Christopher Heil

Introduction to Real Analysis

May 14, 2019

Springer

Preface

This text grew out of lecture notes that I developed over the years for the "Real Analysis" graduate sequence here at Georgia Tech. This two-semester sequence is taken by first-year mathematics graduate students, well-prepared undergraduate mathematics majors, and graduate students from a wide variety of engineering and scientific disciplines. Covered in this book are the topics that are taught in the first semester: Lebesgue measure, the Lebesgue integral, differentiation and absolute continuity, the Lebesgue spaces $L^p(E)$, and Hilbert spaces and $L^2(E)$. This material not only forms the basis of a core subject in pure mathematics, but also has wide applicability in science and engineering. A text covering the second semester topics in analysis, including abstract measure theory, signed and complex measures, operator theory, and functional analysis, is in development.

This text is an introduction to real analysis. There are several classic analysis texts that I keep close by on my bookshelf and refer to often. However, I find it difficult to use any of these as the textbook for teaching a first course on analysis. They tend to be dense and, in the classic style of mathematical elegance and conciseness, they develop the theory in the most general setting, with few examples and limited motivation. These texts are valuable resources, but I suggest that they should be the *second* set of books on analysis that you pick up.

I hope that this text will be the analysis text that you read first. The definitions, theorems, and other results are motivated and explained; the why and not just the what of the subject is discussed. Proofs are completely rigorous, yet difficult arguments are motivated and discussed. Extensive exercises and problems complement the presentation in the text, and provide many opportunities for enhancing the student's understanding of the material.

Audience

This text is aimed at students who have taken a standard (proof-based) undergraduate mathematics course on the basics of analysis. A brief review of the needed background material is presented in the **Preliminaries** section of the text. This includes:

- sequences, series, limits, suprema and infima, and limsups and liminfs,
- functions,
- cardinality,
- basic topology of Euclidean space (open, closed, and compact sets),
- continuity and differentiability of real-valued functions,
- the Riemann integral.

Online Resources

A variety of resources are available on the author's website,

`http://people.math.gatech.edu/~heil/`

These include the following.

- A **Chapter 0**, which contains a greatly expanded version of the material that appears in the **Preliminaries** section of this text, along with discussions and exercises.
- An **Alternative Chapter 1**, which is an expanded version of the material presented in **Chapter 1**, including detailed discussion, motivation, and exercises, focused on the setting of normed spaces.
- A **Chapter 10**, which provides an introduction to abstract measure theory.
- An **Instructor's Guide**, with a detailed course outline, commentary, remarks, and extra problems. The exposition and problems in this guide may be useful for students and readers as well as instructors.
- **Selected Solutions for Students**, containing approximately one worked solution of a problem or exercise from each section of the text.
- An **Errata List** that will be updated as I become aware of typographical or other errors in the text.

Additionally, a **Solutions Manual** is available to instructors upon request; instructions for obtaining a copy are given on the Birkhäuser website for this text.

Outline

Chapter 1 presents a short review of metric and normed spaces. Students who have completed an undergraduate analysis course have likely encountered much of this material, although possibly only in the context of the Euclidean space $\mathbb{R}^d$ (or $\mathbb{C}^d$) instead of abstract metric spaces. The instructor has the option of beginning the course here or proceeding directly to **Chapter 2.** The online **Alternative Chapter 1** presents a significantly expanded version of this chapter focused on normed spaces. (A detailed introduction to the more general setting of metric spaces is available in the first chapters of the author's text *Metrics, Norms, Inner Products, and Operator Theory* [Heil18].)

In **Chapter 2** we begin the study of Lebesgue measure. The fundamental question that motivates this chapter is: Can we assign a "volume" or "measure" to every subset of $\mathbb{R}^d$ in such a way that all of the properties that we expect of a "volume" function are satisfied? For example, we want the measure of a cube or a ball in $\mathbb{R}^d$ to coincide with the standard definition of the volume of a cube or ball, and if we translate an object rigidly in space then we want its measure to always remain the same. If we break an object into countably many disjoint pieces, then we want the measure of the original object to be the sum of the measures of the pieces. Surprisingly (at least to me!), this simply can't be done (more precisely, the Axiom of Choice implies that it is impossible). However, if we relax this goal somewhat then we find that we can define a measure that obeys the correct rules for a "large" class of sets (the *Lebesgue measurable sets*). **Chapter 2** constructs and studies this measure, which we call the *Lebesgue measure* of subsets of $\mathbb{R}^d$.

In **Chapters 3 and 4** we define the integral of real-valued and complex-valued functions whose domain is a measurable subset of $\mathbb{R}^d$. Unfortunately, we cannot define the *Lebesgue integral* of every function. **Chapter 3** introduces the class of *measurable functions* and deals with issues related to convergence of sequences of measurable functions, while **Chapter 4** defines and studies the Lebesgue integral of a measurable function. The Lebesgue integral extends the Riemann integral, but is far more general. We can define the Lebesgue integral for functions whose domain is any measurable set. We prove powerful results that allow us, in a large family of cases, to make conclusions about the convergence of a sequence of Lebesgue integrals, or to interchange the order of iterated integrals of functions of more than one variable.

The Fundamental Theorem of Calculus (FTC) is, as its name suggests, central to analysis. **Chapters 5 and 6** explore issues related to differentiation and the FTC in detail. We see that there are surprising examples of *nonconstant* functions whose derivatives are zero at "almost every" point (and therefore fail the FTC). In our quest to fully understand the FTC we define functions of *bounded variation* and study averaging operations in **Chapter 5**. Then in **Chapter 6** we introduce the class of *absolutely continuous functions*, which turn out to be the functions for which the FTC holds. The

Banach–Zaretsky Theorem plays a prominent role in **Chapter 6**, and it is central to our understanding of absolute continuity and its impact.

In **Chapter 7** our focus turns from individual functions to spaces of functions. The *Lebesgue spaces* $L^p(E)$ group functions by integrability properties, giving us a family of spaces indexed by an extended real number p with $0 < p \le \infty$. For $p \ge 1$ these are normed vector spaces of functions, while for $0 < p < 1$ they are metric spaces whose metric is not induced from a norm. The case $p = 2$ is especially important, because we can define an *inner product* on $L^2(E)$, which makes it a *Hilbert space*. This topic is explored in **Chapter 8**. In a metric space, all that we can do is define the *distance* between points in the space. In a normed space we can additionally define the *length* of each vector in the space. But in a Hilbert space, we furthermore have a notion of *angles between vectors* and hence can define orthogonality. This leads to many powerful results, including the existence of an *orthonormal basis* for every separable Hilbert space. Even though a Hilbert space can be infinite-dimensional, in many respects our intuitions from Euclidean space hold when we deal with a Hilbert space.

Chapter 9 contains "extra" material that is usually not covered in our real analysis sequence here at Georgia Tech, but which has many striking applications of the techniques developed in the earlier chapters. First we define the operation of *convolution*. Then we introduce and study the *Fourier transform* and *Fourier series*. These results form the core of the field of *harmonic analysis*, which has wide applicability throughout mathematics, physics, and engineering. Convolution is a generalization of the averaging operations that were used in **Chapters 5** and **6** to characterize the class of functions for which the Fundamental Theorem of Calculus holds. The Fourier transform and Fourier series allow us to both construct and deconstruct a wide class of functions, signals, or operators in terms of much simpler building blocks based on complex exponentials (or sines and cosines in the real case). Although **Chapter 9** presents only a taste of the theorems of harmonic analysis (which deserves another course, and a future text, to do it justice), we do get to see many applications of all of the tools that we derived in earlier chapters, including convergence of sequences of integrals (via the Dominated Convergence Theorem), interchange of iterated integrals (via Fubini's Theorem), and the Fundamental Theorem of Calculus (via the Banach–Zaretsky Theorem).

Many exercises and problems appear in each section of the text. The *Exercises* are directly incorporated into the development of the theory in each section, while the additional *Problems* given at the end of each section provide further practice and opportunities to develop understanding.

Course Options

There are many options for building a course around this text. The course that I teach at Georgia Tech is fast-paced, but covers most of the text in one semester. Here is a brief outline of such a one-semester course; a more detailed outline with much additional information (and extra problems) is contained in the **Instructor's Guide** that is available on the author's website.

Chapter 1: Assign for student reading, not covered in lecture.
Chapter 2: Sections 2.1–2.4.
Chapter 3: Sections 3.1–3.5. Omit Section 3.6.
Chapter 4: Sections 4.1–4.6.
Chapter 5: Sections 5.1–5.2, and selected portions of Sections 5.3–5.5.
Chapter 6: Sections 6.1–6.4. Omit Sections 6.5-6.6.
Chapter 7: Sections 7.1–7.4.
Chapter 8: Sections 8.1–8.4 (as time allows).
Chapter 9: Bonus material, not covered in lecture.

Another option is to begin the course with **Chapter 1** (or the online **Alternative Chapter 1**). A fast-paced course could cover most of **Chapters 1–8**. A moderately paced course could cover the first half of the text in detail in one semester, while a moderately paced two-semester course could cover all of **Chapters 1–9** in considerable detail.

Acknowledgments

Every text builds on those that have come before it, and this one is no exception. Many classic and recent volumes have influenced the writing, the choice of topics, the proofs, and the selection of problems. Among those that have had the most profound influence on my writing are Benedetto and Czaja [BC09], Bruckner, Bruckner, and Thomson [BBT97], Folland [Fol99], Rudin [Rud87], Stein and Shakarchi [SS05], and Wheeden and Zygmund [WZ77]. I greatly appreciate all of these texts and encourage the reader to consult them. Additional texts and papers are listed in the references.

Various versions of the material in this volume have been used over the years in the real analysis courses that were taught at Georgia Tech, and I thank all of the *many* students and colleagues who have provided feedback. Special thanks are due to Shahaf Nitzan, who taught the course out of earlier versions of the text and provided invaluable feedback.

Christopher Heil
Atlanta, Georgia
April 27, 2019