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Introduction to Real Analysis: Selected Solutions for Students

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Solutions to Selected Exercises and Problems

These are selected solutions to some of the exercises and problems from "An Introduction to Real Analysis." Approximately one exercise or problem from each section is included, sometimes more from longer sections. I have tried to select solutions that give some insight, but at the same time I have deliberately not included those problems where I feel that there is a particular benefit from finding the solution on your own. It is always best to find your own solution, but it is also true that sometimes you need a little something to compare to.

The problems in the text vary considerably in difficulty. Some have solutions that are quite short, and others take a lot of work (or maybe I just haven't found an efficient solution yet). Of course, many problems have solutions other than the ones I sketch here. In particular, there could very well be easier solutions than the ones I give (and over the years, my students have often shown me better solutions!). These solutions have not been proofread as carefully as has the text proper, so the probability of errors is correspondingly higher. Please send comments and corrections to "`heil@math.gatech.edu`".

Solutions to Selected Exercises and Problems

1.1.7 (a) Let F be the set of all possible limits of elements of E:

$$F \;=\; \bigl\{y \in X : \text{there exist } x_n \in E \text{ such that } x_n \to y\bigr\}.$$

We must show that $F = \overline{E}$.

Choose any point $y \in F$. Then, by definition, there exist points $x_n \in E$ such that $x_n \to y$. Since $E \subseteq \overline{E}$, the points x_n all belong to $\overline{E}$. Hence y is a limit of elements of $\overline{E}$. But $\overline{E}$ is a closed set, so it must contain all of these limits. Therefore y belongs to $\overline{E}$, so we have shown that $F \subseteq \overline{E}$.

In order to prove that $\overline{E}$ is a subset of F, we will first prove that F^{C} is an open set. To do this, choose any point $y \in F^{\mathrm{C}}$. We must show that there is a ball centered at y that is entirely contained in F^{C}. That is, we must show that there is some $r > 0$ such that $B_r(y)$ contains no limits of elements of E.

Suppose that for each $k \in \mathbb{N}$, the ball $B_{1/k}(y)$ contained some point from E, say $x_k \in B_{1/k}(y) \cap E$. Then these x_k are points of E that converge to y (why?). Hence y is a limit of points of E, which contradicts the fact that $y \notin F$. Hence there must be at least one k such that $B_{1/k}(y)$ contains no points of E. We will show that $r = 1/k$ is the radius that we seek. That is, we will show that the ball $B_r(y)$, where $r = 1/k$, not only contains no elements of E but furthermore contains no *limits* of elements of E.

Suppose that $B_r(y)$ did contain some point z that was a limit of elements of E, i.e., suppose that there did exist some $x_n \in E$ such that $x_n \to z \in B_r(y)$. Then, since $\mathrm{d}(y,z) < r$ and since $\mathrm{d}(z,x_n)$ becomes arbitrarily small, by choosing n large enough we will have $\mathrm{d}(y,x_n) \le \mathrm{d}(y,z) + \mathrm{d}(z,x_n) < r$. But then this point x_n belongs to $B_r(y)$, which contradicts the fact that $B_r(y)$ contains no points of E.

Thus, $B_r(y)$ contains no limits of elements of E. Since F is the set of all limits of elements of E, this means that $B_r(y)$ contains no points of F. That is, $B_r(y) \subseteq F^{\mathrm{C}}$.

In summary, we have shown that each point $y \in F^{\mathrm{C}}$ has some ball $B_r(y)$ that is entirely contained in F^{C}. Therefore F^{C} is an open set. Hence, by definition, F is a closed set. We also know that $E \subseteq F$ (why?), so F is one of the closed sets that contains E. But $\overline{E}$ is the smallest closed set that contains E, so we conclude that $\overline{E} \subseteq F$.

(b) "$\Rightarrow$." Assume that E is dense in X. Then $\overline{E} = X$, so part (a) implies that every point $x \in X$ is a limit of elements of E.

"$\Leftarrow$." Suppose that every point $x \in X$ is a limit of elements of E. Then part (a) implies that $\overline{E} = X$, so E is dense in X.

1.2.13 The partial sums $s_N = \sum_{n=1}^{N} x_n$ of the series converge to x in norm, so by the continuity of the norm and the Triangle Inequality (which by induction applies to any *finite sum*), we see that

$$\begin{aligned} \Bigl\|\sum_{n=1}^{\infty} x_n\Bigr\| \;=\; \|x\| \;&=\; \lim_{N\to\infty} \|s_N\| && \text{(continuity of the norm)} \\ &=\; \lim_{N\to\infty} \Bigl\|\sum_{n=1}^{N} x_n\Bigr\| && \text{(definition of } s_N\text{)} \end{aligned}$$

$$\leq \lim_{N\to\infty} \sum_{n=1}^{N} \|x_n\| \qquad \text{(Triangle Inequality)}$$
$$= \sum_{n=1}^{\infty} \|x_n\| \qquad \text{(definition of infinite series).}$$

1.3.6 "⇒." Suppose that $f\colon \mathbb{R}^d \to \mathbb{C}$ is uniformly continuous, and fix $\varepsilon > 0$. Then there exists some $\delta > 0$ such that

$$\|x-y\| < \delta \implies |f(x)-f(y)| < \varepsilon.$$

Consequently, if $\|a\| < \delta$ then $\|x-(x-a)\| = \|a\| < \delta$ for every x, so

$$\|f - T_a f\|_{\mathrm{u}} = \sup_{x\in\mathbb{R}} |f(x) - f(x-a)| \leq \varepsilon.$$

This says that $\|f - T_a f\|_{\mathrm{u}} \to 0$ as $a \to 0$.

"⇐." Suppose that $\|f - T_a f\|_{\mathrm{u}} \to 0$, and fix $\varepsilon > 0$. Then there exists some $\delta > 0$ such that $\|f - T_a f\|_{\mathrm{u}} < \varepsilon$ whenever $\|a\| < \delta$. Consequently, if $\|x-y\| < \delta$ and we set $a = x-y$, then

$$|f(x)-f(y)| = |f(x)-f(x-a)| \leq \|f - T_a f\|_{\mathrm{u}} < \varepsilon.$$

Hence f is uniformly continuous.

1.4.2 *Case 1: Real-Valued Functions.* Assume that f is a differentiable real-valued function on I whose derivative is bounded, and set

$$K = \|f'\|_{\mathrm{u}} = \sup_{t\in I} |f'(t)|.$$

is finite. Choose any two points $x < y$ in I. Then f is differentiable everywhere on the interval (x,y) and is continuous on $[x,y]$. Because f is real-valued, the Mean Value Theorem therefore implies that there exists a point c between x and y such that

$$f'(c) = \frac{f(y)-f(x)}{y-x}.$$

Rearranging, we see that

$$|f(y)-f(x)| = |f'(c)|\,|y-x| \leq K\,|y-x|.$$

Case 2: Complex-Valued Functions. Suppose that f is a differentiable complex-valued function on I whose derivative is bounded, write $f = g + ih$ where g and h are real-valued. Since f' is bounded and $f' = g' + ih'$, the functions g' and h' are bounded. Since g is real-valued, Case 1 implies that g is Lipschitz and $\|g'\|_{\mathrm{u}}$ is a Lipschitz constant for g. Similarly h is Lipschitz with Lipschitz constant $\|h'\|_{\mathrm{u}}$. Therefore, given any points x, $y \in I$, we compute that

$$\begin{aligned} |f(x)-f(y)| \;&=\; \Big(|g(x)-g(y)|^2 \;+\; |h(x)-h(y)|^2\Big)^{1/2} \\ &\le\; \Big(\|g'\|_{\mathrm{u}}^2\,|x-y|^2 \;+\; \|h'\|_{\mathrm{u}}^2\,|x-y|^2\Big)^{1/2} \\ &=\; K\,|x-y|, \end{aligned}$$

where $K = \|g'\|_{\mathrm{u}} + \|h'\|_{\mathrm{u}}$.

2.1.20 Let $E = R_1 \cup \cdots \cup R_n$ where $R_1, \dots, R_n$ are nonoverlapping boxes. By definition of exterior measure, or by applying monotonicity, we have

$$|E|_e \;\le\; \sum_{j=1}^{n} |R_j|_e \;=\; \sum_{j=1}^{n} \mathrm{vol}(R_j).$$

Let $\{Q_k\}$ be any countable covering of E by countably many boxes, and fix $\varepsilon > 0$. Given $k \in \mathbb{N}$, let Q_k^* be a box that contains Q_k in its interior but is only slightly larger in the sense that

$$\mathrm{vol}(Q_k^*) \;\le\; (1+\varepsilon)\,\mathrm{vol}(Q_k).$$

Since $Q_k \subseteq (Q_k^*)^\circ$, the interiors of the boxes Q_k^* form an open covering of E: As E is compact, this covering must have a finite subcovering. Hence there exists some integer $N > 0$ such that

$$\bigcup_{j=1}^{N} R_j \;=\; E \;\subseteq\; \bigcup_{k=1}^{N} (Q_k^*)^\circ \;\subseteq\; \bigcup_{k=1}^{N} Q_k^*.$$

We wish to show that

$$\sum_{j=1}^{n} \mathrm{vol}(R_j) \;\le\; \sum_{k=1}^{N} \mathrm{vol}(Q_k^*).$$

We apply the same idea as in the proof of Lemma 2.1.6. That is, we extend the sides of the boxes Q_k^* to obtain a grid-like covering of E by smaller boxes. There can be duplicates in this covering. We can ignore any smaller boxes whose interiors lie completely outside of E. Hence we have a collection of smaller boxes that cover E. Recall that E is the union of the finitely many nonoverlapping boxes $R_1, \dots, R_n$. Each box R_j is covered by a distinct subset of these smaller boxes, and those smaller boxes make a grid-like cover of R_j, possibly with overlaps. Hence the sum of the volumes of the boxes R_j is bounded by the sum of all the volumes of the smaller boxes. That sum is itself bounded by the sum of the volumes of the boxes Q_k^*. This gives us the desired inequality

$$\sum_{j=1}^{n} \mathrm{vol}(R_j) \;\le\; \sum_{k=1}^{N} \mathrm{vol}(Q_k^*).$$

Putting this all together, we obtain

$$\sum_{j=1}^{n} \mathrm{vol}(R_j) \;\le\; \sum_{k=1}^{N} \mathrm{vol}(Q_k^*) \;\le\; (1+\varepsilon) \sum_{k=1}^{N} \mathrm{vol}(Q_k) \;\le\; (1+\varepsilon) \sum_{k} \mathrm{vol}(Q_k).$$

Taking the infimum over all such coverings by boxes, we see that

$$\sum_{j=1}^{n} \mathrm{vol}(R_j) \;\le\; (1+\varepsilon)\, |E|_e.$$

Finally, since ε is arbitrary, this yields

$$\sum_{j=1}^{n} \mathrm{vol}(R_j) \;\le\; |E|_e.$$

Alternative proof. Let $Q_1, \dots, Q_N$ be nonoverlapping boxes, and set

$$E \;=\; Q_1 \cup \cdots \cup Q_N.$$

By subadditivity,

$$|E|_e \;=\; \biggl|\bigcup_{k=1}^{N} Q_k\biggr| \;\le\; \sum_{k=1}^{N} |Q_k|_e \;=\; \sum_{k=1}^{N} \mathrm{vol}(Q_k),$$

so our task is to prove the opposite inequality.

Let $\{R_\ell\}$ be a cover of $E = Q_1 \cup \cdots \cup Q_N$ by countably many boxes. For each fixed k, the collection $\{R_\ell \cap Q_k\}_\ell$ is a covering of Q_k by boxes, so

$$\mathrm{vol}(Q_k) \;=\; |Q_k|_e \;\le\; \sum_{\ell} \mathrm{vol}(R_\ell \cap Q_k), \qquad k = 1, \dots, N.$$

Also, $\{R_\ell \cap Q_k\}_{k=1}^N$ is a finite collection of nonoverlapping boxes contained in R_ℓ. A variation on the ideas in Lemma 2.1.6 or Exercise 2.1.7 shows that

$$\sum_{k=1}^{N} \mathrm{vol}(R_\ell \cap Q_k) \;\le\; \mathrm{vol}(R_\ell).$$

Therefore

$$\sum_{k=1}^{N} \mathrm{vol}(Q_k) \;\le\; \sum_{k=1}^{N} \sum_{\ell} \mathrm{vol}(R_\ell \cap Q_k) \;\le\; \sum_{\ell} \mathrm{vol}(R_\ell).$$

Since this is true for every covering, we conclude that

$$\sum_{k=1}^{N} \operatorname{vol}(Q_k) \;\le\; \inf\Big\{\sum_{\ell} \operatorname{vol}(R_\ell)\Big\} \;=\; \Big|\bigcup_{k=1}^{N} Q_k\Big| \;=\; |E|_e,$$

where the infimum is taken over all possible coverings of $E = Q_1 \cup \cdots \cup Q_N$ by countably many boxes R_ℓ.

2.2.37 (a) $\Rightarrow$ (b). Suppose that E is measurable. Then, by definition of measurability, there exists an open set $U \supseteq E$ such that $|U \backslash E|_e < \varepsilon$. By Lemma 2.2.15, there exist an closed set $F \subseteq E$ such that $|E \backslash F|_e < \varepsilon$. Subadditivity therefore implies that

$$|U \backslash F| \;\le\; |U \backslash E|_e \;+\; |E \backslash F|_e \;<\; 2\varepsilon.$$

(b) $\Rightarrow$ (c). Assume that statement (b) holds. Then for each $k \in \mathbb{N}$, there exists an open set U_k and a closed set F_k such that $F_k \subseteq E_k \subseteq U_k$ and $|U_k \backslash F_k| < \frac{1}{k}$. Let $U = \cap U_k$ and $F = \cup F_k$. Then U is a G_δ-set and F is an F_σ-set and $F \subseteq E \subseteq U$. Further, for each k we have

$$|U \backslash F| \;\le\; |U_k \backslash F_k| \;<\; \frac{1}{k},$$

so $|U \backslash F| = 0$. Thus statement (c) holds.

(c) $\Rightarrow$ (a). Suppose that there exists a G_δ-set G and an F_σ-set H such that $H \subseteq E \subseteq G$ and $|G \backslash H| = 0$. Then by monotonicity,

$$|G \backslash E|_e \;\le\; |G \backslash H| \;=\; 0.$$

Hence G is a G_δ-set that contains E and satisfies $|G \backslash E|_e = 0$. Therefore E is measurable by Lemma 2.2.21.

2.3.12 Let Q be a cube in $\mathbb{R}^d$ with sides of length s. By the Pythagorean Theorem, the diameter of this cube is $d^{1/2}s$. That is, x and y are any two points in Q, then $\|x-y\| \le d^{1/2}s$. Since f is Lipschitz, it follows that

$$\|f(x)-f(y)\| \;\le\; K\,\|x-y\| \;\le\; Kd^{1/2}s.$$

Thus, the diameter of the set $f(Q)$ is at most $Kd^{1/2}s$. Consequently $f(Q)$ is contained in a closed ball of diameter at most $Kd^{1/2}s$, and hence is contained in a cube with sidelengths $Kd^{1/2}s$. Therefore the measure of $f(Q)$ is at most

$$|f(Q)|_e \;\le\; (Kd^{1/2}s)^d \;=\; K^d d^{d/2} s^d \;=\; C\,|Q|,$$

where $C = 2^d K^d d^{d/2}$ is a fixed constant that does not depend on the box Q.

2.3.17 Since $|A_n| \to |E|$, we can choose $n_1 < n_2 < \cdots$ so that

$$|E \setminus A_{n_k}| = |E| - |A_{n_k}| < 2^{-k}|E|, \qquad k \in \mathbb{N}.$$

Note that the first equality on the preceding line holds because E has finite measure (see Lemma 2.3.1).

Let $A = \cap A_{n_k}$. Applying Lemma 2.3.1 again, we compute that

$$\begin{aligned} |E| - |A| = |E \setminus A| &= \Bigl| E \setminus \bigcap_{k=1}^{\infty} A_{n_k} \Bigr| \\ &= \Bigl| \bigcup_{k=1}^{\infty} (E \setminus A_{n_k}) \Bigr| \\ &\le \sum_{k=1}^{\infty} |E \setminus A_{n_k}| \\ &< \sum_{k=1}^{\infty} 2^{-k}|E| = |E|. \end{aligned}$$

Rearranging, we see that $|A| > 0$.

To see that this can fail if the measure of E is infinite, set $E = \mathbb{R}$ and $A_n = [2^n, 2^{n+1}]$. Then $|A_n| = 2^n \to \infty = |E|$, but $\cap A_{n_k} = \varnothing$ for every choice of indices $n_1 < n_2 < \cdots$.

2.3.21 Suppose that no such point x exists. Then for each $x \in E$ there is some $\delta_x > 0$ such that

$$|E \cap B_{\delta_x}(x)| = 0.$$

By Exercise 2.3.20(d), there exists a compact set $K \subseteq E$ such that $|K| > 0$. Then $\{B_{\delta_x}(x)\}_{x \in E}$ is an open cover of K, so there must exist finitely many points $x_1, \dots, x_N \in E$ such that

$$K \subseteq \bigcup_{k=1}^{N} B_{\delta_k}(x_k), \qquad \text{where } \delta_k = \delta_{x_k}.$$

But then

$$K = K \cap E \subseteq \bigcup_{k=1}^{N} \bigl(B_{\delta_k}(x_k) \cap E\bigr),$$

so

$$|K| \le \sum_{k=1}^{N} \bigl|B_{\delta_k}(x_k) \cap E\bigr| = \sum_{k=1}^{N} 0 = 0,$$

which is a contradiction.

2.4.10 We will take the (longer) approach of generalizing the Steinhaus Theorem to higher dimensions.

Step 1. Let $Q = [0, s]^d$. We claim that if $t = (t_1, \dots, t_d) \in \mathbb{R}^d$, then

$$|Q \cap (Q+t)| \;\le\; \sum_{k=0}^{d} \binom{d}{k} s^k \, \|t\|^{d-k}.$$

First assume that $0 \le t_k$ for every k. In this case we have $0 \le t_k \le \|t\|$ for every k, so

$$Q \cup (Q+t) \;\subseteq\; [0, s+\|t\|]^d,$$

and therefore, by the Binomial Theorem,

$$|Q \cup (Q+t)| \;\le\; (s+\|t\|)^d \;=\; \sum_{k=0}^{d} \binom{d}{k} s^k \, \|t\|^{d-k}.$$

A similar argument applies if any t_k is negative, so the claim follows.

Step 2. Now we generalize the Steinhaus Theorem. We claim that if $E \subseteq \mathbb{R}^d$ is Lebesgue measurable and $|E| > 0$, then the set of differences

$$E - E \;=\; \bigl\{x - y : x, y \in E\bigr\}$$

contains an open ball $B_r(0)$ for some $r > 0$.

To see this, we apply Problem 2.2.39 and conclude that there exists a cube Q such that the measure of the set $F = E \cap Q$ satisfies

$$|F| \;=\; |E \cap Q| \;>\; \frac{3}{4}\,|Q|. \qquad \text{(A)}$$

The statement of Steinhaus' Theorem is invariant under translations, so by translating E, F, and Q we can assume that $Q = [0,s]^d$ where $s > 0$.

Choose any $t \in \mathbb{R}^d$. If F and $F+t$ are disjoint, then we must have

$$\begin{aligned}
2s^d \;=\; 2\,|Q| \;&<\; 2 \cdot \frac{4}{3}\,|F| && \text{(by equation (A))}\\
&=\; \frac{4}{3}\,|F \cup (F+t)| && \text{(since } F \text{ and } F+t \text{ are disjoint)}\\
&\le\; \frac{4}{3}\,|Q \cup (Q+t)| && \text{(by monotonicity)}\\
&\le\; \frac{4}{3} \sum_{k=0}^{d} \binom{d}{k} s^k \, \|t\|^{d-k} && \text{(by the Lemma).} \qquad \text{(B)}
\end{aligned}$$

However,

$$\lim_{\|t\|\to 0} \frac{4}{3} \sum_{k=0}^{d} \binom{d}{k} s^k \, \|t\|^{d-k} \;=\; \frac{4}{3}\, s^d \;<\; 2s^d.$$

Therefore if $\|t\|$ is small enough then equation (B) cannot hold. Hence there is some $r > 0$ such that

$$\|t\| < r \implies F \text{ and } F+t \text{ are not disjoint.}$$

Therefore, if $\|t\| < r$ then there is some point $x \in F \cap (F+t)$. So, $x = y + t$ for some $y \in F$, which implies that $t = x - y \in F - F$. This shows that $F - F$ contains the open ball $B_r(0)$, and therefore $E - E$ must contain this ball as well.

Step 3. Define a relation on $\mathbb{R}^d$ by declaring that $x \sim y$ if and only if every component of $x - y$ is rational. This is an equivalence relation, so by the Axiom of Choice there exists a set N that contains exactly one element of each distinct equivalence class of this relation.

The distinct equivalence classes partition $\mathbb{R}^d$, so their union is $\mathbb{R}^d$. Therefore

$$\mathbb{R}^d = \bigcup_{x\in N} (\mathbb{Q}^d + x) = \bigcup_{x\in N} \bigcup_{r\in\mathbb{Q}^d} \{r+x\} = \bigcup_{r\in\mathbb{Q}^d} (N+r).$$

Since exterior Lebesgue measure is translation-invariant, the exterior measure of $N + r$ is exactly the same as the exterior measure of N. Combining this fact with countable subadditivity, we see that

$$\infty = |\mathbb{R}|_e = \biggl|\bigcup_{r\in\mathbb{Q}^d} (N+r)\biggr|_e \le \sum_{r\in\mathbb{Q}^d} |N+r|_e = \sum_{r\in\mathbb{Q}^d} |N|_e.$$

Consequently, we must have $|N|_e > 0$. However, any two distinct points $x \neq y$ in N belong to distinct equivalence classes of the relation $\sim$, so some component of x and y must differ by an irrational amount. Therefore $N - N$ contains no open balls, so the Steinhaus Theorem implies that N cannot be Lebesgue measurable.

3.1.18 (a) "$\Leftarrow$." Suppose that $f^{-1}(U)$ is measurable for each open set $U \subseteq \mathbb{R}$. Then for each $a \in \mathbb{R}$ we have that

$$\{f > a\} = \{x \in \mathbb{R}^d : a < f(x)\} = f^{-1}(a,\infty)$$

is measurable, so f is measurable.

"$\Rightarrow$." Suppose that $f\colon \mathbb{R}^d \to \mathbb{R}$ is measurable, and let $U \subseteq \mathbb{R}$ be any open set. Then we can write U as a countable disjoint union of open intervals (possibly including infinite open intervals), say $U = \cup(a_j, b_j)$. Since

$$f^{-1}(a_j, b_j) = \{a_j < f < b_j\} = \{a_j < f\} \cap \{f < b_j\},$$

we conclude that $f^{-1}(a_j, b_j)$ is measurable for each j, and hence $f^{-1}(U) = \cup f^{-1}(a_j, b_j)$ is measurable.

(b) "$\Rightarrow$." Suppose that $f\colon \mathbb{R}^d \to \mathbb{C}$ is measurable. Then its real part f_r and its imaginary part f_i are both measurable. For simplicity let us identify $\mathbb{C}$ with $\mathbb{R}^2$. In particular, with this identification we write $f(x) = (f_r(x), f_i(x))$.

Given an open strip $(a,b)\times\mathbb{R}$ in $\mathbb{C}$, we have

$$f^{-1}\big((a,b)\times\mathbb{R}\big) \;=\; f_r^{-1}(a,b),$$

which is measurable since f_r is measurable.. Similarly,

$$f^{-1}\big(\mathbb{R}\times(c,d)\big) \;=\; f_i^{-1}(c,d)$$

is measurable. Consequently the inverse image of the open rectangle

$$(a,b)\times(c,d) \;=\; \big((a,b)\times\mathbb{R}\big) \,\cap\, \big(\mathbb{R}\times(c,d)\big)$$

is measurable. Every open subset of $\mathbb{C}$ can be written as a countable union of open rectangles, so it follows that $f^{-1}(U)$ is measurable for every open set $U\subseteq\mathbb{C}$.

"$\Leftarrow$." Suppose that the inverse image of any open subset of $\mathbb{C}$ is measurable. Again identifying $\mathbb{C}$ with $\mathbb{R}^2$, if we fix $a\in\mathbb{R}$ then the set $(a,\infty)\times\mathbb{R}$ is open in $\mathbb{C}$. Hence

$$\{f_r>a\} \;=\; f_r^{-1}(a,\infty) \;=\; f^{-1}\big((a,\infty)\times\mathbb{R}\big)$$

is measurable. Therefore f_r is a measurable function, and similarly f_i is measurable, so we conclude that f is measurable.

3.2.16 *Case 1:* $c\in\mathbb{R}$. Without loss of generality, consider $c=0$. Define

$$Z_1 \;=\; \{f=\infty\} \,\cap\, \{g=-\infty\},$$
$$Z_2 \;=\; \{f=-\infty\} \,\cap\, \{g=\infty\}.$$

The sets Z_1 and Z_2 are measurable since f and g are measurable. Therefore

$$Z \;=\; Z_1\cup Z_2$$

is a measurable set as well.

Define

$$F(x) \;=\; (f\cdot\chi_{Z^{\mathrm{C}}})(x) \;=\; \begin{cases} f(x), & x\notin Z,\\ 0, & x\in Z,\end{cases}$$

and

$$G(x) \;=\; (g\cdot\chi_{Z^{\mathrm{C}}})(x) \;=\; \begin{cases} g(x), & x\notin Z,\\ 0, & x\in Z.\end{cases}$$

The function h defined in the problem statement is $h=F+G$.

Fix $a\in\mathbb{R}$. If $a>0$ then

$$\{F>a\} \;=\; \{f>a\}\setminus Z.$$

If $a \le 0$ then

$$\{F > a\} \;=\; \{f > a\} \,\cup\, Z.$$

In any case, $\{F > a\}$ is measurable, so F is a measurable function. Similarly, G is measurable. Hence $a - G = a + (-1)G$ is measurable, and therefore

$$\{h > a\} \;=\; \{F + G > a\} \;=\; F > a - G$$

is measurable. Consequently h is a measurable function.

Case 2: $c = \infty$. Define

$$F(x) \;=\; \begin{cases} f(x), & x \notin Z, \\ \infty, & x \in Z, \end{cases}$$

and

$$G(x) \;=\; \begin{cases} g(x), & x \notin Z, \\ \infty, & x \in Z, \end{cases}$$

so $h = F + G$. If $a \in \mathbb{R}$ then

$$\{F > a\} \;=\; \{f > a\} \,\cup\, Z,$$

which is measurable. Hence F is measurable and likewise G is measurable. A lemma from the text therefore implies as before that $\{F > a - G\}$ is measurable, and this is the same set as $\{h > a\}$.

Case 3: $c = -\infty$. This is similar to Case 2.

3.3.8 "$\Rightarrow$." Suppose that $f_n \to f$ in L^∞-norm. For each n, let

$$Z_n \;=\; \{|f - f_n| > \|f - f_n\|_\infty\}.$$

By definition of the L^∞-norm, each set Z_n has measure zero. Therefore the set

$$Z \;=\; \bigcup_{n=1}^{\infty} Z_n$$

also has measure zero.

If $x \notin Z$, then $x \notin Z_n$ for any n, so $|f(x) - f_n(x)| \le \|f - f_n\|_\infty$ for every n. Letting $\|f - f_n\|_{\mathrm{u}}$ denote the uniform norm on the set $E \backslash Z$, we therefore have

$$\|f - f_n\|_{\mathrm{u}} \;=\; \sup_{x \in E \backslash Z} |f(x) - f_n(x)| \;\le\; \|f - f_n\|_\infty \;\to\; 0 \quad \text{as } n \to \infty.$$

Thus f_n converges uniformly to f on $E \backslash Z$.

"$\Leftarrow$." Suppose that Z is a set of measure zero such that $f_n \to f$ uniformly on $E \backslash Z$. Letting $\|f - f_n\|_{\mathrm{u}}$ denote the uniform norm on the set $E \backslash Z$, we have

$$|f(x) - f_n(x)| \le \|f - f_n\|_{\mathrm{u}}, \qquad \text{all } x \in E \backslash Z.$$

As Z has measure zero, we therefore have

$$|f(x) - f_n(x)| \le \|f - f_n\|_{\mathrm{u}}, \qquad \text{a.e. } x \in E.$$

Hence

$$\|f - f_n\|_\infty \le \|f - f_n\|_{\mathrm{u}}.$$

The opposite inequality follows by definition. Therefore

$$\|f - f_n\|_\infty = \|f - f_n\|_{\mathrm{u}} \to 0 \quad \text{as } n \to \infty.$$

Thus f_n converges to f in L^∞-norm on E.

3.4.5 (a) $f_n = \chi_{[-n,n]}$ converges pointwise to the constant function $f = 1$, but the convergence is not uniform on any unbounded subset of $\mathbb{R}$.

Another example is $f_n(x) = x/n$, which converges pointwise to the zero function, but the convergence is not uniform on any unbounded subset of $\mathbb{R}$.

(b) Suppose that $|E| > 0$. Even if each f_n is finite a.e., we must require f to be finite a.e. For example, if $f_n(x) = n$ for $x \in E$, then f_n is finite everywhere and f_n converges pointwise to the function $f = \infty$, but the convergence is not uniform on any subset of E.

3.5.16 The proof is similar to that of Problem 1.1.20.

Fix $\varepsilon > 0$ and $\eta > 0$, and let

$$\gamma = \min\{\varepsilon, \eta\}.$$

Since $f_{n_k} \stackrel{\mathrm{m}}{\to} f$, there exists some $K > 0$ such that

$$n_k > K \quad \Longrightarrow \quad \bigl|\{|f - f_{n_k}| > \gamma\}\bigr| < \gamma.$$

Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure, there exists an N such that

$$m, n > N \quad \Longrightarrow \quad \bigl|\{|f_m - f_n| > \gamma\}\bigr| < \gamma.$$

Suppose that $n > N$. Then since the n_k are strictly increasing, there exists some n_k that is greater than both K and N. Since

$$\{|f - f_n| > 2\varepsilon\} \subseteq \{|f - f_n| > 2\gamma\} \subseteq \{|f - f_{n_k}| > \gamma\} \cup \{|f_{n_k} - f_n| > \gamma\},$$

we have

$$\bigl|\{|f - f_n| > 2\varepsilon\}\bigr| \le \bigl|\{|f - f_{n_k}| > \gamma\}\bigr| + \bigl|\{|f_{n_k} - f_n| > \gamma\}\bigr| < \gamma + \gamma \le 2\eta.$$

This is true for all $n > N$, so

$$\lim_{n\to\infty} \bigl|\{|f - f_n| > 2\varepsilon\}\bigr| = 0.$$

That is, $f_n \overset{\mathrm{m}}{\to} f$.

4.1.3 (e) If $\int_{A_k} \phi = \infty$ for some k then there is nothing to prove, so we may assume that $\int_{A_k} \phi < \infty$ for every k. If we set $A_0 = \varnothing$, then

$$\bigcup_{k=1}^{\infty} A_k \;=\; \bigcup_{j=1}^{\infty} (A_j \setminus A_{j-1}),$$

and the sets on the right-hand side above are disjoint. Further,

$$A_{j+1} \;=\; A_j \,\cup\, (A_{j+1} \setminus A_j),$$

and the sets on the right are disjoint. Lemma 4.1.2 therefore implies that

$$\int_{A_{j+1}} \phi \;=\; \int_{A_j} \phi \;+\; \int_{A_{j+1} \setminus A_j} \phi,$$

where all of these integrals are finite. Applying Lemma 4.1.2 again, we see that

$$\begin{aligned}
\int_A \phi \;&=\; \sum_{j=1}^{\infty} \int_{A_j \setminus A_{j-1}} \phi \\
&=\; \lim_{N\to\infty} \sum_{j=1}^{N} \biggl(\int_{A_j} \phi \;-\; \int_{A_{j-1}} \phi \biggr) \\
&=\; \lim_{N\to\infty} \int_{A_N} \phi \;-\; \int_{A_0} \phi \\
&=\; \lim_{N\to\infty} \int_{A_N} \phi.
\end{aligned}$$

4.2.11 Let $E_n = E \cap [-n,n]^d$, and set $f_n = f\chi_{E_n}$. Then $0 \le f_n \le f$ and $f_n \to f$ pointwise, so Problem 4.2.9 implies that

$$\lim_{n\to\infty} \int_{E_n} f \;=\; \lim_{n\to\infty} \int_E f_n \;=\; \int_E f.$$

The result therefore follows by taking $A = E_n$ with n large enough.

4.3.9 Given a measurable set $E \subseteq \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} \chi_E(x-a)\,dx \;=\; \int_{\mathbb{R}^d} \chi_{E+a}(x)\,dx \;=\; |E+a| \;=\; |E| \;=\; \int_{\mathbb{R}^d} \chi_E(x)\,dx.$$

Hence the integral of a characteristic function is invariant under translations. Taking linear combinations, this fact extends to simple functions. Given a nonnegative function $f\colon \mathbb{R}^d \to [0,\infty]$, there exist simple functions ϕ_n that increase pointwise to f. The functions $\phi_n(x-a)$ increase pointwise to $f(x-a)$, so by applying the Monotone Convergence Theorem we see that

$$\begin{aligned} \int_{\mathbb{R}^d} f(x-a)\,dx &= \lim_{n\to\infty} \int_{\mathbb{R}^d} \phi_n(x-a)\,dx \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^d} \phi_n(x)\,dx \\ &= \int_{\mathbb{R}^d} f(x)\,dx. \end{aligned}$$

Now suppose that $f\colon \mathbb{R}^d \to [-\infty,\infty]$ is an arbitrary extended real-valued function whose integral exists. Then the integrals of f^+ and f^- both exist, with at most one of these being infinite. Applying the translation-invariance proved for nonnegative functions, it follows that

$$\begin{aligned} \int_{\mathbb{R}^d} f(x-a)\,dx &= \int_{\mathbb{R}^d} f^+(x-a)\,dx - \int_{\mathbb{R}^d} f^-(x-a)\,dx \\ &= \int_{\mathbb{R}^d} f^+(x)\,dx - \int_{\mathbb{R}^d} f^-(x)\,dx \\ &= \int_{\mathbb{R}^d} f(x)\,dx. \end{aligned}$$

Finally, if f is complex-valued then we write $f = f_r + if_i$ and use the fact that the integrals of f_r and f_i are invariant under translations.

The proof for invariance under reflection is similar, starting with the calculation

$$\int_{\mathbb{R}^d} \chi_E(-x)\,dx = \int_{\mathbb{R}^d} \chi_{-E}(x)\,dx = |-E| = |E| = \int_{\mathbb{R}^d} \chi_E(x)\,dx.$$

This equality then extends by cases to generic functions.

An alternative approach is to use Problem 4.2.17, i.e., compare the regions under the graph of $f(x-a)$ or $f(-x)$ to the region under the graph of $f(x)$.

4.4.17 (a) Since f is integrable, it is finite a.e. Let $h(x) = f(x)$ whenever $f(x)$ is finite, and set $h(x) = 0$ when $f(x) = \pm\infty$. Then h is measurable and finite at every point. Therefore $g(x) - h(x)$ never takes an indeterminate form. As h and g are both measurable, it follows from Lemma 3.2.1 that $g-h$ is measurable. As $g - f = g - h$ a.e., it follows that $g - f$ is also measurable.

Since $g \ge f$ a.e., we have $g^- \le f^-$ a.e. As f is integrable, it follows that

$$0 \le \int_E g^- \le \int_E f^- \le \int_E |f| < \infty.$$

Further, $\int_E g^+$ exists as a nonnegative, extended real number. Therefore the integral of g on E exists, and

$$-\infty \;<\; \int_E g \;\le\; \infty.$$

Also, $g - f \ge 0$ a.e., so the integral of $g - f$ exists and is a nonnegative, extended real number.

If g is integrable, then by the linearity of the integral for integrable functions, we immediately obtain

$$\int_E (g-f) \;=\; \int_E g - \int_E f,$$

so in this case we are finished. On the other hand, if g is not integrable, then we must have $\int_E g^+ = \infty$ and therefore

$$\int_E g \;=\; \int_E g^+ \;-\; \int_E g^- \;=\; \infty.$$

Since f is integrable, it follows that

$$\int_E g \;-\; \int_E f \;=\; \infty.$$

As g is not integrable, f is integrable, and the sum of integrable functions is integrable, the function $g - f$ cannot be integrable. Since $g - f \ge 0$ a.e., we therefore have

$$\int_E (g-f) \;=\; \infty.$$

Consequently,

$$\int_E (g-f) \;=\; \infty \;=\; \int_E g - \int_E f.$$

(b) First we consider the extension of the MCT.

Suppose that $f_n \ge g$ a.e., where g is integrable, and $f_n \nearrow f$ on E. Then $f_n - g \ge 0$ for every n, and $f_n - g \nearrow f - g$ a.e. Applying part (a) and the Monotone Convergence Theorem (or more precisely the MCT variation derived in Theorem 4.3.7), we therefore obtain

$$\begin{aligned}\int_E f \;-\; \int_E g \;&=\; \int_E (f-g)\\ &=\; \lim_{n\to\infty} \int_E (f_n - g)\end{aligned}$$

$$= \lim_{n\to\infty} \left(\int_E f_n - \int_E g \right)$$

$$= \left(\lim_{n\to\infty} \int_E f_n \right) - \int_E g.$$

As $\int_E g$ is finite, it follows that

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

Now we consider the extension of Fatou's Lemma. The argument is similar. Assume that $f_n \ge g$ a.e., where g is integrable, and let $f = \liminf f_n$. Then $f_n - g \ge 0$ a.e., so by Fatou's Lemma and part (a) we have

$$\begin{aligned}
\int_E f - \int_E g &= \int_E (f-g) \\
&= \int_E \liminf_{n\to\infty} (f_n - g) \\
&\le \liminf_{n\to\infty} \int_E (f_n - g) \\
&= \liminf_{n\to\infty} \left(\int_E f_n - \int_E g \right) \\
&= \left(\liminf_{n\to\infty} \int_E f_n \right) - \int_E g.
\end{aligned}$$

As g is integrable, it follows that

$$\int_E f \le \liminf_{n\to\infty} \int_E f_n.$$

To see that the assumption that g is integrable is necessary, let $E = \mathbb{R}$ and consider $f_n = -\frac{1}{n}$. Then $f_n \ge -1$ for every n and $f_n \nearrow 0$, but

$$\int_{\mathbb{R}} f_n = -\infty \not\to 0 = \int_{\mathbb{R}} 0.$$

Also,

$$\int_{\mathbb{R}} \liminf_{n\to\infty} f_n = \int_{\mathbb{R}} 0 = 0$$

which is strictly greater than

$$\liminf_{n\to\infty} \int_{\mathbb{R}} f_n = -\infty.$$

4.5.17 *First proof.* Fix $\varepsilon > 0$. Since f is integrable, Exercise 4.5.5 implies that there exists a constant $\delta > 0$ such that $\int_E |g| < \varepsilon$ for every measurable set E satisfying $|E| < \delta$. If $x \in \mathbb{R}$ and $0 \le h < \delta$, then the measure of the interval $[x, x+h]$ is less than δ, so

$$|F(x+h) - F(x)| \;=\; \left|\int_0^{x+h} f - \int_0^x f\right| \;=\; \left|\int_x^{x+h} f\right| \;\le\; \int_x^{x+h} |f| \;\le\; \varepsilon.$$

A similar argument applies if $-\delta < h \le 0$, so we conclude that F is uniformly continuous.

Second proof. Given $x,\ a \in \mathbb{R}$, we compute that

$$\begin{aligned}
|F(x-a) - F(x)| \;&=\; \left|\int_{-\infty}^{x-a} f(t)\,dt \;-\; \int_{-\infty}^{x} f(t)\,dt\right| \\
&=\; \left|\int_{-\infty}^{x} f(t-a)\,dt \;-\; \int_{-\infty}^{x} f(t)\,dt\right| \\
&\le\; \int_{-\infty}^{x} \bigl|T_a f(t) - f(t)\bigr|\,dt \\
&\le\; \|T_a f - f\|_1.
\end{aligned}$$

Therefore

$$\|T_a F - F\|_{\mathrm{u}} \;=\; \sup_{x\in\mathbb{R}} |F(x-a) - F(x)| \;\le\; \|T_a f - f\|_1 \;\to\; 0,$$

so F is uniformly continuous by Problem 1.3.6.

4.6.24 We are given that

$$M \;=\; \sum_{n=1}^{\infty} \|f_n\|_1 \;<\; \infty,$$

i.e., the series $\sum f_n$ converges absolutely in L^1-norm. Since each f_n is integrable, it follows that

$$\sum_{n=1}^{\infty} \left|\int_E f_n\right| \;\le\; \sum_{n=1}^{\infty} \int_E |f_n| \;=\; \sum_{n=1}^{\infty} \|f_n\|_1 \;<\; \infty.$$

Therefore the series

$$\sum_{n=1}^{\infty} \int_E f_n$$

is an absolutely convergent series of scalars, so it converges to some finite scalar.

To show that the series $\sum_{n=1}^{\infty} f_n(x)$ converges a.e., set

$$g_N(x) \;=\; \sum_{n=1}^{N} |f_n(x)| \qquad \text{and} \qquad g(x) \;=\; \sum_{n=1}^{\infty} |f_n(x)|.$$

These are series of nonnegative extended real numbers, so they converge pointwise a.e. in the extended real sense. By the Triangle Inequality, for each N we have

$$\|g_N\|_1 \;\le\; \sum_{n=1}^{N} \|f_n\|_1 \;\le\; M.$$

Since $g_N \nearrow g$, the Monotone Convergence Theorem implies that

$$\begin{aligned} \|g\|_1 \;=\; \int_E g \;&=\; \lim_{N\to\infty} \int_E g_N \\ &=\; \lim_{N\to\infty} \|g_N\|_1 \\ &\le\; \lim_{N\to\infty} \sum_{n=1}^{N} \|f_n\|_1 \;=\; M \;<\; \infty. \end{aligned}$$

Therefore $g \in L^1(E)$, i.e., g is integrable on E. Hence g is finite a.e. At any point x where $g(x) < \infty$, we have

$$g(x) \;=\; \sum_{n=1}^{\infty} |f_n(x)| \;<\; \infty.$$

Consequently, the series

$$f(x) \;=\; \sum_{n=1}^{\infty} f_n(x)$$

converges absolutely at almost every point x. Since $|f| \le g$, we have $f \in L^1(E)$. Also, if we set

$$h_N(x) \;=\; \sum_{n=1}^{N} f_n(x),$$

then $h_N \to f$ pointwise a.e. For every N we have

$$|h_N| \;\le\; g_N \;\le\; g \;\in\; L^1(\mathbb{R}),$$

so we can apply the Dominated Convergence Theorem. The DCT tells us that $h_N \to f$ in L^1-norm, and the integral of h_N converges to the integral of f. Thus

$$\int_E \sum_{n=1}^{\infty} f_n \;=\; \int_E f \;=\; \lim_{N\to\infty} \int_E h_N$$

$$= \lim_{N\to\infty} \int_E \sum_{n=1}^{N} f_n$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \int_E f_n \;=\; \sum_{n=1}^{\infty} \int_E f_n.$$

5.1.5 If $|x-y| \le 3^{-k}$, then we have $|\varphi(x)-\varphi(y)| \le 2^{-k}$. Let $k \ge 0$ be the unique integer such that

$$\frac{1}{3^{k+1}} \;<\; |x-y| \;\le\; \frac{1}{3^k}.$$

Then

$$|\varphi(x)-\varphi(y)| \;\le\; \frac{2}{2^{k+1}} \;=\; 2\left(\frac{1}{3^{k+1}}\right)^{\log_3 2} \;\le\; 2\,|x-y|^{\log_3 2}.$$

Hence φ is Hölder continuous with exponent $\alpha = \log_3 2$.

Fix $0 < \beta < \alpha = \log_3 2$. If $x,\, y \in [0,1]$ then $|x-y| \le 1$, so

$$|\varphi(x)-\varphi(y)| \;\le\; 2\,|x-y|^{\alpha} \;=\; 2\,|x-y|^{\alpha-\beta}\,|x-y|^{\beta} \;\le\; 2\,|x-y|^{\beta}.$$

Hence φ is Hölder continuous with exponent β.

On the other hand,

$$|\varphi(3^{-k})-\varphi(0)| \;=\; |2^{-k}-0| \;=\; 2^{-k} \;=\; (3^{-k})^{\log_3 2}.$$

It follows φ cannot be Hölder continuous for any exponent $\alpha > -\log_3 2$.

5.2.4 (a) Since $f(x) = x\sin(1/x)$, we have for $h \ne 0$ that

$$\frac{f(0+h)-f(0)}{h-0} \;=\; \frac{h\sin\frac{1}{h}}{h} \;=\; \sin\frac{1}{h}.$$

Since this quantity does not converge as $h \to 0$, we see that f is not differentiable at $x = 0$.

For $n \in \mathbb{N}$, we have

$$\left|f\Big(\frac{2}{n\pi}\Big)\right| \;=\; \frac{2}{n\pi}\left|\sin\Big(\frac{n\pi}{2}\Big)\right| \;=\; \begin{cases} \frac{2}{n\pi}, & n \text{ odd},\\ 0, & n \text{ even}. \end{cases}$$

Therefore,

$$\sum_{k=1}^{N}\left|f\bigg(\frac{2}{(k+1)\pi}\bigg) - f\Big(\frac{2}{k\pi}\Big)\right| \;\ge\; \sum_{\substack{j=1,\dots,N\\ j \text{ odd}}} \frac{2}{j\pi}.$$

Hence, if we set

$$\Gamma_N = \Bigl\{-1,\, 0,\, \frac{2}{N\pi},\, \frac{2}{(N-1)\pi},\, \dots,\, \frac{2}{\pi}, 1\Bigr\},$$

then

$$S_{\Gamma_N} \ge \sum_{\substack{j=1,\dots,N \\ j \text{ odd}}} \frac{2}{j\pi}.$$

Since $\sup S_{\Gamma_N} = \infty$, we see that f does not have bounded variation.

5.2.23 (a) We are given complex-valued functions f_n on $[a,b]$ that converge pointwise to a limit f. Let $\Gamma = \{a = x_0 < \cdots < x_n = b\}$ be any partition of $[a,b]$. Then, using the discrete version of Fatou's Lemma, we compute that

$$\begin{aligned}
S_\Gamma[f;a,b] &= \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \\
&= \sum_{j=1}^{n} \liminf_{n\to\infty} |f_n(x_j) - f_n(x_{j-1})| \\
&\le \liminf_{n\to\infty} \sum_{j=1}^{n} |f_n(x_j) - f_n(x_{j-1})| \\
&= \liminf_{n\to\infty} S_\Gamma[f_n;a,b] \\
&\le \liminf_{n\to\infty} V[f_n;a,b].
\end{aligned}$$

Taking the supremum over all partitions Γ, it follows that

$$V[f;a,b] \le \liminf_{n\to\infty} V[f_n;a,b].$$

5.3.5 Let the balls B_k be constructed just as in the proof of Theorem 5.3.3. If the construction process stops after a finite number of steps then we are done, so suppose that the process does not end.

Choose any point $x \in E \setminus \bigcup_{k=1}^\infty B_k$. Then given $N \in \mathbb{N}$ we have $x \in E \setminus \bigcup_{k=1}^N B_k$. The argument of the proof of Theorem 5.3.3 shows that we then have $x \in B_n^*$ for some $n > N$. Therefore

$$\Bigl|E \setminus \bigcup_{k=1}^{\infty} B_k\Bigr|_e \le \Bigl|E \setminus \bigcup_{k=1}^{N} B_k\Bigr|_e \le \sum_{k=N+1}^{\infty} |B_k^*| = 5^d \sum_{k=N+1}^{\infty} |B_k|.$$

But N is arbitrary and $\sum |B_k| < \infty$, so this implies that $E \setminus \bigcup_{k=1}^\infty B_k$ has measure zero.

5.4.6 First assume that there is some $\delta > 0$ such that $D^+ f \ge \delta$ on (a,b). Fix $a < x < y < b$. Since f is continuous on the closed bounded interval $[x,y]$, it has a max at some point in that interval, say at x_0. Suppose that

$x \le x_0 < y$. Then for all $x_0 < t < y$ we have

$$\frac{f(t) - f(x_0)}{t - x_0} \;\le\; 0,$$

and therefore

$$D^+ f(x_0) \;=\; \limsup_{t \to x_0^+} \frac{f(t) - f(x_0)}{t - x_0} \;\le\; 0.$$

This is a contradiction. Therefore f must achieve its maximum on $[x, y]$ at the point y. Consequently $f(x) \le f(y)$. This shows that f is monotone increasing on (a, b). Since f is continuous, it follows that f is monotone increasing on all of $[a, b]$.

Now suppose that we just have $D^+ f \ge 0$ on (a, b). Fix $\delta > 0$, and let $g(x) = f(x) + \delta x$, which is continuous. The limsup of a sum is not the sum of limsups in general, but it is if one of the limsups is a limit. That is the case here:

$$\begin{aligned} D^+ g(x) \;&=\; \limsup_{t \to x^+} \frac{g(t) - g(x)}{t - x} \\ &=\; \limsup_{t \to x^+} \frac{f(t) - f(x)}{t - x} \;+\; \lim_{t \to x^+} \frac{\delta t - \delta x)}{t - x} \\ &=\; D^+ f(x) \;+\; \delta \;\ge\; \delta. \end{aligned}$$

Our work above therefore implies that g is monotone increasing on $[a, b]$. Hence, given any $a \le x < y \le b$ we have

$$f(x) + \delta x \;=\; g(x) \;\le\; g(y) \;=\; f(y) + \delta y.$$

Rearranging,

$$f(y) - f(x) \;\ge\; \delta(y - x).$$

Letting $\delta \to 0$, we obtain $f(y) - f(x) \ge 0$. Thus $f(x) \le f(y)$, so f is monotone increasing on $[a, b]$.

5.5.17 Since $B_h(x)$ is one of the open balls that contain x, we immediately obtain $Mf(x) \le M^* f(x)$.

Suppose that B is any open ball that contains x. Then $B = B_r(y)$ for some point y and some radius $r > 0$, and $\|x - y\| < r$.

Let z be any point in B. Then $\|y - z\| < r$, so

$$\|x - z\| \;\le\; \|x - y\| + \|y - z\| \;<\; r + r \;=\; 2r.$$

Hence $z \in B_{2r}(x)$. This shows that $B \subseteq B_{2r}(x)$. Note that

$$|B| \;=\; C_d r^d \qquad \text{and} \qquad |B_{2r}(x)| \;=\; C_d (2r)^d \;=\; C_d 2^d r^d$$

where C_d is a constant that depends only on the dimension. Therefore $|B| = 2^{-d}|B_{2r}(x)|$, so

$$\frac{1}{|B|}\int_B |f| \;\le\; \frac{1}{2^{-d}|B_{2r}(x)|}\int_{B_{2r}(x)} |f| \;\le\; \frac{1}{2^{-d}}\,Mf(x) \;=\; 2^d Mf(x).$$

Taking the supremum over all open balls B that contain x, it follows that

$$M^*f(x) \;\le\; 2^d Mf(x).$$

6.1.7 "⇒." Assume that $f \in \mathrm{AC}[a,b]$ and fix $\varepsilon > 0$. Let δ be the number whose existence is given in definition of absolute continuity. Assume that $\{[a_j, b_j]\}$ is any countable collection of nonoverlapping subintervals of $[a,b]$ such that

$$\sum_j (b_j - a_j) \;<\; \delta.$$

Then, by definition of absolute continuity, we have

$$\sum_j |f(b_j) - f(a_j)| \;<\; \varepsilon.$$

For any complex number $z = z_r + iz_i$ we have

$$|z_r| \;\le\; \Bigl(|z_r|^2 + |z_i|^2\Bigr)^{1/2} \;=\; |z|.$$

Therefore

$$\sum_j |f_r(b_j) - f_r(a_j)| \;\le\; \sum_j |f(b_j) - f(a_j)| \;<\; \varepsilon.$$

Therefore f_r is absolutely continuous, and a similar argument shows that f_i is absolutely continuous.

"⇐." Assume that f_r and f_i are each absolutely continuous. Fix $\varepsilon > 0$, and let δ_r and δ_i be the numbers whose existence is given by applying the definition of absolute continuity to f_r and f_i, respectively. Let

$$\delta \;=\; \min\{\delta_r, \delta_i\}.$$

Assume that $\{[a_j, b_j]\}$ is any countable collection of nonoverlapping subintervals of $[a,b]$ such that

$$\sum_j (b_j - a_j) \;<\; \delta.$$

For any complex number $z = z_r + iz_i$ we have

$$|z| = |z_r + iz_i| \le |z_r| + |z_i|.$$

Therefore

$$\sum_j |f(b_j) - f(a_j)| \le \sum_j \Big(|f_r(b_j) - f_r(a_j)| + |f_i(b_j) - f_i(a_j)| \Big) < \varepsilon + \varepsilon = 2\varepsilon.$$

Therefore f is absolutely continuous.

6.2.5 Suppose that f is differentiable a.e. on $E \subseteq [a, b]$, and $A \subseteq E$ is such that $f(x) = c$ for every $x \in A$. Write $f = g + ih$ where g and h are real-valued. Then g is differentiable a.e. on A and $g(x) = \mathrm{Re}(c)$ for $x \in A$. Therefore $g(A) = \{\mathrm{Re}(c)\}$, which has measure zero. Corollary 6.2.3 therefore implies that $g' = 0$ a.e. on A. A similar argument shows that $h' = 0$ a.e. on A, so it follows that $f' = g' + ih' = 0$ a.e. on A.

6.3.5 The measure of

$$f(X) = \{f(x) : x \in X\} = \{f_r(x) + if_i(x) : x \in X\}$$

as a subset of $\mathbb{C}$ is defined to be the measure of the set

$$A = \Big\{ \big(f_r(x), f_i(x)\big) : x \in X \Big\}$$

as a subset of $\mathbb{R}^2$. Note that

$$A \subseteq \Big\{ \big(f_r(x), f_i(y)\big) : x, y \in X \Big\} = f_r(X) \times f_i(X).$$

Therefore

$$|f(X)| = |A| \le |f_r(X)|\,|f_i(X)|.$$

Consequently, if $f_r(X)$ and $f_i(X)$ each have measure zero, then $f(X)$ has measure zero.

To show that the converse implication can fail, define $f\colon [0,1] \to \mathbb{C}$ by

$$f(x) = x = x + 0i, \qquad x \in [0,1].$$

Then $f[0,1]$ is contained in the real axis in $\mathbb{C}$, so it has measure zero. Yet

$$|f_r[0,1]| = |[0,1]| = 1.$$

Another example is $f\colon [0,1] \to \mathbb{C}$ defined by $f(x) = x + ix$. In this case we have $|f[0,1]| = 0$ while $|f_r[0,1]| = |f_i[0,1]| = 1$.

6.3.9 *First proof.* Since g is continuous, its indefinite integral $F(x) = \int_a^x g(t)\,dt$ is absolutely continuous. Further, since g is continuous, Exercise 5.2.8 tells us that F is differentiable everywhere and $F'(x) = g(x)$ for every

$x \in [a,b]$. Hence $(F-f)' = F' - f' = g - g = 0$ a.e. Corollary 6.3.4 therefore implies that $F - f$ is constant. Hence $f = F + c$, so f is differentiable at all points and $f'(x) = g(x)$ for all $x \in [a,b]$.

Second proof. This proof uses the Fundamental Theorem of Calculus for absolutely continuous functions.

Since g is continuous, every point is a Lebesgue point of g. Suppose that $x \in (a,b)$. We have $x + h \in (a,b)$ for all $|h|$ is small enough, so

$$\begin{aligned} g(x) &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} g(t)\,dt && \text{Fund. Thm. Calculus} \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f'(t)\,dt && \text{since } f' = g \text{ a.e.} \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} && \text{since } f \in \mathrm{AC}[a,b]. \end{aligned}$$

This shows that f is differentiable at all points in (a,b), and it also shows that $f'(x) = g(x)$ for all $x \in (a,b)$.

If $x = a$ then the same calculation is valid if we take limits from the right:

$$\begin{aligned} g(a) &= \lim_{h\to 0^+} \frac{1}{h}\int_a^{a+h} g(t)\,dt && \text{Fund. Thm. Calculus} \\ &= \lim_{h\to 0^+} \frac{1}{h}\int_a^{a+h} f'(t)\,dt && \text{since } f' = g \text{ a.e.} \\ &= \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} && \text{since } f \in \mathrm{AC}[a,b]. \end{aligned}$$

Therefore f is differentiable from the right at the point a and we have $f'(a) = g(a)$. A similar argument shows that $f'(b) = g(b)$.

6.4.14 Each f_n is absolutely continuous, so the Fundamental Theorem of Calculus implies that

$$\int_0^x f_n' \;=\; f_n(x) - f_n(0) \;=\; f_n(x), \qquad x \in [0,1].$$

Since $x^{-1/2}$ is integrable on $[0,1]$, we have $h \in L^1[0,1]$. Therefore, the function

$$f(x) \;=\; \int_0^x h, \qquad x \in [0,1],$$

is well-defined and is absolutely continuous on $[0,1]$. Further, $h - f_n' \to 0$ and

$$|h - f_n'| \;\le\; |h| + |f_n'| \;\le\; 2x^{-1/2} \;\in\; L^1[0,1].$$

It therefore follows from the Dominated Convergence Theorem that

$$\lim_{n\to\infty} \int_0^1 |h - f_n'| \;=\; 0.$$

Hence,

$$\begin{aligned}
\sup_{0\le x\le 1} |f(x) - f_n(x)| \;&=\; \sup_{0\le x\le 1} \left| \int_0^x (h - f_n') \right| \\
&\le\; \sup_{0\le x\le 1} \int_0^x |h - f_n'| \\
&\le\; \int_0^1 |h - f_n'| \\
&\to\; 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus we have shown that $f_n \to f$ uniformly.

6.5.10 The change of variable formulas that we would like to apply is formulated in terms of monotone increasing functions, yet we would like to use it with the monotone decreasing function $1/t$. For each monotone increasing result there is an analogous monotone decreasing result, but for preciseness we will will formulate this proof so that it uses the monotone increasing function $-1/t$ instead of $1/t$.

Fix $k \in \mathbb{N}$, and let

$$f_k(x) \;=\; f(x)\, x^{-2k}.$$

We know that f_k is integrable on $[1,\infty)$ since f is integrable and x^{-2k} is bounded on that interval.

Define $g(t) = -1/t$ for $t \neq 0$, and for $t \le -1$ set

$$h_k(t) \;=\; f_k\bigl(-\tfrac{1}{t}\bigr)\, t^{-2}, \;=\; f\bigl(-\tfrac{1}{t}\bigr)\, t^{2k}\, t^{-2}, \qquad t \le -1.$$

Fix $0 < \delta < 1$, and set $d = 1/\delta$. Then g is monotone increasing on the interval $[-1,-\delta]$, and g maps $[-1,-\delta]$ onto $[1,1/\delta]$. The function $|f_k|$ is integrable on $[1,1/\delta]$. Corollary 6.5.8 therefore implies that

$$|h_k(t)| \;=\; |f_k\bigl(-\tfrac{1}{t}\bigr)|\, t^{-2} \;=\; |f_k(g(t))|\, g'(t)$$

is measurable, and

$$\int_{-1}^{-\delta} |h_k(t)|\, dt \;=\; \int_{-1}^{-\delta} |f_k(g(t))|\, g'(t)\, dt \;=\; \int_1^{1/\delta} |f_k(x)|\, dx \;<\; \infty.$$

Therefore h_k is integrable on $[-1,-\delta]$. Further, the Monotone Convergence Theorem implies that

$$\int_{-1}^{0} |h_k(t)|\,dt \;=\; \lim_{\delta\to 0}\int_{-1}^{-\delta} |h_k(t)|\,dt$$
$$\;=\; \lim_{\delta\to 0}\int_{1}^{1/\delta} |f_k(x)|\,dx \;=\; \int_{1}^{\infty} |f_k(x)|\,dx \;<\; \infty.$$

Therefore h_k is integrable on $[-1,0]$.

Applying the same argument without absolute values we obtain

$$\int_{-1}^{-\delta} h_k(t)\,dt \;=\; \int_{-1}^{-\delta} f_k(g(t))\,g'(t)\,dt \;=\; \int_{1}^{1/\delta} f_k(x)\,dx.$$

Now that we know that h_k is integrable, we can apply the DCT to obtain

$$\int_{-1}^{0} h_k(t)\,dt \;=\; \int_{1}^{\infty} f_k(x)\,dx.$$

Now,

$$h_k(t) \;=\; f_k\bigl(-\tfrac{1}{t}\bigr)\,t^{-2}, \;=\; f\bigl(-\tfrac{1}{t}\bigr)\,t^{2k}\,t^{-2}, \;=\; h_0(t)\,t^{2n}.$$

Therefore h_0 is an integrable function on $[-1,0]$ that satisfies

$$\int_{-1}^{0} h_0(t)\,t^{2k}\,dt \;=\; \int_{-1}^{0} h_k(t)\,dt \;=\; \int_{1}^{\infty} f_k(x)\,dx \;=\; \int_{1}^{\infty} f(x)\,x^{-2k}\,dx \;=\; 0.$$

This is true for every $k \in \mathbb{N}$. Consequently,

$$\int_{-1}^{0} t^2\,h_0(t)\,t^{2k}\,dt \;=\; 0, \qquad k = 0,1,2,\dots.$$

Therefore $t^2\,h_0(t) = 0$ a.e. by Problem 6.4.21(b). (Technically, that problem uses the interval $[0,1]$ instead of $[-1,0]$, but an entirely symmetrical argument shows that we can replace $[0,1]$ by $[-1,0]$ in Problem 6.4.21.) Therefore $h = 0$ a.e.

6.6.13 The series

$$\sum_{n=1}^{\infty} 2^{-n} a_n \qquad \text{(A)}$$

converges to a nonnegative real number since we have $0 < a_n \le 1$ for every n.

The fact that $0 < a_n \le 1$ also impleis that $\ln a_n \le 0$ for every n. Therefore every term of the series

$$\sum_{n=1}^{\infty} 2^{-n} \ln a_n \qquad \text{(B)}$$

is negative. Consequently the series converges in the extended real sense, though the sum could be either $-\infty$ or a nonpositive real number. If the sum is $-\infty$ then there is nothing to prove, so we may assume that the series in equation (B) converges to a finite, nonpositive, real number.

Fix any $N \in \mathbb{N}$. Since $-\ln x$ is convex on $(0,\infty)$, the Discrete Jensen Inequality tells us that

$$-\ln\left(\frac{\sum_{n=1}^N 2^{-n} a_n}{\sum_{n=1}^N 2^{-n}}\right) \;\le\; -\frac{\sum_{n=1}^N 2^{-n}\ln a_n}{\sum_{n=1}^N 2^{-n}}.$$

Rearranging and evaluating the sum of 2^{-n}, we obtain

$$\sum_{n=1}^N 2^{-n}\ln a_n \;\le\; (1-2^{-N})\,\ln\left(\frac{\sum_{n=1}^N 2^{-n} a_n}{1-2^{-N}}\right).$$

Since the series in equations (A) and (B) both converge and since $\ln x$ is a continuous function, it follows that

$$\begin{aligned}
\sum_{n=1}^\infty 2^{-n}\ln a_n &= \lim_{N\to\infty}\sum_{n=1}^N 2^{-n}\ln a_n \\
&\le \lim_{N\to\infty}\left((1-2^{-N})\,\ln\left(\frac{\sum_{n=1}^N 2^{-n} a_n}{1-2^{-N}}\right)\right) \\
&= 1\cdot\ln\left(\frac{\sum_{n=1}^\infty 2^{-n} a_n}{1}\right) \\
&= \ln\left(\sum_{n=1}^\infty 2^{-n} a_n\right).
\end{aligned}$$

Remark: An alternative approach is use the convexity of e^x to prove that

$$\exp\left(\sum_{n=1}^\infty 2^{-n}\ln a_n\right) \;\le\; \sum_{n=1}^\infty 2^{-n} a_n.$$

7.1.19 If $p=1$, then

$$\sum_{k=1}^\infty \frac{|x_k|}{k} \;\le\; \sum_{k=1}^\infty |x_k| \;=\; \|x\|_1 \;<\; \infty.$$

If $1<p<\infty$ then we have $1<p'<\infty$. Applying Hölder's Inequality, it follows that

$$\sum_{k=1}^\infty \frac{|x_k|}{k} \;\le\; \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p}\left(\sum_{k=1}^\infty \frac{1}{k^{p'}}\right)^{1/p'} \;=\; \|x\|_p\left(\sum_{k=1}^\infty \frac{1}{k^{p'}}\right)^{1/p'} \;<\; \infty.$$

If $p = \infty$ and we set $x_k = 1$ for every k, then

$$\sum_{k=1}^{\infty} \frac{|x_k|}{k} \;=\; \sum_{k=1}^{\infty} \frac{1}{k} \;=\; \infty.$$

7.1.23 *Case* $1 < p < \infty$. By Exercise 7.1.5, equality holds in $ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$ if and only if $b = a^{p-1}$. For the normalized case $\|x\|_p = \|y\|_{p'} = 1$, equality in Hölder's Inequality requires that we have equality in equation (7.9), and this will happen if and only if $|y_k| = |x_k|^{p-1}$ for each k. This is equivalent to

$$|y_k|^{p'} \;=\; |y_k|^{p/(p-1)} \;=\; |x_k|^p.$$

For the nonnormalized case, if $x,\, y \ne 0$ then equality holds in Hölder's Inequality if and only if it holds when we replace x and y by $x/\|x\|_p$ and $y/\|y\|_{p'}$. Therefore, we must have

$$\frac{|y_k|^{p'}}{\|y\|_{p'}^{p'}} \;=\; \left(\frac{|y_k|}{\|y\|_{p'}}\right)^{p'} \;=\; \left(\frac{|x_k|}{\|x\|_p}\right)^{p} \;=\; \frac{|x_k|^p}{\|x\|_p^p}, \qquad k \in I.$$

Hence $\alpha\,|x_k|^p = \beta\,|y_k|^{p'}$ with $\alpha = \|y\|_{p'}^{p'}$ and $\beta = \|x\|_p^p$. On the other hand, if either $x = 0$ or $y = 0$, then we have equality in Hölder's Inequality, and we also have $\alpha\,|x_k|^p = \beta\,|y_k|^{p'}$ with $\alpha,\, \beta$ not both zero.

For the converse direction, suppose that $\alpha\,|x_k|^p = \beta\,|y_k|^{p'}$ for each $k \in I$, where $\alpha,\, \beta \in \mathbb{C}$ are not both zero. If $\alpha = 0$, then $y_k = 0$ for every k, and hence we trivially have $\|xy\|_1 = 0 = \|x\|_p\,\|y\|_{p'}$. Likewise, equality holds trivially if $\beta = 0$. Therefore, we can assume both $\alpha,\, \beta \ne 0$, and by dividing both sides by β, we may assume that $\beta = 1$ and $\alpha > 0$. Then we have $|y_k|^{p'} = \alpha\,|x_k|^p$, so

$$\|y\|_{p'}^{p'} \;=\; \sum_{k\in I} |y_k|^{p'} \;=\; \alpha \sum_{k\in I} |x_k|^p \;=\; \alpha\,\|x\|_p^p.$$

If either $x = 0$ or $y = 0$ then equality holds trivially in Hölder's Inequality, so let us assume both $x,\, y \ne 0$. Then we have

$$\frac{|y_k|^{p'}}{\|y\|_{p'}^{p'}} \;=\; \frac{\alpha|x_k|^p}{\alpha\|x\|_p^p} \;=\; \frac{|x_k|^p}{\|x\|_p^p}.$$

By the work above, this implies that equality holds in Hölder's Inequality.

Case $p = 1$, $p' = \infty$. Set $M = \sup_k |y_k|$. Suppose equality holds in Hölder's Inequality, i.e.,

$$\sum_{k\in I} |x_k y_k| \;=\; \biggl(\sum_{k\in I} |x_k|\biggr)\Bigl(\sup_k |y_k|\Bigr).$$

Then

$$\sum_{k\in I} |x_k y_k| \;=\; \sum_{k\in I} M\,|x_k|.$$

Hence

$$\sum_{k\in I} (M - |y_k|)\,|x_k| \;=\; 0,$$

but $0 \le M - |y_k|$ for every k, so we must have $(M - |y_k|)\,|x_k| = 0$ for every k. Thus whenever $x_k \neq 0$, we must have $|y_k| = M$.

Conversely, if $|y_k| = M$ for all k such that $x_k \neq 0$, equality holds in Hölder's Inequality.

7.2.12 (a) *Case* $1 \le p < \infty$. Assume that $f_n \in L^p(E)$, $f_n \to f$ a.e., and

$$C \;=\; \sup_n \|f_n\|_p \;<\; \infty.$$

Then by Fatou's Lemma,

$$\|f\|_p^p \;=\; \int_E |f|^p \;=\; \int_E \liminf_{n\to\infty} |f_n|^p \;\le\; \liminf_{n\to\infty} \int_E |f_n|^p \;\le\; C^p,$$

so $f \in L^p(E)$.

Case $p = \infty$. Assume that $f_n \in L^p(E)$, $f_n \to f$ a.e., and

$$C \;=\; \sup_n \|f_n\|_\infty \;<\; \infty.$$

Then for almost every x we have

$$|f(x)| \;=\; \lim_{n\to\infty} |f_n(x)| \;\le\; \sup_n \,\|f\|_\infty \;\le\; C,$$

and therefore $\|f\|_\infty \le C$.

(b) *Case* $1 \le p < \infty$. Let $E = [0,1]$, and set

$$f_n(x) \;=\; x^{-1/p}\, \chi_{[\frac{1}{n},1]}(x).$$

Then f_n is bounded and hence belongs to $L^p[0,1]$. Further, f_n converges pointwise a.e. to

$$f(x) \;=\; x^{-1/p}.$$

However,

$$\|f\|_p^p \;=\; \int_0^1 |f|^p \;=\; \int_0^1 \frac{1}{x}\,dx \;=\; \infty,$$

so $f \notin L^p[0,1]$.

Case $p = \infty$. Let $E = [0,1]$, and set

$$f_n(x) \;=\; \frac{1}{x}\, \chi_{[\frac{1}{n},1]}(x).$$

Then f_n is bounded and hence belongs to $L^\infty[0,1]$. Further, f_n converges pointwise a.e. to

$$f(x) \;=\; \frac{1}{x}.$$

However, $\|f\|_\infty = \infty$, so $f \notin L^\infty[0,1]$.

7.3.10 Suppose that $f \in L^p(E)$ where $1 \le p < \infty$, and fix $\varepsilon > 0$. By Theorem 7.3.9, there exists a compactly supported function $g \in L^p(E)$ such that $\|f-g\|_p < \varepsilon$.

By Corollary 3.2.15, there exist simple functions ϕ_n that converge pointwise to f and satisfy $|\phi_n| \le |g|$ a.e. Hence $|g-\phi_n|^p \to 0$ a.e., and

$$|g-\phi_n|^p \;\le\; \big(|g| + |\phi_n|\big)^p \;\le\; \big(2|g|\big)^p \;=\; 2^p\,|g|^p \;\in\; L^1(E).$$

The Dominated Convergence Theorem therefore implies that

$$\|g-\phi_n\|_p^p \;=\; \int_E |g-\phi_n|^p \;\to\; 0 \quad \text{as } n \to \infty.$$

Hence if we choose n large enough then we will have

$$\|f-\phi_n\|_p \;\le\; \|f-g\|_p + \|g-\phi_n\|_p \;\le\; 2\varepsilon.$$

Each function ϕ_n belong to S_c, so we conclude that S_c is dense in $L^p(E)$.

Suppose $p = \infty$ and $f \in L^\infty(E)$. Then $|f(x)| \le \|f\|_\infty$ except for points x in a set Z that has measure zero. Hence f is bounded on Z^{C}, and therefore Corollary 3.2.15 implies that there exist simple functions that converge uniformly to f on Z^{C}. Consequently these simple functions converge in L^∞-norm to f, so the set of simple functions is dense in $L^\infty(\mathbb{R})$.

7.3.18 Fix $f \in L^r(E)$. For each $k \in \mathbb{N}$, define

$$g_k(x) \;=\; \begin{cases} f(x), & |f(x)| \le k, \\ 0, & |f(x)| > k, \end{cases}$$

and set

$$h_k \;=\; g_k \cdot \chi_{E \cap [-k,k]^d}.$$

Note that

$$\|h_k\|_p^p \;=\; \int_E |h_k|^p \;=\; \int_{E\cap[-k,k]^d} |h_k|^p \;\le\; \int_{E\cap[-k,k]^d} k^p \;\le\; (2k)^d\, k^p \;<\; \infty.$$

Therefore $h_k \in L^p(E)$. If q is finite, then a similar argument shows that $h_k \in L^q(E)$. On the other hand, if $q = \infty$ then we have $h_k \in L^\infty(E)$ since h_k is bounded. In any case, we see that $h_k \in L^p(E) \cap L^q(E)$.

Additionally, $h_k \to f$ pointwise, so $|f - h_k| \to 0$ pointwise, and we have $|f-h_k|^r \le |f|^r \in L^1(E)$. Because r is finite, we can apply the Dominated

Convergence Theorem to obtain

$$\lim_{k\to\infty} \|f-h_k\|_r^r \;=\; \lim_{k\to\infty} \int_E |f-h_k|^r \;=\; 0.$$

That is, $h_k \to f$ in L^r-norm. Since $h_k \in L^p(E) \cap L^q(E)$ for every k, it follows that $L^p(E) \cap L^q(E)$ is dense in $L^r(E)$.

7.4.6 Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a complete sequence in a normed space X, and let

$$S \;=\; \biggl\{\sum_{n=1}^N r_n f_n : N > 0,\ \mathrm{Re}(r_n),\, \mathrm{Im}(r_n) \in \mathbb{Q}\biggr\}.$$

Then S is countable, and we claim it is dense in X. Without loss of generality, we may assume that each vector f_n is nonzero.

Choose any $f \in X$ and fix $\varepsilon > 0$. Since $\mathrm{span}\{f_n\}$ is dense in X, there exists a vector

$$g \;=\; \sum_{n=1}^N c_n f_n \in \mathrm{span}\{f_n\}$$

such that $\|f-g\| < \varepsilon$. For each $n \in \mathbb{N}$, choose a scalar r_n with real and imaginary parts such that

$$|c_n - r_n| \;<\; \frac{\varepsilon}{N\,\|f_n\|},$$

and set

$$h \;=\; \sum_{n=1}^N r_n f_n.$$

Then $h \in S$ and

$$\|g-h\| \;\le\; \sum_{n=1}^N |c_n - r_n|\,\|f_n\| \;<\; \sum_{n=1}^N \frac{\varepsilon}{N\,\|f_n\|} \;=\; \varepsilon.$$

Hence

$$\|f-h\| \;\le\; \|f-g\| + \|g-h\| \;<\; 2\varepsilon.$$

This shows that S is dense in X.

8.1.2 (a) This follows from linearity in the first variable and the fact that $\langle x,y\rangle = \overline{\langle y,x\rangle}$.

(b) Given $x,\, y \in H$,

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2.\end{aligned}$$

(c) This follows immediately from part (b).

(d) Given x, $y \in H$,

$$\begin{aligned}&\|x+y\|^2 + \|x-y\|^2 \\ &= \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2 + \|x\|^2 - \langle x,y\rangle - \langle y,x\rangle + \|y\|^2 \\ &= 2\,\|x\|^2 + 2\,\|y\|^2.\end{aligned}$$

8.1.13 It is clear that $\langle\cdot,\cdot\rangle$ is an inner product on H.

To show that H is complete, suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in H. Then $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2[a,b]$, so there is some function $f \in L^2[a,b]$ such that $f_n \to f$ in L^2-norm. Also, $\{f_n'\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2[a,b]$, so there is some function $g \in L^2[a,b]$ such that $f_n' \to g$ in L^2-norm.

For future reference, note that since $[a,b]$ has finite measure, we can use the CBS Inequality to compute that for each function $F \in L^2[a,b]$ we have

$$\|F\|_1 = \int_a^b |F| \le \biggl(\int_a^b 1^2\biggr)^{1/2} \biggl(\int_a^b |F|^2\biggr)^{1/2} = (b-a)^{1/2}\,\|F\|_2.$$

This problem would be significantly easier if we knew that $\{f_n(a)\}_{n\in\mathbb{N}}$ was a Cauchy sequence. However, we have not yet established that (and I do not see an easy way to infer it). So, recall that f_n is absolutely continuous, and define

$$h_n(x) = f_n(x) - f_n(a) = \int_a^x f_n', \qquad x \in [a,b].$$

For each x we therefore have

$$\begin{aligned}|h_m(x) - h_n(x)| &= \biggl|\int_a^x (f_m' - f_n')\biggr| \\ &\le \int_a^x |f_m' - f_n'| \\ &\le \|f_m' - f_n'\|_1 \le (b-a)^{1/2}\,\|f_m' - f_n'\|_2.\end{aligned}$$

Consequently

$$\|h_m - h_n\|_{\mathrm{u}} = \sup_x |f_m(x) - f_n(x)| \le (b-a)^{1/2}\,\|f_m' - f_n'\|_2.$$

Since $\{f_n'\}_{n\in\mathbb{N}}$ is Cauchy in L^2-norm, we conclude that $\{h_n\}_{n\in\mathbb{N}}$ is Cauchy with respect to the uniform norm. Since each f_n is continuous and $C[a,b]$

is a Banach space with respect to $\|\cdot\|_{\mathrm{u}}$, this implies that there exists some continuous function h such that $h_n \to h$ uniformly.

Since $f_n \to f$ in L^2-norm, there is a subsequence such that $f_{n_k} \to f$ a.e. If x is such that $f_{n_k}(x) \to f(x)$, then we also have $h_{n_k}(x) \to h(x)$. Therefore $f_{n_k}(a)$ converges, because

$$\begin{aligned} C \;=\; \lim_{k\to\infty} f_{n_k}(a) \;&=\; \lim_{k\to\infty} \Big(f_{n_k}(x) - f_{n_k}(x) + f_{n_k}(a)\Big) \\ &=\; \lim_{k\to\infty} \Big(f_{n_k}(x)\Big) \;-\; \lim_{k\to\infty} \Big(f_{n_k}(x) - f_{n_k}(a)\Big) \\ &=\; f(x) - h(x). \end{aligned}$$

Hence $f = h + C$ a.e. Thus f is equal a.e. to the continuous function $h + C$. Since f is only defined up to a set of measure zero, we can redefine f on a set of measure zero and take $f = h + C$. That is, we choose a representative of f such that $f = h + C$.

Since $[a, b]$ has finite measure, g is integrable and we can define

$$G(x) \;=\; \int_a^x g(t)\,dt \;+\; C, \qquad x \in [a, b].$$

This function G is absolutely continuous and $G' = g$ a.e. Also, given $x \in [a, b]$ we have

$$\begin{aligned} |f(x) - G(x)| \;&=\; \lim_{k\to\infty} |f_{n_k}(x) - G(x)| \\ &=\; \lim_{k\to\infty} \left| \int_a^x f'_{n_k} \;+\; f_{n_k}(a) \;-\; \int_a^x f \;-\; C \right| \\ &\le\; \lim_{k\to\infty} \left| \int_a^x (f'_n - g) \right| \;+\; \lim_{k\to\infty} \big|f_{n_k}(a) - C\big| \\ &\le\; \lim_{k\to\infty} \|f'_n - g\|_1 \;+\; 0 \\ &\le\; \lim_{k\to\infty} (b-a)^{1/2}\, \|f'_n - g\|_2 \;=\; 0. \end{aligned}$$

Hence $G = f$. Therefore f is absolutely continuous, $f' = G' = g$ a.e., and

$$\begin{aligned} \|f - f_n\| \;&=\; \|f - f_n\|_2 \;+\; \|f' - f'_n\|_2 \\ &=\; \|f - f_n\|_2 \;+\; \|g - f'_n\|_2 \\ &\to\; 0 \quad \text{as } n \to 0. \end{aligned}$$

Therefore $f_n \to f$ in the norm of H, and hence H is complete.

8.2.5 E and O are clearly subspaces. If $f_n \in E$ and $f_n \to f$ in L^2-norm, then there exists a subsequence such that $f_{n_k} \to f$ a.e. Therefore for almost every x we have

$$f(-x) \;=\; \lim_{k\to\infty} f_{n_k}(-x) \;=\; \lim_{k\to\infty} f_{n_k}(x) \;=\; f(x),$$

so $f \in E$. Therefore E is closed, and similarly O is closed.

If $f \in E$ and $g \in O$, then $f\,\overline{g}$ is odd, and therefore $\langle f, g\rangle = 0$. This shows that $E \subseteq O^\perp$ and $O \subseteq E^\perp$.

Suppose that $f \in O^\perp$. Given $t \in \mathbb{R}$ and $h > 0$, the function

$$\chi_t \;=\; \chi_{[t,t+h]} - \chi_{[-t-h,-t]}$$

is odd, so

$$0 \;=\; \langle f, \chi_t\rangle \;=\; \int_t^{t+h} f \;-\; \int_{-t-h}^{-t} f.$$

Therefore

$$\frac{1}{h}\int_t^{t+h} f \;=\; \frac{1}{h}\int_{-t-h}^{-t} f.$$

Applying the Lebesgue Differentiation Theorem, it follows that for almost every t we have

$$f(t) \;=\; \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f \;=\; \lim_{h\to 0}\frac{1}{h}\int_{-t-h}^{-t} f \;=\; f(-t).$$

Therefore f is even. This shows that $O^\perp \subseteq E$, and a similar argument shows that $E^\perp \subseteq O$.

8.2.13 Suppose that x, $y \in \overline{\operatorname{span}}(A)$ are given. Then there exist vectors x_n, $y_n \in \operatorname{span}(A)$ such that $x_n \to x$ and $y_n \to y$ in norm. Therefore $x_n + y_n \to x+y$ in norm. As $x_n+y_n \in \operatorname{span}(A)$ for every n, it follows that $x+y$ belongs to the closure of $\operatorname{span}(A)$, which is $\overline{\operatorname{span}}(A)$. Therefore $\overline{\operatorname{span}}(A)$ is closed under vector addition, and a similar argument shows that it is closed under scalar multiplication. Therefore $\overline{\operatorname{span}}(A)$ is a subspace of X. By definition $\overline{\operatorname{span}}(A)$ is a closed set, so it is a closed subspace.

(b) Suppose that M is a closed subspace of X and $A \subseteq M$. Since M is closed under vector addition and scalar multiplication, it follows that $\operatorname{span}(A) \subseteq M$. Since M is closed under limits, it follows that M contains every limit of elements of $\operatorname{span}(A)$. The set of all such limits is the closure of the span, so we have shown that $\overline{\operatorname{span}}(A) \subseteq M$.

8.2.21 Suppose that $g_n \in M$ and $g_n \to g \in L^2(\mathbb{R}^d)$. Then there exists a subsequence $g_{n_k} \to g$ pointwise a.e. Each g_{n_k} is zero a.e. outside of M, so it follows that $g = 0$ a.e. outside of M as well. Therefore $g \in M$, and hence M is closed.

If $f \in L^2(\mathbb{R}^d)$, then $p = f\chi_E \in M$. Let $e = f - p$, and choose any function $g \in M$. Note that $e(x) = 0$ for a.e. $x \in E$. On the other hand, if $g \in M$ then

$g(x) = 0$ for a.e. $x \notin E$. Therefore $e(x)\,\overline{g(x)} = 0$ for a.e. x. Hence

$$\langle e, g\rangle \;=\; \int_{\mathbb{R}^d} e(x)\,\overline{g(x)}\,dx \;=\; 0.$$

This shows that $e \in M^\perp$. Since we have written $f = p + e$ where $p \in M$ and $e \in M^\perp$, one of the characterizations of orthogonal projections tells us that p is *the* orthogonal projection of f onto M.

8.3.19 We follow the idea of the Gram–Schmidt procedure.

Suppose that H is infinite-dimensional. We proceed inductively. Since $H \neq \{0\}$, it contains some unit vector x_1.

Once orthonormal vectors $x_1, \dots, x_n$ have been constructed, let $H_n = \operatorname{span}\{x_1, \dots, x_n\}$. Then $H_n \neq H$, since H is infinite-dimensional. Hence there exists a vector $y_{n+1} \notin H_n$. Let p_{n+1} be the orthogonal projection of y_{n+1} onto H_n. Then $p_{n+1} \neq y_{n+1}$, so $e_{n+1} = y_{n+1} - p_{n+1}$ is not the zero vector. Letting

$$x_{n+1} \;=\; \frac{e_{n+1}}{\|e_{n+1}\|},$$

we see that $\{x_1, \dots, x_n, x_{n+1}\}$ is an orthonormal sequence.

By repeating this forever we obtain an infinite orthonormal sequence $x_1, x_2, \dots$.

8.3.23 (a) Suppose that $\sum \|x_n\| < \infty$. This is a series of nonnegative real numbers, so Theorem 8.3.3 implies that for any bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ we have $\sum \|x_{\sigma(n)}\| < \infty$. Therefore the series $\sum x_{\sigma(n)}$ converges absolutely. Since X is a Banach space, every absolutely convergent series in X converges (see Theorem 1.2.8). Therefore $\sum x_{\sigma(n)}$ converges in X. Since this is true for every bijection σ, the series $\sum x_n$ is unconditionally convergent.

(b) If H is infinite-dimensional, then it contains an infinite orthonormal sequence $\{e_n\}_{n\in\mathbb{N}}$. Then the series $\sum \frac{1}{n} e_n$ converges unconditionally (since $\sum 1/n^2 < \infty$), but it does not converge absolutely.

8.4.7 For $n = 0$ we have

$$\widehat{f}(0) \;=\; \int_0^1 x\,dx \;=\; \frac{1}{2}.$$

For $n \neq 0$, we use integration by parts with $u = x$ and $dv = e^{-2\pi i n x}\,dx$ to compute that

$$\begin{aligned}
\widehat{f}(n) \;=\; \int_0^1 x\,e^{-2\pi i n x}\,dx \;&=\; \frac{x e^{-2\pi i n x}}{-2\pi i n}\bigg|_0^1 \;-\; \int_0^1 \frac{e^{-2\pi i n x}}{-2\pi i n}\,dx \\
&=\; \frac{e^{-2\pi i n} - 0}{-2\pi i n} \;-\; \frac{e^{-2\pi i n}}{(-2\pi i n)^2}\bigg|_0^1 \;=\; \frac{1}{-2\pi i n} \;-\; 0.
\end{aligned}$$

Now,

$$\|f\|_2^2 = \int_0^1 x^2\,dx = \frac{x^3}{3}\bigg|_0^1 = \frac{1}{3}.$$

Applying the Plancherel Equality, we therefore have

$$\begin{aligned}\frac{1}{3} = \|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 &= |\widehat{f}(0)|^2 + \sum_{n\neq 0} |\widehat{f}(n)|^2 \\ &= \frac{1}{4} + \sum_{n\neq 0} \frac{1}{4\pi^2 n^2} \\ &= \frac{1}{4} + 2\sum_{n=1}^{\infty} \frac{1}{4\pi^2 n^2}.\end{aligned}$$

Therefore

$$\frac{1}{12} = \frac{1}{3} - \frac{1}{4} = \frac{2}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2},$$

which after rearranging yields

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{4\pi^2}{2\cdot 12} = \frac{\pi^2}{6}.$$

9.1.20 We use Minkowski's Integral Inequality to compute that

$$\begin{aligned}\|f*g\|_p &= \biggl(\int_{-\infty}^{\infty}\biggl|\int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy\biggr|^p\,dx\biggr)^{1/p} \\ &= \biggl\|\int_{-\infty}^{\infty} f(y)\,g(\cdot - y)\,dy\biggr\|_p \\ &\le \int_{-\infty}^{\infty} |f(y)|\,\|T_y g\|_p\,dy \\ &= \int_{-\infty}^{\infty} |f(y)|\,\|g\|_p\,dy \\ &= \|f\|_1\,\|g\|_p.\end{aligned}$$

9.1.32 First proof, covering $1 \le p < \infty$.

The convolution-based solution to Problem 7.4.5 for the case $p = 1$ can be extended to finite p as follows.

We are given $f \in L^p(\mathbb{R})$ such that $\int f\phi = 0$ for every $\phi \in C_c^\infty(\mathbb{R})$. Let $k \in C_c^\infty(\mathbb{R})$ be such that $\int k = 1$, and set $k_N(x) = Nk(Nx)$. If we fix $t \in \mathbb{R}$, then $k_N(t-x) \in C_c^\infty(\mathbb{R})$, so by hypothesis we have

$$(f * k_N)(t) \;=\; \int_{-\infty}^{\infty} f(x)\, k_N(t-x)\, dx \;=\; 0.$$

But $f * k_N \to f$ in L^p-norm, so this implies that $f = 0$ a.e.

Second proof, covering $1 < p \le \infty$.

The solution to Problem 7.4.5 can essentially be repeated for this range of p. For completeness, we give the details below.

We are given $f \in L^p(\mathbb{R})$ such that $\int f\phi = 0$ for every $\phi \in C_c^\infty(\mathbb{R})$. Suppose that g is any function in $L^{p'}(\mathbb{R})$. Since $1 \le p' < \infty$ (this is why we assumed $p > 1$), we know that $C_c^\infty(\mathbb{R})$ is dense in $L^{p'}(\mathbb{R})$. Therefore there exist functions $\phi_k \in C_c^\infty(\mathbb{R})$ such that $\|g - \phi_k\|_{p'} \to 0$ as $k \to \infty$. Applying the hypotheses and Hölder's Inequality, we compute that

$$\begin{aligned} 0 \;\le\; \left|\int_{-\infty}^{\infty} fg\right| \;&\le\; \left|\int_{-\infty}^{\infty} f\phi_k\right| \;+\; \left|\int_{-\infty}^{\infty} f\,(g-\phi_k)\right| \\ &\le\; 0 \;+\; \|f\|_p\, \|g-\phi_k\|_{p'} \\ &\to\; 0 \quad \text{as } k \to \infty. \end{aligned}$$

Therefore $\int fg = 0$ for every $g \in L^{p'}(\mathbb{R})$. Applying the Converse to Hölder's Inequality, we conclude that

$$\|f\|_p \;=\; \sup_{\|g\|_{p'}=1} \left|\int_{-\infty}^{\infty} fg\right| \;=\; 0.$$

Therefore $f = 0$ a.e.

9.2.18 (a) If $f \in L^1(\mathbb{R})$ is even, then (by making the change of variables $x \mapsto -x$) we compute that

$$\begin{aligned} \widehat{f}(-\xi) \;&=\; \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i(-\xi)x}\, dx \\ &=\; -\int_{\infty}^{-\infty} f(-x)\, e^{-2\pi i(-\xi)(-x)}\, dx \\ &=\; \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\xi x}\, dx \;=\; \widehat{f}(\xi). \end{aligned}$$

(b) Suppose that $f \in L^1(\mathbb{R})$ and $\widehat{f}$ is even. Set $g(x) = \big(f(x) + f(-x)\big)/2$. Then, since $\widehat{f}$ is even,

$$\widehat{g}(\xi) \;=\; \frac{\widehat{f}(\xi) + \widehat{f}(-\xi)}{2} \;=\; \frac{\widehat{f}(\xi) + \widehat{f}(\xi)}{2} \;=\; \widehat{f}(\xi).$$

The Uniqueness Theorem therefore implies that $f = g$ a.e. Hence f is even almost everywhere, and therefore has a representative that is even.

9.2.22 Suppose $f \in L^1(\mathbb{R})$ satisfies $f = f * f$. Then $\widehat{f}(\xi) = \widehat{f}(\xi)^2$, so $\widehat{f}(\xi)$ takes only the values 0 or 1 for every ξ. But $\widehat{f}$ is continuous, so this implies that $\widehat{f}$ is either identically 0 or identically 1. The latter is impossible by the Riemann–Lebesgue Lemma, so $\widehat{f} = 0$. By the Uniqueness Theorem, it follows that $f = 0$ a.e.

9.3.20 (a) Fix $f \in C(\mathbb{T})$ and $\varepsilon > 0$. Since f is uniformly continuous, there exists $0 < \delta < 1$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Fix $|a| < \delta$. Then

$$|f(x) - T_a f(x)| \;=\; |f(x) - f(x-a)| \;<\; \varepsilon.$$

Thus $\|f - T_a f\|_\infty \le \varepsilon$ whenever $|a| < \delta$, so $\|T_a f - f\|_\infty \to 0$.

(b) We will reduce the problem to the point where we can apply facts about the denseness of $C_c(\mathbb{R})$ in $L^p(\mathbb{T})$.

Fix $1 \le p < \infty$, and choose $f \in L^p(\mathbb{T})$ and $\varepsilon > 0$. Applying the Dominated Convergence Theorem, there must exist some $0 < \delta < \frac{1}{2}$ such that

$$g \;=\; f \cdot \chi_{[2\delta, 1-2\delta]}$$

satisfies

$$\|f - g\|_p \;=\; \biggl(\int_0^1 |f - g|^p\biggr)^{1/p} \;<\; \varepsilon.$$

Although f is 1-periodic, the function g is identically zero outside of the interval $[2\delta, 1-2\delta]$, so g belongs to $L^p(\mathbb{R})$. Since $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, there exists a function $\theta \in C_c(\mathbb{R})$ such that

$$\|g - \theta\|_p \;=\; \biggl(\int_{-\infty}^{\infty} |g - \theta|^p\biggr)^{1/p} \;<\; \varepsilon.$$

Since g is identically zero outside of $[2\delta, 1-2\delta]$, we can modify θ so that

- θ is unchanged on $[2\delta, 1-2\delta]$,
- $\theta = 0$ outside of $[\delta, 1-\delta]$,
- $\|g - \theta\|_p < \varepsilon$.

Since $\theta(0) = \theta(1)$, we can take θ on the interval $[0,1)$ and extend it 1-periodically to $\mathbb{R}$ to obtain a continuous, 1-periodic function on $\mathbb{R}$. This function θ belongs to $C(\mathbb{T})$, and, computing the integrals on the domain $[0,1)$,

$$\|f - \theta\|_p \;\le\; \|f - g\|_p + \|g - \theta\|_p \;<\; 2\varepsilon.$$

Therefore $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$.

Now we will show that translation is strongly continuous on $L^p(\mathbb{T})$. Fix $1 \le p < \infty$, and choose $f \in L^p(\mathbb{T})$. Given $\varepsilon > 0$, we can find $g \in C(\mathbb{T})$ such that $\|f - g\|_p < \varepsilon$. Since g is uniformly continuous, there exists a $\delta > 0$ such that

$$|a| < \delta \implies \|g - T_a g\|_\infty < \varepsilon.$$

Therefore, for such a we have

$$\|g - T_a g\|_p^p \;=\; \int_0^1 |g(x) - T_a g(x)|^p \, dx \;\le\; \int_0^1 \varepsilon^p \, dx \;=\; \varepsilon^p.$$

Since translation is isometric on $L^p(\mathbb{T})$, we therefore have for $|a| < \delta$ that

$$\begin{aligned} \|f - T_a f\|_p \;&\le\; \|f - g\|_p + \|g - T_a g\|_p + \|T_a g - T_a f\|_p \\ &\le\; \varepsilon + \varepsilon + \varepsilon \;=\; 3\varepsilon. \end{aligned}$$

Hence $T_a f \to f$ in $L^p(\mathbb{T})$ as $a \to 0$.

9.4.9 The fact that A is linear immediately implies that range(A) is a subspace of Y.

Suppose that vectors $y_n \in$ range(A) converge to a vector $y \in Y$. By the definition of the range, for each n there is some vector $x_n \in X$ such that $Ax_n = y_n$. As A is linear and isometric, we therefore have

$$\|x_m - x_n\| \;=\; \|A(x_m - x_n)\| \;=\; \|Ax_m - Ax_n\| \;=\; \|y_m - y_n\|. \qquad \text{(A)}$$

But $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy in Y (because it converges), so equation (A) implies that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in X. Since X is complete, there is some $x \in X$ such that $x_n \to x$. As A is bounded and therefore continuous, this implies that $Ax_n \to Ax$. By assumption we also have $Ax_n = y_n \to y$, so the uniqueness of limits implies that $y = Ax$. Thus $y \in$ range(A). This shows that range(A) contains every limit of points from range(A), so it is a closed set.