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Introduction to Real Analysis

Chapter 10

Online Extra Chapter on
Abstract Measure Theory

Last Updated: January 25, 2020

Chapter 10
Abstract Measure Theory

Lebesgue measure is one of the premier examples of a measure on $\mathbb{R}^d$, but it is not the only measure and certainly not the only important measure on $\mathbb{R}^d$. Further, $\mathbb{R}^d$ is not the only domain on which we encounter measures. This chapter develops the theory of measures from an abstract viewpoint. The fact that we have already examined Lebesgue measure in detail will simplify this task considerably, as there are many aspects of abstract measure theory that are precisely analogous to results for Lebesgue measure. However, other portions of the theory require extra care or new approaches. In return for developing this abstract theory of measure, it is possible to construct a powerful and useful theory of integration with respect to measures (although that will require more chapters that are not yet written).

Acknowledgment: This chapter is strongly inspired by Folland's text [Fol99]. I encourage you to consult that text as a comparison.

10.1 Sigma Algebras

There are three components to any measure. First there is the set X whose subsets we wish to measure. Unfortunately, we cannot always construct a measure that is suitably well-behaved on all of the subsets of X (consider Lebesgue measure on $\mathbb{R}^d$!). Therefore the second component of a measure is the collection Σ of subsets of X that we will actually be allowed to measure. Finally, there is the measure itself, which is a mapping $\mu\colon \Sigma \to [0, \infty]$.

We cannot choose Σ at random; it must satisfy the properties of a σ-algebra. We state the definition here, and also introduce some terminology for referring to a set together with a σ-algebra.

Definition 10.1.1 (Sigma Algebra). Let X be a set.

(a) An *algebra* or *field* on X is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under complements and *finite* unions.

(b) A *σ-algebra* or *σ-field* on X is a nonempty collection Σ of subsets of X that is closed under complements and *countable* unions.

(c) If Σ is a σ-algebra on X, then we say that (X, Σ) is a *measurable space*, and in this case we refer to the elements of Σ as the *measurable subsets of* X. ♢

By definition, *countable* means either finite or countably infinite, so a σ-algebra is closed under both finite and countably infinite unions. Since a σ-algebra is closed under complements, it follows from De Morgan's Laws that it is also closed under countable intersections. Additionally, a σ-algebra is also closed under *relative complements*, for if $A, B \in \Sigma$ then

$$A \backslash B = A \cap B^{\mathrm{C}} \in \Sigma.$$

If Σ is a σ-algebra then it is nonempty and hence must contain at least one subset E of X. Therefore $E^{\mathrm{C}} = X \backslash E$ also belongs to Σ, so

$$X = E \cup E^{\mathrm{C}} \quad \text{and} \quad \varnothing = X \backslash X$$

are elements of Σ.

Example 10.1.2. (a) Trivial examples of σ-algebras on a set X are $\Sigma = \{\varnothing, X\}$ and the power set $\mathcal{P}(X) = \{E : E \subseteq X\}$.

(b) We proved in Chapter 2 that the class of all Lebesgue measurable subsets of $\mathbb{R}^d$, which we will denote as $\mathcal{L}$ or $\mathcal{L}_{\mathbb{R}^d}$, is a σ-algebra on $\mathbb{R}^d$. ♢

We often encounter situations where we have a particular family $\mathcal{E}$ of subsets of X that we want to measure, but $\mathcal{E}$ is not a σ-algebra. In this case there will usually be many larger collections that are σ-algebras and include all of the sets from $\mathcal{E}$. For example, the power set $\mathcal{P}(X)$ is a σ-algebra that contains $\mathcal{E}$. However, we typically need the "smallest possible" σ-algebra that contains $\mathcal{E}$. The next exercise constructs this smallest σ-algebra.

Exercise 10.1.3. Let $\mathcal{E}$ be a collection of subsets of a set X. Show that

$$\Sigma(\mathcal{E}) = \bigcap \big\{\Sigma : \Sigma \text{ is a } \sigma\text{-algebra and } \mathcal{E} \subseteq \Sigma\big\} \tag{10.1}$$

is a σ-algebra on X. We call $\Sigma(\mathcal{E})$ the *σ-algebra generated by* $\mathcal{E}$. ♢

Note that if Σ_1 and Σ_2 are σ-algebras on X, then $\Sigma_1 \cap \Sigma_2$ is not formed by intersecting the elements of Σ_1 with those of Σ_2. That is, $\Sigma_1 \cap \Sigma_2$ does *not* mean $\{A \cap B : A \in \Sigma_1,\, B \in \Sigma_2\}$. Rather, $\Sigma_1 \cap \Sigma_2$ is the collection of all sets that are *common* to both Σ_1 and Σ_2:

$$\Sigma_1 \cap \Sigma_2 = \big\{A \subseteq X : A \in \Sigma_1 \text{ and } A \in \Sigma_2\big\}. \tag{10.2}$$

Similarly, the collection $\Sigma(\mathcal{E})$ defined in equation (10.1) consists of all those sets A that belong to every σ-algebra Σ that satisfies $\mathcal{E} \subseteq \Sigma$.

The following exercise explains why we call $\Sigma(\mathcal{E})$ the *smallest σ-algebra that contains $\mathcal{E}$*.

Exercise 10.1.4. Let $\mathcal{E}$ be a collection of subsets of a set X. Show that the σ-algebra $\Sigma(\mathcal{E})$ contains $\mathcal{E}$, and if Σ is any other σ-algebra that contains $\mathcal{E}$ then $\Sigma(\mathcal{E}) \subseteq \Sigma$. $\diamondsuit$

Here is an example of a σ-algebra and a generating family for that σ-algebra.

Exercise 10.1.5. Given a set X, let Σ consist of all sets $E \subseteq X$ such that at least one of E or $X \backslash E$ is countable.

(a) Show that Σ is a σ-algebra on X.

(b) Let $\mathcal{S} = \big\{\{x\} : x \in X\big\}$ be the set of all singletons of elements of X. Show that $\Sigma = \Sigma(\mathcal{S})$, i.e., Σ is the σ-algebra generated by $\mathcal{S}$. $\diamondsuit$

When working with $\mathbb{R}^d$, we inevitably must deal with the topology of $\mathbb{R}^d$. The Lebesgue σ-algebra $\mathcal{L}_{\mathbb{R}^d}$ contains all of the open subsets of $\mathbb{R}^d$, but is it the *smallest* σ-algebra that contains the open sets? Problem 10.1.18 shows that the answer to this is no—the σ-algebra generated by the open subsets of $\mathbb{R}^d$ is a *proper* subset of $\mathcal{L}_{\mathbb{R}^d}$. We have a special name for this σ-algebra.

Definition 10.1.6 (Borel σ-algebra on $\mathbb{R}^d$). The *Borel σ-algebra $\mathcal{B}_{\mathbb{R}^d}$ on* $\mathbb{R}^d$ is the smallest σ-algebra that contains all of the open subsets of $\mathbb{R}^d$. That is, if we set $\mathcal{U} = \{U \subseteq \mathbb{R}^d : U \text{ is open}\}$, then

$$\mathcal{B}_{\mathbb{R}^d} \;=\; \Sigma(\mathcal{U}).$$

The elements of $\mathcal{B}_{\mathbb{R}^d}$ are called the *Borel subsets of* $\mathbb{R}^d$. $\diamondsuit$

In particular, $\mathcal{B}_{\mathbb{R}^d}$ includes all of the open and closed subsets of $\mathbb{R}^d$, as well as the G_δ and F_σ subsets of $\mathbb{R}^d$ (see Definition 2.2.18). However, not every Lebesgue measurable subset of $\mathbb{R}^d$ is a Borel set, and not every Borel set is a G_δ or an F_σ set. On the other hand, if E is a Lebesgue measurable subset of $\mathbb{R}^d$ then there exists a G_δ set H that contains E and satisfies $|H \backslash E| = 0$. Hence we can obtain the Lebesgue σ-algebra by "adjoining" sets of measure zero to the Borel σ-algebra. Specifically, a set E belongs to $\mathcal{L}_{\mathbb{R}^d}$ if and only if it can be written as $E = H \cup Z$ where $H \in \mathcal{B}_{\mathbb{R}^d}$ and $|Z| = 0$.

The Borel subsets of $\mathbb{R}^d$ are generated from the open subsets of $\mathbb{R}^d$. There is nothing special about $\mathbb{R}^d$ in this regard—whenever we have a space that has a topology, we can define Borel sets in a similar manner. This is stated precisely in the following definition.

Definition 10.1.7 (Borel σ-algebra on X). Let X be a topological space. Then the *Borel σ-algebra $\mathcal{B}_X$ on X* is the smallest σ-algebra that contains all of the open subsets of X. The elements of $\mathcal{B}_X$ are called the *Borel subsets of X*. $\diamondsuit$

We will often be working with countable collections of sets. A countable collection can be either finite or countably infinite, and we will need to deal with both possibilities simultaneously. Therefore we introduce the following notational convention.

Notation 10.1.8 (Countable Collections of Subsets). When we are dealing with subsets E_k of a given set X, the notations $\{E_k\}$ or $\{E_k\}_k$ will implicitly mean that we have a *countable* collection of subsets. That is, $\{E_k\}$ will denote a family that has one of the forms $\{E_k\}_{k\in\mathbb{N}}$ or $\{E_k\}_{k=1}^N$, where N is a positive integer. ♢

We end this section by pointing out a simple but useful fact about countable unions of sets.

Exercise 10.1.9 (The Disjointization Trick). If Σ is a σ-algebra of subsets of X and $E_1, E_2, \dots \in \Sigma$, then the sets F_k defined by

$$F_1 = E_1, \quad F_2 = E_2 \setminus E_1, \quad F_3 = E_3 \setminus (E_1 \cup E_2), \quad \dots$$

are disjoint, belong to Σ, and satisfy

$$\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k. \qquad \diamondsuit$$

Problems

10.1.10. Let X be a set, and suppose that $\mathcal{E}, \mathcal{F} \subseteq \mathcal{P}(X)$. Show that if $\mathcal{E} \subseteq \Sigma(\mathcal{F})$ then $\Sigma(\mathcal{E}) \subseteq \Sigma(\mathcal{F})$.

10.1.11. Let (X, Σ) be a measure space. Given a subset Y of X, set $\Sigma_Y = \{E \cap Y : E \in \Sigma\}$. Show that Σ_Y is a σ-algebra on Y.

10.1.12. Show that $\Sigma = \{E \subseteq \mathbb{R}^d : |E| = 0 \text{ or } |\mathbb{R}^d \setminus E| = 0\}$ is a σ-algebra on $\mathbb{R}^d$.

10.1.13. Show that if $\mathcal{A}$ is an algebra of subsets of a set X, then the following three statements are equivalent.

(a) $\mathcal{A}$ is a σ-algebra.

(b) $\mathcal{A}$ is closed under countable disjoint unions.

(c) $\mathcal{A}$ is closed under countable increasing unions.

10.1.14. Let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra on $\mathbb{R}$. Show that each of the following collections generate $\mathcal{B}_{\mathbb{R}}$.

(a) $\mathcal{E}_1 = \{(a, b) : a < b\}$.

(b) $\mathcal{E}_2 = \{[a, b] : a < b\}$.

(c) $\mathcal{E}_3 = \{[a, b) : a < b\}$.
(d) $\mathcal{E}_4 = \{(a, \infty) : a \in \mathbb{R}\}$.
(e) $\mathcal{E}_5 = \{[a, \infty) : a \in \mathbb{R}\}$.
(f) $\mathcal{E}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$.
(g) $\mathcal{E}_7 = \{(-\infty, a] : a \in \mathbb{R}\}$.
(h) $\mathcal{E}_8 = \{(r, \infty) : r \in \mathbb{Q}\}$.
(i) $\mathcal{E}_9 = \{(-\infty, r) : r \in \mathbb{Q}\}$.

10.1.15. Let $\mathcal{B}_{\mathbb{R}^2}$ be the Borel σ-algebra on $\mathbb{R}^2$. Show that each of the following collections generates $\mathcal{B}_{\mathbb{R}^2}$.

(a) $\mathcal{E}_1 = \{(a, b) \times (c, d) : a < b, c < d\}$.

(b) $\mathcal{E}_2 = \{(a, b) \times \mathbb{R} : a < b\} \cup \{\mathbb{R} \times (c, d) : c < d\}$.

10.1.16. Given a subset A of a set X, show that

$$\Sigma \;=\; \{S : S \subseteq A\} \;\cup\; \{X \setminus S : S \subseteq A\}$$

is a σ-algebra on X.

10.1.17. Suppose that $\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$ are nested increasing σ-algebras on X. Must $\Sigma = \cup \Sigma_k$ be a σ-algebra?

Hint: Let $X = \{x_n\}_{n \in \mathbb{N}}$. Let Σ_N consist of every set $A \in \mathcal{P}(\{x_1, \dots, x_N\})$ together with the complement of each such set A.

10.1.18. Let C be the Cantor set, let φ the Cantor–Lebesgue function, and set $g(x) = \varphi(x) + x$ for $x \in [0, 1]$.

(a) Prove that $g\colon [0, 1] \to [0, 2]$ and its inverse $h = g^{-1}\colon [0, 2] \to [0, 1]$ are continuous, strictly increasing bijections.

(b) Show that $g(C)$ is a closed subset of $[0, 2]$, and $|g(C)| = 1$.

(c) Since $g(C)$ has positive measure, it contains a nonmeasurable set N. Show that $A = h(N)$ is a Lebesgue measurable subset of $[0, 1]$. (Note that $N = h^{-1}(A)$ is not measurable, so this shows that the inverse image of a Lebesgue measurable set under a continuous function need not be Lebesgue measurable.)

(d) Show that, although A is Lebesgue measurable, it is not a Borel set.

(e) Set $f = \chi_A$. Prove that $f \circ h$ is not a Lebesgue measurable function, even though f is Lebesgue measurable and h is continuous.

10.1.19.* Show that any σ-algebra Σ that contains infinitely many subsets of X must be uncountable.

Hint: First show that if Σ contains infinitely many disjoint sets $E_1, E_2, \dots$ then Σ must be uncountable. One method of showing the existence of such sets is to define a relation on X by declaring $x \sim y$ if and only if for each $A \in \Sigma$ we have $x \in A \Leftrightarrow y \in A$. Prove that $\sim$ is an equivalence relation, and show that if Σ is countable then the equivalence classes $[x] = \cap\{A \in \Sigma : x \in A\}$ all belong to Σ.

10.2 Measures

Now we turn to measures. Here is the formal definition.

Definition 10.2.1 (Measure). Let (X, Σ) be a measurable space. We say that a function $\mu\colon \Sigma \to [0, \infty]$ is a *measure* on (X, Σ) if the following two statements hold:

(a) $\mu(\varnothing) = 0$, and

(b) μ is *countably additive*, i.e.,

$$E_1, E_2, \ldots \in \Sigma \text{ are disjoint} \implies \mu\Big(\bigcup_k E_k\Big) = \sum_k \mu(E_k). \tag{10.3}$$

In this case we call (X, Σ, μ) a *measure space*, and we refer to the elements of Σ as the *μ-measurable subsets* of X. If the measure μ is clear from context, then we may simply call them the *measurable subsets* of X. The number $\mu(E)$ is the *μ-measure* or simply the *measure* of $E \in \Sigma$. ♢

Remark 10.2.2. (a) We use the phrases "(X, Σ, μ) is a measure space" and "μ is a measure on (X, Σ)" interchangeably. If the σ-algebra Σ is understood then we may simply write "μ is a measure on X."

(b) To avoid multiplicities of parentheses and braces, we usually write $\mu\{x\}$ instead of $\mu(\{x\})$. Similarly, if μ is a measure on $\mathbb{R}$ then we usually write $\mu[a, b)$ instead of $\mu([a, b))$, and so forth.

(c) Since $\mu(E_k) \geq 0$ for every k, the series $\sum \mu(E_k)$ that appears in equation (10.3) always exists in the extended real sense, i.e., it either converges to a finite nonnegative number, or it is ∞. Further, if we do have $\sum \mu(E_k) < \infty$ then, since all of the terms are nonnegative, this series converges absolutely and hence unconditionally.

(d) Countable additivity implies finite additivity, but Problem 10.3.14 shows that finite additivity need not imply countable additivity. ♢

We introduce some terminology for measures with special properties.

Definition 10.2.3. Let (X, Σ, μ) be a measure space.

(a) If $\mu(X) < \infty$ then we say that μ is a *bounded measure* or a *finite measure*, and in this case we call (X, Σ, μ) a *finite measure space*.

(b) A measure that is not bounded is an *unbounded measure*. In this case we say that (X, Σ, μ) is an *infinite measure space*.

(c) If there exist countably many sets $E_1, E_2, \ldots \in \Sigma$ such that $\mu(E_k) < \infty$ for every k and $X = \cup E_k$, then μ is a *σ-finite measure*.

(d) If every set $E \in \Sigma$ with $\mu(E) = \infty$ has a subset $F \subseteq E$ such that $F \in \Sigma$ and $0 < \mu(F) < \infty$, then μ is a *semifinite measure*.

(e) A set $A \in \Sigma$ is an *atom* if $\mu(A) > 0$ but A contains no measurable subsets with positive measure. If Σ has no atoms then we say that μ is *nonatomic*. In this case every set $E \in \Sigma$ with $\mu(E) > 0$ contains a subset B such that $B \in \Sigma$ and $0 < \mu(B) < \mu(E)$. ♢

Suppose that μ is a measure on a measurable space (X, Σ), and Σ' is a σ-algebra contained in Σ. (We could call Σ' a sub-σ-algebra, but that terminology seems rather awkward, so we will usually avoid it.) In this case, $\mu|_{\Sigma'}$ (μ restricted to Σ') is a measure on (X, Σ'). Technically, μ and $\mu|_{\Sigma'}$ are two different measures, but often we refer to the restriction of μ as "μ on Σ'," or even just as μ. Lebesgue measure is a typical illustration of this situation, as we describe next.

Example 10.2.4 (Lebesgue Measure). We proved in Chapter 2 that Lebesgue measure is a measure on the measurable space $(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d})$. This measure is σ-finite, but it is not bounded (because $|\mathbb{R}^d| = \infty$), and furthermore it is nonatomic. Some texts assign a symbol such as m or λ to represent Lebesgue measure, in which case we would denote the Lebesgue measure of $E \in \mathcal{L}_{\mathbb{R}^d}$ by $m(E)$ or $\lambda(E)$. However, for our purposes it is usually more convenient to simply write $|E|$ for the Lebesgue measure of E, just as we did in Chapter 2, and to speak of "Lebesgue measure" without assigning a symbol to represent it. In those cases where it is convenient to have a symbol that represents Lebesgue measure, we will write "Lebesgue measure dx."

When we deal with Lebesgue measure in this chapter, we will usually want to take the σ-algebra to be the Lebesgue σ-algebra. Therefore, when we speak of Lebesgue measure without qualification we assume that the σ-algebra is $\mathcal{L}_{\mathbb{R}^d}$. However, we sometimes need to restrict to a smaller σ-algebra. After the Lebesgue σ-algebra $\mathcal{L}_{\mathbb{R}^d}$, the σ-algebra on $\mathbb{R}^d$ that we encounter most often is the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^d}$. We refer to the restriction of Lebesgue measure to the Borel subsets of $\mathbb{R}^d$ as *Lebesgue measure on* $\mathcal{B}_{\mathbb{R}^d}$. ♢

Here are some new examples of measures.

Exercise 10.2.5 (The Delta Measure). Let X be a set, and fix an element $a \in X$. For each set $E \subseteq X$ define

$$\delta_a(E) \;=\; \begin{cases} 1, & \text{if } a \in E, \\ 0, & \text{if } a \notin E. \end{cases}$$

Show that δ_a is a bounded measure with respect to the σ-algebra $\mathcal{P}(X)$ (and hence also with respect to every σ-algebra on X). $\diamondsuit$

The measure δ_a has many names, including the *δ or delta measure at a*, the *Dirac measure at a*, and the *point mass at a*. If X is a vector space and $a = 0$ then we often use the shorthand $\delta = \delta_0$ for the point mass at the origin, i.e.,

$$\delta(E) \;=\; \begin{cases} 1, & 0 \in E, \\ 0, & 0 \notin E. \end{cases}$$

If we compare Lebesgue measure on $\mathbb{R}^d$ with a delta measure δ_a, where $a \in \mathbb{R}^d$, then we see many differences, such as the following.

- $\delta_a(E)$ is defined for every subset E of $\mathbb{R}^d$. That is, the (default) σ-algebra for δ is $\mathcal{P}(\mathbb{R}^d)$.
- δ_a is a bounded measure: $\delta_a(\mathbb{R}^d) = 1$.
- There are singletons that have nonzero measure. For example, $\delta_a\{a\} = 1$. Consequently $\{a\}$ is an atom for δ.
- Sets with infinite Lebesgue measure can have zero measure with respect to δ_a. For example, $\delta_a(\mathbb{R}^d\backslash\{a\}) = 0$.
- δ_a is not translation-invariant, since $\delta_a(E)$ need not equal $\delta_a(E+h)$.

Next we define a measure that is quite different from either Lebesgue measure or a delta measure. In this exercise the notation $\#E$ means the number of elements in the finite set E.

Exercise 10.2.6 (Counting Measure). Let X be a set. For each set $E \subseteq X$ define

$$\mu(E) \;=\; \begin{cases} \#E, & \text{if } E \text{ is finite}, \\ \infty, & \text{if } E \text{ is infinite}. \end{cases}$$

Show that μ is an unbounded measure with respect to the σ-algebra $\mathcal{P}(X)$ (and hence also with respect to every σ-algebra on X). This measure is called *counting measure* on X. $\diamondsuit$

Focusing on counting measure on $\mathbb{R}^d$, here are some ways in which counting measure is similar to Lebesgue measure or a delta measure, but also more ways in which it is different.

- μ is defined on every subset of $\mathbb{R}^d$ (the σ-algebra is $\mathcal{P}(\mathbb{R}^d)$).
- μ is unbounded and semifinite, but it is not σ-finite.
- Only finite sets have finite measure with respect to μ.
- Every singleton $\{x\}$ is an atom for μ.
- μ is translation-invariant.

Since the definition of a σ-algebra involves countable unions, it should not be surprising that it is often unpleasant to work with measures that are not σ-finite. Fortunately, most of the measures that we encounter in practice are σ-finite.

Remark 10.2.7. Although counting measure on $\mathbb{R}^d$ is not σ-finite, counting measure on a countable set such as $\mathbb{N}$ or $\mathbb{Z}^d$ is σ-finite. In fact, this is an important measure because integration with respect to counting measure is directly related to infinite series. For example, if μ is counting measure on the natural numbers $\mathbb{N}$ then the integral of a function $f\colon \mathbb{N} \to [0,\infty]$ with respect to μ turns out to be

$$\int_{\mathbb{N}} f\,d\mu \;=\; \sum_{n=1}^{\infty} f(n).$$

Thus, results about series are special cases of results about integration with respect to counting measure. $\diamondsuit$

As we have seen, there can be many different measures on a given space. In Chapters 2–9 we focused on Lebesgue measure on $\mathbb{R}^d$. There are many other measures on $\mathbb{R}^d$, but we adopt the following convention that Lebesgue measure is our "default" measure on $\mathbb{R}^d$.

Notation 10.2.8 (Default Measure on $\mathbb{R}^d$). Unless we specifically state otherwise, whenever we deal with $\mathbb{R}^d$ we will implicitly assume that we have chosen to work with Lebesgue measure. Further, unless specifically stated otherwise, we will assume that the σ-algebra associated with Lebesgue measure is the Lebesgue σ-algebra $\mathcal{L}_{\mathbb{R}^d}$. For example, using this convention, the phrase "let E be a measurable subset of $\mathbb{R}^d$" is interpreted to mean that E is a subset of $\mathbb{R}^d$ that is measurable with respect to Lebesgue measure (i.e., E belongs to $\mathcal{L}_{\mathbb{R}^d}$). $\diamondsuit$

Here is one way that we can derive new measures on $\mathbb{R}^d$ from Lebesgue measure.

Exercise 10.2.9. Assume that $f\colon \mathbb{R}^d \to [0,\infty]$ is a Lebesgue measurable function, and define $\mu_f(E) = \int_E f(x)\,dx$ for $E \in \mathcal{L}_{\mathbb{R}^d}$.

(a) Prove that μ_f is a measure on $(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d})$.

(b) If f is locally integrable (see Definition 5.5.4), then $\mu_f(E) < \infty$ for every bounded measurable set E.

(c) μ_f is bounded if and only if $f \in L^1(\mathbb{R}^d)$.

(d) For which f is μ_f translation-invariant?

(e) Implicitly restricting δ to the Lebesgue σ-algebra $\mathcal{L}_{\mathbb{R}^d}$, is there any function f such that $\mu_f = \delta$? Is there any f such that μ_f is counting measure on $\mathbb{R}^d$? $\diamondsuit$

Problems

10.2.10. (a) Prove that every nonnegative finite linear combination of measures is a measure, i.e., if $\mu_1, \dots, \mu_N$ are finitely many measures on (X, Σ) and $c_1, \dots, c_N \ge 0$, then $\mu(E) = \sum_{n=1}^{N} c_k \, \mu_k(E)$ for $E \in \Sigma$ defines a measure on (X, Σ).

(b) For each $k \in \mathbb{N}$ let μ_k be a measure on (X, Σ). What conditions (if any) on extended real numbers c_k are needed so that $\mu(E) = \sum_{n=1}^{\infty} c_k \, \mu_k(E)$ defines a measure on (X, Σ)?

10.2.11. Let (X, Σ, μ) be a measure space, and let $Y \in \Sigma$ be fixed. Show that $\mu_Y \colon \Sigma \to [0, \infty]$ defined by $\mu_Y(E) = \mu(E \cap Y)$ is a measure on (X, Σ).

10.2.12. Let μ be a measure on $\bigl(\mathbb{N}, \mathcal{P}(\mathbb{N})\bigr)$. Show that μ is completely determined by the values $(\mu\{k\})_{k \in \mathbb{N}}$. In other words, show that $\mu \mapsto (\mu\{k\})_{k \in \mathbb{N}}$ is an injective map of the measures on $\bigl(\mathbb{N}, \mathcal{P}(\mathbb{N})\bigr)$ into the space of sequences of extended real numbers. Identify the range of this map, and identify the sequences that correspond to bounded measures on $\mathbb{N}$.

10.2.13. Let X be an infinite set.

(a) For each subset E of X, set $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is infinite. Show that μ is finitely additive but not countably additive.

(b) Now define $\mu(\varnothing) = 0$, and set $\mu(E) = \infty$ for all nonempty subsets of X. Is μ finitely additive? Is μ a measure?

10.2.14. Let X be an uncountable set. Let Σ consist of all subsets A of X such that either A or $X \setminus A$ is at most countable. Given $E \in \Sigma$, define $\mu(E) = 0$ if E is countable, and $\mu(E) = 1$ otherwise. Show that Σ is a σ-algebra and μ is a measure on (X, Σ).

Remark: A set whose complement is countable is said to be *cocountable*.

10.2.15. Let X be a set, let $\mathcal{E}$ be a collection of subsets of X, and let Σ be the σ-algebra generated by $\mathcal{E}$. If μ and ν are two measures on (X, Σ) that agree on $\mathcal{E}$, must it be true that $\mu = \nu$ on all of Σ?

10.3 Basic Properties of Measures

This section presents some properties of abstract measures that are analogous to properties of Lebesgue measure that we encountered in previous chapters. As most of the proofs of these properties are almost identical to those for Lebesgue measure, we state these results as exercises.

Exercise 10.3.1. Let (X, Σ, μ) be a measure space. Prove the following statements.

(a) **Monotonicity:** If $A, B \in \Sigma$ and $A \subseteq B$ then $\mu(A) \le \mu(B)$.

(b) If $A, B \in \Sigma$, $A \subseteq B$, and $\mu(A) < \infty$ then $\mu(B \setminus A) = \mu(B) - \mu(A)$. ◇

Remark 10.3.2. A *signed measure* satisfies countable additivity but is allowed to take values in the range $-\infty \le \mu(E) \le \infty$ (though at most one of $-\infty$ or ∞ can be achieved, so we avoid indeterminate forms). An important difference between *measures* and *signed measures* is that monotonicity need not hold for signed measures. If μ is a signed measure and $A \subseteq B$ then, since $\mu(B \setminus A)$ might be negative, we cannot infer from the statement $\mu(A) + \mu(B \setminus A) = \mu(B)$ that $\mu(A) \le \mu(B)$. ◇

Since *measures* are monotonic, we immediately obtain the following corollary of Exercise 10.3.1.

Corollary 10.3.3. *Let* (X, Σ, μ) *be a measure space, and suppose that* $E \in \Sigma$ *satisfies* $\mu(E) = 0$. *If* $A \subseteq E$ *and* $A \in \Sigma$, *then* $\mu(A) = 0$. □

Thus, all of the *measurable* subsets of a zero measure set have zero measure. In general, however, a set with zero measure may contain subsets that are not measurable. We give the following name to measures that have the property that *every* subset of a measurable set with zero measure are measurable.

Definition 10.3.4 (Complete Measure). Assume that (X, Σ, μ) is a measure space. We say that μ is *complete* if

$$E \in \Sigma,\ \mu(E) = 0 \quad \Longrightarrow \quad A \in \Sigma \text{ for all } A \subseteq E. \qquad \diamondsuit$$

The reader should be aware that the terms "complete" and "algebra" are heavily overused in mathematics and appear in many unrelated definitions and contexts.

Example 10.3.5. If Z is a Lebesgue measurable subset of $\mathbb{R}^d$ that has measure zero, then every subset of Z is Lebesgue measurable (see Lemma 2.2.4). Therefore Lebesgue measure is complete, although we should emphasize that we are implicitly taking the σ-algebra here to be $\mathcal{L}_{\mathbb{R}^d}$. If we change the σ-algebra, then Lebesgue measure restricted to this new σ-algebra may not be complete. For example, consider Lebesgue measure restricted to the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^d}$. If $Z \in \mathcal{B}_{\mathbb{R}^d}$ is a Borel set and $|Z| = 0$, then it is possible that Z may contain a subset A that is not a Borel set. This set A is not measurable *with respect to* $\mathcal{B}_{\mathbb{R}^d}$, so Lebesgue measure is not a complete measure *on the measure space* $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$. ◇

We will consider completeness of measures in more detail in Section 10.4. For now, we give a name to those sets E that have zero measure.

Notation 10.3.6 (Null Sets). If (X, Σ, μ) is a measure space, then a set $E \in \Sigma$ that satisfies $\mu(E) = 0$ is called a *μ-null set*, a set with *μ-measure zero*, or simply a *zero measure set*.

A property that holds for all $x \in X$ except possibly for x in a μ-null set E is said to hold *μ-almost everywhere* (abbreviated μ-a.e.). ◊

We often omit writing the symbol μ if it is clear from context. For example, we may simply write that *E is a null set* instead spelling out that *E is a μ-null set*. Similarly, we often say that a property holds *almost everywhere* instead of *μ-almost everywhere*.

Remark 10.3.7. Null sets can be "very large" in senses other than their measure. For example, consider the δ-measure on $\mathbb{R}^d$. If we let $E = \mathbb{R}^d \setminus \{0\}$ then $0 \notin E$, so $\delta(E) = 0$. Therefore E is a δ-null set, even though its Lebesgue measure is infinite. Since E has measure zero with respect to δ, we write $\chi_E = 0$ δ-a.e. ◊

Abstract measures satisfy continuity from above and below in the following sense.

Exercise 10.3.8. Let (X, Σ, μ) be a measure space. Given countably many measurable sets $E_1, E_2, \dots$ in X, prove that the following statements hold.

(a) **Continuity from below:** If $E_1 \subseteq E_2 \subseteq \cdots$, then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k).$$

(b) **Continuity from above:** If $E_1 \supseteq E_2 \supseteq \cdots$ and $\mu(E_k) < \infty$ for some k, then

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k). \qquad \diamondsuit$$

The final property that we will discuss in this section is countable subadditivity. Here we require a different approach than we took when we considered Lebesgue measure. Before defining Lebesgue measure, we first constructed *exterior Lebesgue measure*, which is subadditive, and then restricted to the Lebesgue measurable sets to obtain Lebesgue measure. Hence Lebesgue measure simply inherits subadditivity from exterior Lebesgue measure. With Lebesgue measure, the difficulty lay in proving that countable additivity holds on the Lebesgue measurable sets. In contrast, an abstract measure μ is countably additive by definition, and the difficulty is that we must *deduce* countable subadditivity from countable additivity.

Theorem 10.3.9 (Countable Subadditivity). *Let (X, Σ, μ) be a measure space. If $E_1, E_2, \ldots \in \Sigma$, then*

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} \mu(E_k).$$

Proof. Using the disjointization trick (Exercise 10.1.9), we can write $\cup E_k = \cup F_k$ where the sets F_k are μ-measurable and disjoint, and $F_k \subseteq E_k$ for each k. Combining countable additivity and monotonicity, it follows that

$$\mu\biggl(\bigcup_{k=1}^{\infty} E_k\biggr) \;=\; \mu\biggl(\bigcup_{k=1}^{\infty} F_k\biggr) \;=\; \sum_{k=1}^{\infty} \mu(F_k) \;\le\; \sum_{k=1}^{\infty} \mu(E_k). \qquad \square$$

Problems

10.3.10. Show that every σ-finite measure is semifinite.

10.3.11. Suppose that (X, Σ, μ) is a measure space and $\{x\} \in \Sigma$ for every $x \in X$. Show that if μ is a finite measure, then $E = \{x \in X : \mu\{x\} > 0\}$ is countable.

10.3.12. Let (X, Σ, μ) be a measure space and choose sets $E_k \in \Sigma$ for $k \in \mathbb{N}$. Define

$$\limsup_{k\to\infty} E_k \;=\; \bigcap_{j=1}^{\infty}\biggl(\bigcup_{k=j}^{\infty} E_k\biggr), \qquad \liminf_{k\to\infty} E_k \;=\; \bigcup_{j=1}^{\infty}\biggl(\bigcap_{k=j}^{\infty} E_k\biggr).$$

By Exercise 2.1.15, $\limsup E_k$ consists of those points $x \in X$ that belong to infinitely many of the E_k, while $\liminf E_k$ consists of those $x \in X$ that belong to all but finitely many E_k.

(a) Show that

$$\mu\Bigl(\liminf_{k\to\infty} E_k\Bigr) \;\le\; \liminf_{k\to\infty} \mu(E_k).$$

(b) Prove that if $\mu(\cup E_k) < \infty$, then

$$\mu\Bigl(\limsup_{k\to\infty} E_k\Bigr) \;\ge\; \limsup_{k\to\infty} \mu(E_k).$$

(c) Prove the *Borel–Cantelli Lemma*: If $\sum \mu(E_k) < \infty$, then $\liminf E_k$ and $\limsup E_k$ each have μ-measure zero.

10.3.13. Let (X, Σ, μ) be a measure space. Suppose that μ is semifinite, $E \in \Sigma$, and $\mu(E) = \infty$. Prove that if $C > 0$, then there exists some measurable set $A \subseteq E$ that satisfies $C < \mu(A) < \infty$.

Hint: Define $C = \sup\{\mu(F) : F \in \Sigma,\ F \subseteq E,\ \mu(F) < \infty\}$. If $C < \infty$ then there exist measurable sets $F_k \subseteq E$ with finite measure such that $\mu(F_k) \to C$.

10.3.14. Suppose that μ is a *finitely* additive function on a σ-algebra Σ of subsets of X, i.e., $\mu(\varnothing) = 0$ and $\mu\bigl(\cup_{k=1}^{N} E_k\bigr) = \sum_{k=1}^{N} \mu(E_k)$ for any finite collection of disjoint sets $E_1, \dots, E_N \in \Sigma$. Prove the following statements.

(a) μ is a measure if and only if μ satisfies continuity from below.

(b) If $\mu(X) < \infty$, then μ is a measure if and only μ it satisfies continuity from above.

10.3.15. Let (X, Σ) be a measure space, and suppose that Σ is generated by $\mathcal{E} \subseteq \mathcal{P}(X)$. If $\mu(E) = \nu(E)$ for every set $E \in \mathcal{E}$, must it be true that $\mu = \nu$?

Hint: Keep it simple, consider $X = \mathbb{Z}$.

10.4 The Completion of a Measure

By definition, $E \subseteq X$ is a null set for a measure μ on X if $E \in \Sigma$ and $\mu(E) = 0$. In general, an arbitrary subset A of E need not be measurable, but if A happens to be measurable then monotonicity implies that $\mu(A) = 0$. A *complete measure* is one such that every subset A of every null set E is measurable (Definition 10.3.4).

Complete measures are often more convenient to work with than incomplete measures. Fortunately, if we have a incomplete measure μ in hand, there is a way to extend μ to a larger σ-algebra $\overline{\Sigma}$ in such a way that the extended measure is complete. This new extended measure $\overline{\mu}$ is called the *completion* of μ, and its construction is given in the next exercise.

Exercise 10.4.1. Let (X, Σ, μ) be a measure space, and let $\mathcal{N}$ be the collection of all μ-null sets in X:

$$\mathcal{N} \; = \; \bigl\{N \in \Sigma : \mu(N) = 0\bigr\}.$$

Define

$$\overline{\Sigma} \; = \; \{E \cup Z : E \in \Sigma,\, Z \subseteq N \in \mathcal{N}\},$$

and for each set $E \cup Z \in \overline{\Sigma}$ let

$$\overline{\mu}(E \cup Z) \; = \; \mu(E).$$

Prove the following statements.

(a) $\overline{\Sigma}$ is a σ-algebra on X.

(b) $\overline{\mu}$ is well-defined on $\overline{\Sigma}$. That is, if $E_1 \cup Z_1 = E_2 \cup Z_2$ where $E_1, E_2 \in \Sigma$, $Z_1 \subseteq N_1 \in \mathcal{N}$, and $Z_2 \subseteq N_2 \in \mathcal{N}$, then $\overline{\mu}(E_1 \cup Z_1) = \overline{\mu}(E_2 \cup Z_2)$.

(c) $\overline{\mu}$ is a measure on $(X, \overline{\Sigma})$.

(d) $\overline{\mu}$ is the *unique* measure on $(X, \overline{\Sigma})$ that coincides with μ on Σ.

(e) $\overline{\mu}$ is complete. $\diamondsuit$

Example 10.4.2. Let $\mathcal{B}_{\mathbb{R}^d}$ be the Borel σ-algebra on $\mathbb{R}^d$, and let μ be Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$. Since every open subset of $\mathbb{R}^d$ is Lebesgue measurable, $\mathcal{B}_{\mathbb{R}^d}$ is contained in the σ-algebra $\mathcal{L}_{\mathbb{R}^d}$ of Lebesgue measurable subsets of $\mathbb{R}^d$.

Using results from Chapter 2, we see that the completion of $\mathcal{B}_{\mathbb{R}^d}$ is precisely $\mathcal{L}_{\mathbb{R}^d}$, and $\overline{\mu}$ is Lebesgue measure $|\cdot|$ on $(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d})$. ◇

Example 10.4.3. Consider the δ-measure as a measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$. In this case

$$\overline{\mathcal{B}_{\mathbb{R}^d}} \;=\; \mathcal{P}(\mathbb{R}^d),$$

and $\overline{\delta} = \delta$ on $(\mathbb{R}^d, \mathcal{P}(\mathbb{R}^d))$. ◇

10.5 Outer Measures

In Chapter 2, we created exterior Lebesgue measure by beginning with a class of sets whose measures were known (boxes), followed by an extension to arbitrary subsets of $\mathbb{R}^d$ via countable coverings by boxes. We could not do this in a "good" way for all sets, so after constructing exterior Lebesgue measure on the measure space $(\mathbb{R}^d, \mathcal{P}(\mathbb{R}^d))$, we restricted to the measure space $(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d})$ to obtain Lebesgue measure, which is countably additive on this smaller measure space.

However, we had a different starting point in this chapter. We assumed that we had an abstract measure μ in hand. By definition, such a measure is countably additive on an associated σ-algebra. Therefore we did not have to prove that countable additivity is satisfied; rather it was assumed as a hypothesis. But this leaves us with a basic existential question: How do we construct such abstract measures? In this this section and the next we will show that we can create measures through a process similar to the one that we used to construct Lebesgue measure from exterior Lebesgue measure. Specifically, given a class of subsets of X that we know how we want to measure, we will construct an exterior or outer measure μ^* that is defined on all subsets of X but is not a true measure, and then construct an actual measure μ by restricting μ^* to an appropriate σ-algebra Σ of "measurable sets."

We will break this process into two parts:

(i) the construction of an outer measure μ^*, and

(ii) the construction of a measure μ from the outer measure μ^*.

We will address the second item in this section, showing how to obtain a measure μ from an outer measure μ^*. In Section 10.6 we will consider the issue of constructing an outer measure μ^* from scratch.

Our first task is to precisely define outer measures. Considering the example of exterior Lebesgue measure, it seems that the most important requirements are that an outer measure μ^* should be defined on all subsets of X and that it should satisfy countable subadditivity. However, there is another important but hidden property, which is *monotonicity.* Although *countable additivity* implies monotonicity, *countable subadditivity* does not. Therefore we need to include monotonicity as an explicit part of the definition of an outer measure.

Definition 10.5.1 (Outer Measure). Let X be a nonempty set. An *outer measure* or *exterior measure* on X is a function $\mu^*\colon \mathcal{P}(X) \to [0,\infty]$ that satisfies the following conditions.

(a) $\mu^*(\varnothing) = 0$.

(b) *Monotonicity*: If $A \subseteq B$, then $\mu^*(A) \le \mu^*(B)$.

(c) *Countable subadditivity*: If $E_1, E_2, \ldots \subseteq X$, then

$$\mu^*\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) \;\le\; \sum_{k=1}^{\infty} \mu^*(E_k). \qquad \diamondsuit$$

By taking $E_k = \varnothing$ for $k > N$, we see that every outer measure is finitely subadditive, i.e., if $N \in \mathbb{N}$ then

$$\mu^*\Bigl(\bigcup_{k=1}^{N} E_k\Bigr) \;\le\; \sum_{k=1}^{N} \mu^*(E_k).$$

Given an outer measure μ^*, our goal is to create a σ-algebra Σ on X such that μ^* *restricted to* Σ is countably additive. The elements of Σ will be our "good sets," the sets that are measurable with respect to μ^*. But how do we define measurability for an arbitrary outer measure? Our set X need not have a topology, so we cannot define measurability in terms of surrounding open sets, as we did for Lebesgue measure. On the other hand, the formulation of Lebesgue measurability given by Carathéodory's Criterion does not explicitly involve topology, and as such it is the appropriate motivation for the following definition.

Definition 10.5.2 (μ^*-Measurable Sets). Let μ^* be an outer measure on a set X. Then a set $E \subseteq X$ is *μ^*-measurable*, or simply *measurable* for short, if for every set $A \subseteq X$ we have

$$\mu^*(A) \;=\; \mu^*(A \cap E) \;+\; \mu^*(A \backslash E). \qquad \diamondsuit \tag{10.4}$$

It is sometimes helpful to note that equation (10.4) is the same as

$$\mu^*(A) \;=\; \mu^*(A \cap E) \;+\; \mu^*(A \cap E^{\mathrm{C}}).$$

The next exercise, which follows by combining subadditivity with monotonicity, asks for a proof that every subset of X that has outer measure zero is measurable.

Exercise 10.5.3. Let μ^* be an outer measure on X. Show that if $E \subseteq X$ and $\mu^*(E) = 0$ then E is μ^*-measurable. $\diamondsuit$

Now we prove that the collection of μ-measurable sets forms a σ-algebra on X, and μ^* restricted to this σ-algebra is a complete measure on X.

Theorem 10.5.4 (Carathéodory's Theorem). *If μ^* is an outer measure on a set X, then the following statements hold.*

(a) $\Sigma = \bigl\{E \subseteq X : E$ *is μ^*-measurable*$\bigr\}$ *is a σ-algebra on X.*

(b) $\mu = \mu^*|_\Sigma$ *is a measure on (X, Σ).*

(c) *μ is a complete measure. In fact,* every *set $Z \subseteq X$ that satisfies $\mu^*(Z) = 0$ is μ^*-measurable.*

Proof. (a) We break the proof into a series of steps.

Step 1. Σ is not empty since the empty set is μ^*-measurable.

Step 2. To show that Σ is closed under complements, fix any set $E \in \Sigma$ and let A be an arbitrary subset of X. Using the fact that E is μ^*-measurable, we compute that

$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap E) + \mu^*(A \setminus E) \\ &= \mu^*(A \cap E) + \mu^*(A \cap E^{\mathrm{C}}) \\ &= \mu^*\bigl(A \cap (E^{\mathrm{C}})^{\mathrm{C}}\bigr) + \mu^*(A \cap E^{\mathrm{C}}) \\ &= \mu^*(A \cap E^{\mathrm{C}}) + \mu^*(A \setminus E^{\mathrm{C}}). \end{aligned}$$

Since this holds for every set A, we conclude that E^{C} is μ^*-measurable, and therefore $E^{\mathrm{C}} \in \Sigma$.

Step 3. Suppose that E and F are any two sets in Σ. We will prove that $E \cup F \in \Sigma$. To do this, choose any set $A \subseteq X$. By subadditivity,

$$\mu^*(A) \le \mu^*\bigl(A \cap (E \cup F)\bigr) + \mu^*\bigl(A \cap (E \cup F)^{\mathrm{C}}\bigr).$$

To prove the opposite inequality, observe that since F is μ^*-measurable and $A \cap E$ is a subset of X, we have

$$\mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^{\mathrm{C}}) = \mu^*(A \cap E).$$

Similarly,

$$\mu^*(A \cap E^{\mathrm{C}} \cap F) + \mu^*(A \cap E^{\mathrm{C}} \cap F^{\mathrm{C}}) = \mu^*(A \cap E^{\mathrm{C}}).$$

Applying these equalities and using the fact that μ^* is finitely subadditive, we see that

$$\begin{aligned} &\mu^*\bigl(A \cap (E \cup F)\bigr) + \mu^*\bigl(A \cap (E \cup F)^{\mathrm{C}}\bigr). \\ &= \mu^*\Bigl((A \cap E \cap F) \cup (A \cap E \cap F^{\mathrm{C}}) \cup (A \cap E^{\mathrm{C}} \cap F)\Bigr) + \mu^*(A \cap (E \cup F)^{\mathrm{C}}) \end{aligned}$$

$$\begin{aligned}
&\le \mu^*(A\cap E\cap F) + \mu^*(A\cap E\cap F^{\mathrm{C}}) + \mu^*(A\cap E^{\mathrm{C}}\cap F) + \mu^*(A\cap E^{\mathrm{C}}\cap F^{\mathrm{C}})\\
&= \mu^*(A\cap E) + \mu^*(A\cap E^{\mathrm{C}})\\
&= \mu^*(A),
\end{aligned}$$

where at the final equality we have used the fact that E is μ^*-measurable. Therefore $E\cup F \in \Sigma$. Applying induction, it follows that Σ is closed under finite unions.

Step 4. Next we will show that μ^* is finitely additive on Σ. Suppose that E and F *disjoint* elements of Σ. Then

$$\begin{aligned}
\mu^*(E\cup F) &= \mu^*\big((E\cup F)\cap E\big) + \mu^*\big((E\cup F)\cap E^{\mathrm{C}}\big) && (E \text{ is measurable})\\
&= \mu^*(E) + \mu^*(F) && (E,\, F \text{ are disjoint}).
\end{aligned}$$

Therefore $E\cup F$ is measurable. By induction, it follows that μ^* is finitely additive when we choose sets from Σ.

Step 5. In this step we will show that Σ is closed under countable unions. By Problem 10.1.13, it suffices to show that Σ is closed under countable *disjoint* unions. So, assume that $E_1, E_2, \dots \in \Sigma$ are disjoint sets and define

$$F = \bigcup_{k=1}^{\infty} E_k \qquad \text{and} \qquad F_n = \bigcup_{k=1}^{n} E_k, \quad n\in\mathbb{N}.$$

Note that $F_n\in\Sigma$ since Σ is closed under finite unions.

Choose any set $A\subseteq X$. We claim that

$$\mu^*(A\cap F_n) = \sum_{k=1}^{n} \mu^*(A\cap E_k). \tag{10.5}$$

Now, if we knew that μ^* was finitely additive on *all* subsets of X then equation (10.5) would be immediate. However, all that we know is that μ^* is finitely additive on the μ^*-measurable sets. Since A is an arbitrary set, this does not help us. Instead, we prove equation (10.5) by induction.

Since $F_1 = E_1$, equation (10.5) is trivial when $n=1$. Suppose that equation (10.5) holds for some integer $n\ge 1$. Then, since E_{n+1} is μ^*-measurable,

$$\begin{aligned}
\mu^*(A\cap F_{n+1}) &= \mu^*\Big(A \cap \bigcup_{k=1}^{n+1} E_k\Big)\\
&= \mu^*\Big(A \cap \bigcup_{k=1}^{n+1} E_k \cap E_{n+1}\Big) + \mu^*\Big(A \cap \bigcup_{k=1}^{n+1} E_k \cap E_{n+1}^{\mathrm{C}}\Big)
\end{aligned}$$

$$= \mu^*(A \cap E_{n+1}) + \mu^*\Big(A \cap \bigcup_{k=1}^{n} E_n\Big) \qquad \text{(by disjointness)}$$

$$= \mu^*(A \cap E_{n+1}) + \sum_{k=1}^{n} \mu^*(A \cap E_k).$$

This completes the induction, so equation (10.5) holds for all n.

Next, we compute that

$$\begin{aligned} &\sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap F^{\mathrm{C}}) \\ &\qquad \le \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap F_n^{\mathrm{C}}) && \text{(since } F^{\mathrm{C}} \subseteq F_n^{\mathrm{C}}) \\ &\qquad = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^{\mathrm{C}}) && \text{(by equation (10.5))} \\ &\qquad = \mu^*(A) && \text{(since } F_n \in \Sigma). \end{aligned} \tag{10.6}$$

Applying subadditivity and taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} \mu^*(A) &\le \mu^*(A \cap F) + \mu^*(A \cap F^{\mathrm{C}}) && \text{(subadditivity)} \\ &\le \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A \cap F^{\mathrm{C}}) && \text{(subadditivity)} \\ &\le \mu^*(A) && \text{(by equation (10.6)).} \end{aligned}$$

Therefore equality holds in the preceding lines:

$$\begin{aligned} \mu^*(A) &= \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A \cap F^{\mathrm{C}}) \\ &= \mu^*(A \cap F) + \mu^*(A \cap F^{\mathrm{C}}). \end{aligned} \tag{10.7}$$

This shows that F is measurable, and so $F \in \Sigma$.

(b) To show that μ^* *restricted to* Σ is countably additive, let $E_1, E_2, \dots$ be disjoint sets in Σ, and set $F = \cup E_k$. Then, since F is μ^*-measurable, by applying equation (10.7) with $A = F$ we see that

$$\mu^*(F) = \sum_{k=1}^{\infty} \mu^*(F \cap E_k) + \mu^*(F \cap F^{\mathrm{C}}) = \sum_{k=1}^{\infty} \mu^*(E_k).$$

Hence μ^* is countably additive on Σ, and therefore $\mu = \mu^*|_\Sigma$ is a measure.

(c) This follows from Exercise 10.5.3. □

Problems

10.5.5. Let μ^* be an outer measure on X, and let A and B be μ^*-measurable subsets of X. Show that $\mu^*(A \cup B) + \mu^*(A \cap B) \; = \; \mu^*(A) + \mu^*(B)$.

10.5.6. Let μ^* be an outer measure on X. Show that if A and B are disjoint subsets of X and A is μ^*-measurable, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

10.5.7. Let X be an uncountable set. Define $\mu^*(E) = 0$ if $E \subseteq X$ is countable, and $\mu^*(E) = 1$ if $E \subseteq X$ is uncountable, Show that μ^* is an outer measure on X, and identify the μ^*-measurable subsets of X.

10.5.8. Let μ^* be an outer measure on a set X, and let $\{E_j\}_{j=1}^{\infty}$ be a sequence of disjoint μ^*-measurable subsets of X. Prove the following statements.

(a) For every set $A \subseteq X$ (measurable or not) and every integer $n > 0$ we have

$$\mu^*\biggl(A \cap \Bigl(\bigcup_{j=1}^{n} E_j\Bigr)\biggr) \; = \; \sum_{j=1}^{n} \mu^*(A \cap E_j).$$

(b) For any $A \subseteq X$ (measurable or not) we have

$$\mu^*\biggl(A \cap \Bigl(\bigcup_{j=1}^{\infty} E_j\Bigr)\biggr) \; = \; \sum_{j=1}^{\infty} \mu^*(A \cap E_j).$$

10.6 The Construction of an Outer Measure

In this section we will show that if we are given a particular collection of "elementary sets" $\mathcal{E} \subseteq \mathcal{P}(X)$ whose measures are specified, then we can extend from these sets to obtain an outer measure on X. We do this by employing the same technique that we used to create exterior Lebesgue measure, i.e., we cover arbitrary sets by countable unions of elementary sets in all possible ways. Note that in this result we place no restrictions on the values $\rho(E)$ that we assign to our "elementary sets" E other than $\rho(\varnothing) = 0$ and $0 \le \rho(E) \le \infty$ for each $E \in \mathcal{E}$.

Theorem 10.6.1. *Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a fixed collection of subsets of X such that*

(a) $\varnothing \in \mathcal{E}$, *and*

(b) *there exist countably many sets $E_k \in \mathcal{E}$ such that $\cup E_k = X$.*

Suppose that $\rho\colon \mathcal{E} \to [0, \infty]$ satisfies $\rho(\varnothing) = 0$. For each $A \subseteq X$, define

$$\mu^*(A) \; = \; \inf\biggl\{\sum_k \rho(E_k)\biggr\}, \tag{10.8}$$

where the infimum is taken over all finite or countable covers of A by sets $E_k \in \mathcal{E}$. Then μ^* is an outer measure on X.

Proof. The hypotheses ensure that every subset of X has at least one covering by elements of $\mathcal{E}$. Hence the infimum in equation (10.8) is not taken over the empty set, and therefore it defines a unique value in $[0,\infty]$ for each $A \subseteq X$.

Since $\{\varnothing\}$ is one covering of $\varnothing$ by elements of $\mathcal{E}$, we have

$$0 \ \le\ \mu^*(\varnothing) \ \le\ \rho(\varnothing) \ =\ 0.$$

Monotonicity follows from the fact that if $A \subseteq B$ then every covering of B by sets $E_k \in \mathcal{E}$ is also a covering of A. Finally, the proof that μ^* is countably subadditive is just like the proof that exterior Lebesgue measure is countably subadditive (Theorem 2.1.13), so we assign the justification of this fact as an exercise. □

We refer to the elements of the collection $\mathcal{E}$ in Theorem 10.6.1 as *elementary sets*. By Theorem 10.5.4, since the function μ^* constructed in Theorem 10.6.1 is an outer measure, we know that there is an associated σ-algebra Σ of μ^*-measurable sets, and we also know that $\mu = \mu^*|_\Sigma$ is a complete measure on (X, Σ). However, there are still two important questions that we have not addressed.

- Are the elementary sets measurable, i.e., do we have $\mathcal{E} \subseteq \Sigma$?
- Is $\mu^*(E) = \rho(E)$ for $E \in \mathcal{E}$?

Unfortunately, the following example shows that the answers to these questions are *no* in general.

Example 10.6.2. Let X be a set that contains at least two elements. Let E be a nonempty proper subset of X, and define $\mathcal{E} = \{\varnothing,\, E,\, E^{\mathrm{C}},\, X\}$.

(a) If we define

$$\rho(\varnothing) \,=\, 0, \qquad \rho(E) \,=\, \frac{1}{4}, \qquad \rho(E^{\mathrm{C}}) \,=\, \frac{1}{4}, \qquad \rho(X) \,=\, 1,$$

then $\mu^*(X) = \frac{1}{2} \neq \rho(X)$, so the outer measure μ^* does not agree with ρ on the elementary sets.

(b) If we define

$$\rho(\varnothing) \,=\, 0, \qquad \rho(E) \,=\, 1 \qquad \rho(E^{\mathrm{C}}) \,=\, 1 \qquad \rho(X) \,=\, 1,$$

then

$$\mu^*(X) \ =\ 1 \ \neq\ 2 \ =\ \mu^*(X \cap E) + \mu^*(X \cap E^{\mathrm{C}}),$$

so E is not μ^*-measurable, even though it is an elementary set. ◇

Thus, we need to impose some extra conditions on the function ρ and the class $\mathcal{E}$ of elementary sets.

Definition 10.6.3 (Premeasure). Let X be a set, and let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an *algebra*, i.e., $\mathcal{A}$ is nonempty and is closed under complements and *finite* unions. A *premeasure* on $\mathcal{A}$ is a function $\rho\colon \mathcal{A} \to [0,\infty]$ that satisfies

(a) $\rho(\varnothing) = 0$, and

(b) if $E_1, E_2, \ldots \in \mathcal{A}$ are disjoint and if $\bigcup_{k=1}^\infty E_k \in \mathcal{A}$, then

$$\rho\biggl(\bigcup_{k=1}^\infty E_k\biggr) \;=\; \sum_{k=1}^\infty \rho(E_k). \qquad \diamondsuit$$

Note that in requirement (b) we are not assuming that $\mathcal{A}$ is closed under countable unions. We only require that *if* the union of the disjoint sets A_k belongs to $\mathcal{A}$ *then* ρ will be countably additive on those sets. On the other hand, the next lemma shows that every premeasure is both *finitely additive* and *monotonic*.

Lemma 10.6.4. *A premeasure ρ on an algebra $\mathcal{A}$ is monotonic and finitely additive on $\mathcal{A}$.* $\diamondsuit$

Proof. To show finite additivity, choose any disjoint sets $A_1, \ldots, A_N \in \mathcal{A}$. Define $A_k = \varnothing$ for $k > N$. Then, since $\mathcal{A}$ is closed under *finite unions*, we have

$$\bigcup_{k=1}^\infty A_k \;=\; \bigcup_{k=1}^N A_k \;\in\; \mathcal{A}.$$

Consequently, by the definition of a premeasure,

$$\rho\biggl(\bigcup_{k=1}^N A_k\biggr) \;=\; \rho\biggl(\bigcup_{k=1}^\infty A_k\biggr) \;=\; \sum_{k=1}^\infty \rho(A_k) \;=\; \sum_{k=1}^N \rho(A_k).$$

Hence μ^* is finitely additive *on the algebra* $\mathcal{A}$.

Now suppose that $A, B \in \mathcal{A}$ and $A \subseteq B$. Then, by finite additivity,

$$\rho(B) \;=\; \rho(A) \;+\; \rho(B\backslash A) \;\geq\; \rho(A).$$

Therefore ρ is monotonic *on the algebra* $\mathcal{A}$. $\square$

Remark 10.6.5. The collection of all boxes in $\mathbb{R}^d$ does not form an algebra, since it is not closed under either complements or finite unions. Thus it is not *entirely* obvious how the construction of Lebesgue measure from Chapter 2 relates to premeasures. This issue will require some attention, but we will not deal with it in this chapter. $\diamondsuit$

If ρ is a premeasure, then we let μ^* denote the associated outer measure

$$\mu^*(E) = \inf\biggl\{\sum_k \rho(E_k) : E_k \in \mathcal{A},\, E \subseteq \bigcup_k E_k\biggr\}, \qquad \text{for } E \subseteq X.$$

Since $\mathcal{A}$ is nonempty, there is some set $A \in \mathcal{A}$, and therefore $A^{\mathrm{C}} \in \mathcal{A}$ as well. Hence $X = A \cup A^{\mathrm{C}}$. This shows that the hypotheses of Theorem 10.6.1 are satisfied, so μ^* is indeed an outer measure. Let Σ denote the corresponding σ-algebra of μ^*-measurable sets. Carathéodory's Theorem (Theorem 10.5.4) implies that $\mu = \mu^*|_\Sigma$ is a complete measure. We show next that this measure is "well-behaved." For simplicity of presentation throughout this proof, we write a countable cover as $\{E_k\}_{k\in\mathbb{N}}$. We can do this without loss of generality because if $\{E_1, \dots, E_N\}$ is a finite cover, then we can just take $E_k = \varnothing$ for $k > N$.

Theorem 10.6.6. *If ρ is a premeasure on an algebra $\mathcal{A}$ and μ^* is the associated outer measure, then the following statements hold.*

(a) *$\mu^*|_{\mathcal{A}} = \rho$, i.e., $\mu^*(E) = \rho(E)$ for every set $E \in \mathcal{A}$.*

(b) *$\mathcal{A} \subseteq \Sigma$, i.e., every set in $\mathcal{A}$ is μ^*-measurable. Consequently, $\mu(E) = \rho(E)$ for every set $E \in \mathcal{A}$.*

(c) *If ν is any measure on Σ that satisfies $\nu|_{\mathcal{A}} = \rho$, then $\nu(E) \le \mu(E)$ for all $E \in \Sigma$, with equality holding if $\mu(E) < \infty$. Furthermore, if ρ is σ-finite, then $\nu = \mu$.*

Proof. (a) Suppose that $E \in \mathcal{A}$. Then $\{E\}$ is a covering of E by a single set from $\mathcal{A}$, so $\mu^*(E) \le \rho(E)$. For the converse inequality, let $\{E_k\}_{k\in\mathbb{N}}$ be any countable collection of sets from $\mathcal{A}$ that covers E. Disjointize the sets E_k by defining

$$F_1 = E \cap E_1 \qquad \text{and} \qquad F_n = E \cap \biggl(E_n \setminus \bigcup_{k=1}^{n-1} E_k\biggr), \quad n > 1.$$

Then $F_1, F_2, \ldots \in \mathcal{A}$ and $\cup F_n = E \in \mathcal{A}$, so by the definition of a premeasure and the fact that ρ is monotonic we have

$$\rho(E) = \sum_{n=1}^{\infty} \rho(F_n) \le \sum_{n=1}^{\infty} \rho(E_n).$$

Taking the infimum over all coverings of E therefore gives us $\rho(E) \le \mu^*(E)$. Hence μ^* agrees with ρ on $\mathcal{A}$.

(b) Suppose that $E \in \mathcal{A}$ and $A \subseteq X$. If we fix $\varepsilon > 0$, then there exists a countable covering $\{E_k\}_{k\in\mathbb{N}}$ of A by sets $E_k \in \mathcal{A}$ such that

$$\sum_{k=1}^{\infty} \rho(E_k) \le \mu^*(A) + \varepsilon.$$

Hence,

$$
\begin{aligned}
\mu^*(A) &\le \mu^*(A \cap E) + \mu^*(A \cap E^{\mathrm{C}}) && \text{(subadditivity)} \\
&\le \mu^*\Bigl(\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) \cap E\Bigr) + \mu^*\Bigl(\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) \cap E^{\mathrm{C}}\Bigr) && \text{(monotonicity)} \\
&= \mu^*\Bigl(\bigcup_{k=1}^{\infty} (E_k \cap E)\Bigr) + \mu^*\Bigl(\bigcup_{k=1}^{\infty} (E_k \cap E^{\mathrm{C}})\Bigr) \\
&\le \sum_{k=1}^{\infty} \mu^*(E_k \cap E) + \sum_{k=1}^{\infty} \mu^*(E_k \cap E^{\mathrm{C}}) && \text{(subadditivity)} \\
&= \sum_{k=1}^{\infty} \Bigl(\rho(E_k \cap E) + \rho(E_k \cap E^{\mathrm{C}})\Bigr) && \text{(part (a))} \\
&= \sum_{k=1}^{\infty} \rho(E_k) && \text{(countable additivity on } \mathcal{A}) \\
&\le \mu^*(A) + \varepsilon.
\end{aligned}
$$

Since this is true for every ε, we conclude that

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\mathrm{C}}).$$

Therefore E is μ^*-measurable.

(c) Suppose that ν is any measure on Σ that extends ρ, and fix $A \in \Sigma$. If $\{E_k\}_{k\in\mathbb{N}}$ is a cover of A by countably many sets from $\mathcal{A}$, then

$$\nu(A) \le \sum_{k=1}^{\infty} \nu(E_k) = \sum_{k=1}^{\infty} \rho(E_k).$$

Since this is true for every covering, we conclude that $\nu(A) \le \mu^*(A) = \mu(A)$.

Suppose in addition that $\mu(A) < \infty$. Then given $\varepsilon > 0$ we can find countably many sets $E_k \in \mathcal{A}$ such that $\cup E_k \supseteq A$ and

$$\sum_{k=1}^{\infty} \rho(E_k) \le \mu^*(A) + \varepsilon.$$

Set $E = \cup E_k$. Then

$$
\begin{aligned}
\mu(E) &\le \sum_{k=1}^{\infty} \mu(E_k) && \text{(subadditivity)} \\
&= \sum_{k=1}^{\infty} \rho(E_k) && \text{(part (a))}
\end{aligned}
$$

$$\leq \mu^*(A) + \varepsilon$$

$$= \mu(A) + \varepsilon \qquad \text{(since } A \in \Sigma).$$

Since all of the quantities above are finite, we can rearrange and use additivity to conclude that

$$\mu(E \setminus A) = \mu(E) - \mu(A) \leq \varepsilon.$$

Now, $\mathcal{A}$ is closed under finite unions, so $\cup_{k=1}^N E_k \in \mathcal{A}$ for every N. By continuity from below and the fact that μ and ν both extend ρ, it follows that

$$\begin{aligned}
\nu(E) &= \lim_{N\to\infty} \nu\biggl(\bigcup_{k=1}^N E_k\biggr) \\
&= \lim_{N\to\infty} \rho\biggl(\bigcup_{k=1}^N E_k\biggr) \\
&= \lim_{N\to\infty} \mu\biggl(\bigcup_{k=1}^N E_k\biggr) = \mu(E).
\end{aligned}$$

Hence

$$\mu(A) \leq \mu(E) = \nu(E) = \nu(A) + \nu(E\setminus A) \leq \nu(A) + \mu(E\setminus A) \leq \nu(A) + \varepsilon.$$

Since ε is arbitrary, it follows that $\mu(A) = \nu(A)$.

Finally, suppose that ρ is σ-finite, i.e., we can write $X = \cup A_k$ where $A_k \in \mathcal{A}$ and $\rho(A_k) < \infty$ for each $k \in \mathbb{N}$. By applying the disjointization trick, we can assume that the sets A_k are disjoint. Then, since each A_k has finite measure, we have for every $E \in \Sigma$ that

$$\nu(E) = \sum_{k=1}^\infty \nu(E \cap A_k) = \sum_{k=1}^\infty \mu(E \cap A_k) = \mu(E). \qquad \diamondsuit$$

Problems

10.6.7. Show that if μ^* is the outer measure induced from a premeasure ρ, then every set $E \subseteq X$ that satisfies $\mu^*(E) = 0$ is μ^*-measurable. Conclude that $\mu = \mu^*|_\Sigma$ is a complete measure on (X, Σ).

10.6.8. Let ρ be a premeasure on an algebra $\mathcal{A}$ of subsets of X. Suppose that ρ is bounded ($\rho(X) < \infty$) and there is a set $A \subseteq X$ such that $\mu^*(A) = \rho(X)$. Show that $\mu^*(E) = \mu^*(E \cap A)$ for every μ^*-measurable set E.

10.6.9. Let $\mathcal{A}$ be an algebra on a set X. Let $\mathcal{A}_\sigma$ be the collection of countable unions of sets from $\mathcal{A}$, and let $\mathcal{A}_{\sigma\delta}$ be the collection of countable intersections of sets from $\mathcal{A}_\sigma$. Given a premeasure ρ on $\mathcal{A}$, prove the following statements.

(a) If $E \subseteq X$ and $\varepsilon > 0$, then there exists a set $A \in \mathcal{A}_\sigma$ such that $E \subseteq A$ and $\mu^*(A) \le \mu^*(E) + \varepsilon$, and there exists a set $H \in \mathcal{A}_{\sigma\delta}$ such that $E \subseteq H$ and $\mu^*(H) = \mu^*(E)$.

(b) If $\mu^*(E) < \infty$, then E is μ^*-measurable if and only if there exists a set $H \in \mathcal{A}_{\sigma\delta}$ such that $E \subseteq H$ and $\mu^*(H \setminus E) = 0$.

(c) If ρ is σ-finite, then an arbitrary set $E \subseteq X$ is μ^*-measurable if and only if there exists a set $H \in \mathcal{A}_{\sigma\delta}$ such that $E \subseteq H$ and $\mu^*(H \setminus E) = 0$.

10.6.10. Let μ^* be an outer measure on X induced from a finite premeasure ρ. Define the *inner measure* of $E \subseteq X$ to be $\mu_*(E) = \rho(X) - \mu^*(E^{\mathrm{C}})$. Prove that E is μ^*-measurable if and only if $\mu^*(E) = \mu_*(E)$.

10.7 Looking Ahead

Now that we have defined abstract measures, what comes next? One goal is to generalize the theory of integration to abstract measures. Some of this development closely parallels the theory for the Lebesgue integral derived in earlier chapters. In particular, the following topics are very similar for both Lebesgue measure and abstract measures.

- Measurability of functions with respect to an abstract measure.
- Properties of measurable functions.
- Convergence in measure.
- Egorov's Theorem for bounded measures.
- The integral of nonnegative, extended real-valued, and complex-valued functions with respect to a measure μ.
- Convergence theorems (MCT, Fatou's Lemma, DCT).
- L^p spaces with respect to μ.

However, some topics are not as simple to generalize, or benefit from new techniques or approaches. These include the following.

- Product measures $\mu \times \nu$ on $X \times Y$.
- Fubini's and Tonelli's Theorems with respect to a product measure.

There are also entirely new topics, such as the following.

- Signed measures (measures that take extended real values instead of just nonnegative values).
- The Jordan decomposition for signed measures.
- Complex measures.

- Differentiation of measures, including what it means for one measure to be *absolutely continuous* or *singular* with respect to another.
- The fundamental *Radon–Nikodym Theorem*, which gives a decomposition of one measure ν with respect to another measure μ in terms of absolutely continuous and singular parts.

Additionally there are more advanced topics, some of which require knowledge of operator theory and functional analysis. These include the following, among others.

- Borel and Radon measures on a locally compact Hausdorff space.
- The space of bounded Radon measures $M_b(X)$.
- The Riesz Representation Theorem, which establishes that the dual space of $C_0(X)$ is isomorphic to $M_b(X)$.

Many of these and other topics will be covered in a second volume for this text (currently in development).