_n\}$ is 1-tight, $\|S^{-1/2}x_n\|^2 \le 1$ for every $n$ (see **Assignment 5**). □

**We write $x_n^\sharp = S^{-1/2}x_n$, and call $\{x_n^\sharp\}$ the *canonical Parseval frame.***

**Theorem 12.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a frame for $H$. If $x = \sum c_n x_n$ for some scalars $(c_n)$, then

$$\sum_{n=1}^{\infty} |c_n|^2 \;=\; \sum_{n=1}^{\infty} \bigl|\langle x, \widetilde{x}_n\rangle\bigr|^2 \;+\; \sum_{n=1}^{\infty} \bigl|\langle x, \widetilde{x}_n\rangle - c_n\bigr|^2.$$

In particular, the "canonical sequence" $\bigl(\langle x, \widetilde{x}_n\rangle\bigr)$ has minimal $\ell^2$-norm among all such $(c_n)$.

*Proof.* $x = \sum a_n x_n$ where $a_n = \langle x, \widetilde{x}_n\rangle$. If $x = \sum c_n x_n$, then

$$\begin{aligned}
\langle x, S^{-1}x\rangle \;&=\; \left\langle \sum_{n=1}^{\infty} a_n x_n,\, S^{-1}x \right\rangle \\
&=\; \sum_{n=1}^{\infty} a_n \langle \widetilde{x}_n, x\rangle \qquad (S^{-1} \text{ is self-adjoint}) \\
&=\; \sum_{n=1}^{\infty} a_n \overline{a_n} \;=\; \bigl\langle (a_n),\, (a_n)\bigr\rangle_{\ell^2}
\end{aligned}$$

and

$$\langle x, S^{-1}x\rangle \;=\; \left\langle \sum_{n=1}^{\infty} c_n x_n,\, S^{-1}x \right\rangle \;=\; \sum_{n=1}^{\infty} c_n \langle \widetilde{x}_n, x\rangle \;=\; \sum_{n=1}^{\infty} c_n \overline{a_n} \;=\; \bigl\langle (c_n),\, (a_n)\bigr\rangle_{\ell^2}.$$

Therefore $(c_n - a_n)$ is orthogonal to $(a_n)$ in $\ell^2$, the Pythagorean Theorem implies that

$$\|(c_n)\|_{\ell^2}^2 \;=\; \|(c_n - a_n) + (a_n)\|_{\ell^2}^2 \;=\; \|(c_n - a_n)\|_{\ell^2}^2 \;+\; \|(a_n)\|_{\ell^2}^2. \qquad \square$$

**Theorem 13.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a frame.

(a) For each $m \in \mathbb{N}$,

$$\sum_{n\neq m} |\langle x_m, \widetilde{x}_n\rangle|^2 \;=\; \frac{1 \;-\; |\langle x_m, \widetilde{x}_m\rangle|^2 \;-\; |1 - \langle x_m, \widetilde{x}_m\rangle|^2}{2}.$$

(b) If $\langle x_m, \widetilde{x}_m\rangle = 1$, then $\langle x_m, \widetilde{x}_n\rangle = 0$ for $n \neq m$.

(c) The removal of a vector from a frame leaves either a frame or an incomplete set. Specifically,

$$\langle x_m, \widetilde{x}_m\rangle \neq 1 \quad\Longrightarrow\quad \{x_n\}_{n\neq m} \text{ is a frame,}$$

$$\langle x_m, \widetilde{x}_m\rangle = 1 \quad\Longrightarrow\quad \{x_n\}_{n\neq m} \text{ is incomplete.}$$

*Proof.* Fix $m$, let $a_n = \langle x_m, \widetilde{x}_n\rangle$. Then $x_m = \sum a_n x_n$ and $x_m = \sum \delta_{mn} x_n$. By Theorem 12,

$$\begin{aligned} 1 \;=\; \sum_{n=1}^{\infty} |\delta_{mn}|^2 \;&=\; \sum_{n=1}^{\infty} |a_n|^2 \;+\; \sum_{n=1}^{\infty} |a_n - \delta_{mn}|^2 \\ &=\; |a_m|^2 \;+\; \sum_{n\neq m} |a_n|^2 \;+\; |a_m - 1|^2 \;+\; \sum_{n\neq m} |a_n|^2. \end{aligned}$$

Rearranging gives statement (a).

(b) If $\langle x_m, \widetilde{x}_m\rangle = 1$, then $\sum_{n\neq m} |\langle x_m, \widetilde{x}_n\rangle|^2 = 0$.

(c) If $\langle x_m, \widetilde{x}_m \rangle = 1$, then $\widetilde{x}_m \perp x_n$ for every $n \neq m$. However, $\widetilde{x}_m \neq 0$ since $\langle \widetilde{x}_m, x_m \rangle = 1 \neq 0$. Therefore $\{x_n\}_{n \neq m}$ is incomplete.

If $\langle x_m, \widetilde{x}_m \rangle \neq 1$, let $a_n = \langle x_m, \widetilde{x}_n \rangle$. Then $x_m = \sum a_n x_n$, but $a_m \neq 1$, so $x_m = \frac{1}{1-a_m} \sum_{n \neq m} a_n x_n$. Hence

$$|\langle x, x_m \rangle|^2 \;=\; \left| \frac{1}{1-a_m} \sum_{n \neq m} a_n \langle x, x_n \rangle \right|^2 \;\leq\; C \sum_{n \neq m} |\langle x, x_n \rangle|^2.$$

Therefore,

$$\sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \;=\; |\langle x, x_m \rangle|^2 \;+\; \sum_{n \neq m} |\langle x, x_n \rangle|^2 \;\leq\; (1+C) \sum_{n \neq m} |\langle x, x_n \rangle|^2.$$

Hence, if $A$, $B$ are frame bounds for $\{x_n\}_{n \in \mathbb{N}}$ then

$$\frac{A}{1+C} \, \|x\|^2 \;\leq\; \frac{1}{1+C} \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \;\leq\; \sum_{n \neq m} |\langle x, x_n \rangle|^2 \;\leq\; B \, \|x\|^2,$$

so $\{x_n\}_{n \neq m}$ has frame bounds $A/(1+C)$, $B$. $\square$

**Assignment 4.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a frame. Prove that the following are equivalent.

(a) $\{x_n\}_{n\in\mathbb{N}}$ is exact.

(b) $\{x_n\}_{n\in\mathbb{N}}$ and $\{\widetilde{x}_n\}$ are *biorthogonal*, i.e., $\langle x_m, \widetilde{x}_n\rangle = \delta_{mn}$.

(c) $\langle x_n, \widetilde{x}_n\rangle = 1$ for all $n$.

Consequently, if $\{x_n\}_{n\in\mathbb{N}}$ is $A$-tight, then the following are equivalent.

(a') $\{x_n\}_{n\in\mathbb{N}}$ is exact.

(b') $\{x_n\}_{n\in\mathbb{N}}$ is an orthogonal basis for $H$.

(c') $\|x_n\|^2 = A$ for all $n$. $\diamondsuit$

**Assignment 5.** Assume $\{x_n\}_{n\in\mathbb{N}}$ is a frame with frame bounds $A$, $B$.

(a) $\|x_n\|^2 \leq B$ for every $n$.

(b) If $\{x_n\}_{n\in\mathbb{N}}$ is exact then $A \leq \|x_n\|^2$ for every $n$. $\diamondsuit$

**Assignment 6.** Frames are preserved by topological isomorphisms: If $\{x_n\}_{n\in\mathbb{N}}$ is a frame and $T\colon H \to K$ is a topological isomorphism, then $\{Tx_n\}$ is a frame for $K$. $\diamondsuit$

**Theorem 14.** Given a sequence $\{x_n\}_{n\in\mathbb{N}}$, the following are equivalent.

(a) $\{x_n\}_{n\in\mathbb{N}}$ is an exact frame.

(b) $\{x_n\}_{n\in\mathbb{N}}$ is a bounded unconditional basis.

(c) $\{x_n\}_{n\in\mathbb{N}}$ is a Riesz basis.

*Proof.* (a) $\Rightarrow$ (b). Assume $\{x_n\}_{n\in\mathbb{N}}$ is an exact frame for $H$. By assignment, $A \le \|x_n\|^2 \le B$ (this is the *bounded* part). Also,

$$x = \sum_{n=1}^{\infty} \langle x, \widetilde{x}_n\rangle\, x_n, \qquad x \in H,$$

with *unconditional* convergence of the series. Also by assignment, $\{x_n\}_{n\in\mathbb{N}}$ and $\{\widetilde{x}_n\}$ are biorthogonal. This implies that the scalars in the preceding equation are unique (this is the *Schauder basis* part).

(b) $\Leftrightarrow$ (c). Fact from Hilbert spaces.

(c) $\Rightarrow$ (a). If $\{x_n\}_{n\in\mathbb{N}}$ is a *Riesz basis* then, by definition, there exists an orthonormal basis $\{e_n\}$ and a topological isomorphism $T\colon H \to H$ such that $Te_n = x_n$. Since $\{e_n\}$ is an exact frame and exact frames are preserved by topological isomorphisms, $\{x_n\}_{n\in\mathbb{N}}$ must also be an exact frame. □

**Useful characterizations of Riesz bases.**

**Theorem 15.** Given a sequence $\{x_n\}_{n\in\mathbb{N}}$, the following statements are equivalent.

(a) $\{x_n\}_{n\in\mathbb{N}}$ is a Riesz basis for $H$ (i.e., it is the image of an ONB under a topological isomorphism).

(b) $\{x_n\}_{n\in\mathbb{N}}$ is a bounded unconditional basis for $H$.

(c) $\{x_n\}_{n\in\mathbb{N}}$ is a basis for $H$, and

$$\sum_{n=1}^{\infty} c_n x_n \text{ converges} \quad \iff \quad \sum_{n=1}^{\infty} |c_n|^2 < \infty.$$

(d) $\{x_n\}_{n\in\mathbb{N}}$ is complete in $H$ and there exist constants $A$, $B > 0$ such that

$$\forall\, c_1, \dots, c_N, \quad A \sum_{n=1}^{N} |c_n|^2 \;\le\; \left\| \sum_{n=1}^{N} c_n x_n \right\|^2 \;\le\; B \sum_{n=1}^{N} |c_n|^2.$$

(e) There is an equivalent inner product $(\cdot,\cdot)$ for $H$ such that $\{x_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $H$ with respect to $(\cdot,\cdot)$.

(f) $\{x_n\}_{n\in\mathbb{N}}$ is a complete Bessel sequence and possesses a biorthogonal system $\{y_n\}_{n\in\mathbb{N}}$ that is also a complete Bessel sequence.

(g) $\{x_n\}_{n\in\mathbb{N}}$ is complete, and multiplication of vectors in $\ell^2$ by the *Gram matrix* $G = \big[\langle x_n, x_m\rangle\big]_{m,n\in\mathbb{N}}$ defines a topological isomorphism of $\ell^2$ onto itself.

**Assignment 7.** Some implications related to bases.

ONB

$\substack{\Longrightarrow \\ \not\Longleftarrow}$ Riesz basis (image of an ONB under invertible map)

$\substack{\Longrightarrow \\ \not\Longleftarrow}$ Schauder basis $\left(x = \sum_{n=1}^{\infty} c_n x_n \text{ uniquely}\right)$

$\substack{\Longrightarrow \\ \not\Longleftarrow}$ $\exists$ biorthogonal system $\big(\langle x_m, y_n \rangle = \delta_{mn}\big)$

$\Longleftrightarrow$ minimal $\big(x_m \notin \overline{\mathrm{span}}\{x_n\}_{n \neq m},\ \text{every } m\big)$

$\substack{\Longrightarrow \\ \not\Longleftarrow}$ $\omega$-independent $\left(\sum c_n x_n = 0 \implies c_n = 0\right)$

$\substack{\Longrightarrow \\ \not\Longleftarrow}$ finitely independent (usual linear algebra independence) $\diamondsuit$

**Assignment 8.** Some implications related to frames.

ONB

$\overset{\Longrightarrow}{\not\Longleftarrow}$ Riesz basis (image of an ONB under invertible map)

$\Longleftrightarrow$ exact frame $\Longleftrightarrow$ bounded unconditional basis

$\overset{\Longrightarrow}{\not\Longleftarrow}$ frame

$\overset{\Longrightarrow}{\not\Longleftarrow}$ complete set (span is dense) $\Diamond$

**Frames provide a *unconditional representations with redundancy*:**

$$x = \sum_{n=1}^{\infty} \langle x, \widetilde{x}_n \rangle \, x_n.$$

**Numbers correspond to "A Basis Theory Primer."**

**7.16. Show directly that if $\{x_n\}_{n\in\mathbb{N}}$ is complete and $\left\|\sum_{n=1}^{N} c_n x_n\right\|^2 = \sum_{n=1}^{N} |c_n|^2$ for all $N \in \mathbb{N}$ and $c_1, \dots, c_N \in \mathbb{F}$, then $\{x_n\}_{n\in\mathbb{N}}$ is an ONB.**

**7.20 (Assume the results of 7.18 and 7.19.) Set $e_b(x) = e^{2\pi ibx}$. Prove the following statements.**

**(a)** $T_k f(x) = x^k f(x)$ **is a bounded map of** $L^2[-\frac{1}{2}, \frac{1}{2}]$ **into itself, with operator norm** $\|T_k\| = 2^{-k}$.

**(b) If** $a_{nk} = -\dfrac{\big(2\pi i(\lambda_n - n)\big)^k}{k!}$, **then** $e_n - e_{\lambda_n} = \displaystyle\sum_{k=1}^{\infty} a_{nk}\, T_k e_n$, **where the series converges absolutely in** $L^2[-\frac{1}{2}, \frac{1}{2}]$.

**(c) If** $\delta = \sup\limits_{n\in\mathbb{Z}} |n - \lambda_n| < (\ln 2)/\pi \approx 0.22\dots$,
**then** $\{e^{2\pi i\lambda_n x}\}_{n\in\mathbb{Z}}$ **is a Riesz basis for** $L^2[-\frac{1}{2}, \frac{1}{2}]$.

**Remark: This result (due to Duffin and Eachus) is not quite optimal. *Kadec's* $\frac{1}{4}$*-Theorem* states that if $\delta < \frac{1}{4}$ then $\{e^{2\pi i\lambda_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$ (and $\frac{1}{4}$ is optimal).**

**8.4. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $H$. Assume $A\,\|x\|^2 \le \sum |\langle x, x_n\rangle|^2 \le B\,\|x\|^2$ for $x$ in some dense subset $E$. Show that $\{x_n\}_{n\in\mathbb{N}}$ is a frame (thus it suffices to establish the frame condition on some dense, "nice" subset).**

**8.6. Let $\{x_n\}$ be an $A$-tight frame. Prove the following.**

**(a) $\|x_n\|^2 \le A$ for every $n$.**

**(b) If $\|x_m\|^2 < A$ for some $m$, then $\{x_n\}_{n\neq m}$ is a frame and the optimal lower frame bound for $\{x_n\}_{n\neq m}$ is $A - \|x_m\|^2$.**

**(c) If $\|x_m\|^2 = A$ for some $m$, then $x_m \perp x_n$ for all $n \neq m$.**

**8.11. Let $v_1, \dots, v_n$ be vectors in $\mathbb{F}^d$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Give direct proofs of the following.**
**(a) $\{v_1, \dots, v_n\}$ is a Bessel sequence in $\mathbb{F}^d$, its synthesis operator corresponds to multiplication by the matrix $R$ that has $v_1, \dots, v_n$ as columns, and its analysis operator is $C = R^*$, the Hermitian of the matrix $R$.**

**(b) The following are equivalent: (i) $\{v_1, \dots, v_n\}$ spans $\mathbb{F}^d$, (ii) $S = C^*C$ is positive definite, (iii) $\{v_1, \dots, v_n\}$ is a frame for $\mathbb{F}^d$. Further, in case these hold, the optimal frame bounds for $\{v_1, \dots, v_n\}$ are $\lambda_1$, $\lambda_d$, where $\lambda_1$ is the smallest eigenvalue of $S$ and $\lambda_d$ is the largest eigenvalue.**

**(c) The following are equivalent: (i) $\{v_1, \dots, v_n\}$ is linearly independent, (ii) $G = CC^*$ is positive definite, (iii) $\{v_1, \dots, v_n\}$ is a Riesz sequence (a Riesz basis for its span).**

**8.15. Alternative proof that the frame operator is a topological isomorphism.**

**(a) Show that if $U, V \in \mathcal{B}(H)$ are positive and $U \le V$ then $\|U\| \le \|V\|$.**

**(b) Let $S$ be the frame operator for a frame $\{x_n\}_{n\in\mathbb{N}}$ that has frame bounds $A$, $B$. Prove the operator inequality**

$$0 \;\le\; I - \frac{2}{A+B}\,S \;\le\; \frac{B-A}{B+A}\,I,$$

**and use this (and Neumann series) to show $S$ is a topological isomorphism.**

**8.18. Let $\{x_n\}_{n\in\mathbb{N}}$ be a frame for $H$. Let $C$, $R$, $S$ be the analysis, synthesis, and frame operators for $\{x_n\}_{n\in\mathbb{N}}$, and let $\widetilde{C}$, $\widetilde{R}$ be the analysis and synthesis operators for the canonical dual frame $\{\widetilde{x}_n\}$. Prove the following.**

**(a) $\widetilde{C} = CS^{-1}$ and $\widetilde{R} = S^{-1}R$.**

**(b) The orthogonal projection $P$ of $\ell^2$ onto $\mathrm{range}(C)$ is**

$$Pc \;=\; C\widetilde{R}c \;=\; CS^{-1}Rc = \left\{\left\langle \sum_n c_n \widetilde{x}_n,\, x_k \right\rangle\right\}_{k\in\mathbb{N}}, \qquad c = (c_n) \in \ell^2.$$