ERRATA TO:

DENSITY, OVERCOMPLETENESS, AND LOCALIZATION OF FRAMES. II. GABOR SYSTEMS

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Note: This errata is not included in the published version of this paper.

Example 10 on page 338 should be replaced by the following (and note that the conclusion is that row decay holds while column decay fails, instead of the converse as claimed in the original version).

Example (Correction to Example 10)**.** Choose an orthonormal basis for H and index it as

$$\mathcal{E} \;=\; \operatorname{span}\{e_j^n\}_{n\in\mathbf{N},j=1,\dots,n}.$$

Set $H_n = \operatorname{span}\{e_j^n\}_{j=1,\dots,n}$. Define

$$f_i^n \;=\; \begin{cases} e_1^n, & i=1,\\ \frac{1}{2\sqrt{n}}e_1^n + e_i^n, & i=2,\dots,n, \end{cases}$$

and

$$\tilde{f}_i^n \;=\; \begin{cases} e_1^n - \frac{1}{2\sqrt{n}}\sum_{j=2}^n e_j^n, & i=1,\\ e_i^n, & i=2,\dots,n. \end{cases}$$

Clearly $f_i^n,\ \tilde{f}_i^n \in H_n$, and a straightforward calculation shows that $\{f_i^n\}_{i=1}^n$ and $\{\tilde{f}_i^n\}_{i=1}^n$ are biorthogonal sequences in H_n. Since H_n is n-dimensional, this shows that these are dual Riesz bases for H_n. Given any scalars $\{a_i\}_{i=1}^n$, we have

$$\begin{aligned} \Big\|\sum_{i=1}^n a_i f_i^n\Big\| \;&\le\; \Big\|\frac{\sum_{i=1}^n a_i}{2\sqrt{n}}\, e_1^n\Big\| + \Big\|\sum_{i=1}^n a_i e_i^n\Big\| \\ &\le\; \frac{\sum_{i=1}^n |a_i|}{2\sqrt{n}} + \Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2} \;\le\; \frac{3}{2}\Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2}, \end{aligned}$$

and similarly $\big\|\sum_{i=1}^n a_i f_i^n\big\| \;\ge\; \frac{1}{2}\big(\sum_{i=1}^n |a_i|^2\big)^{1/2}$. Thus $\{f_i^n\}_{i=1}^n$ has Riesz bounds $\frac{1}{2},\ \frac{3}{2}$. Since H is the orthogonal direct sum of the H_n and the Riesz bounds are independent of n, we conclude that $\mathcal{F} = \{f_i^n\}_{n\in\mathbf{N},i=1,\dots,n}$ and $\tilde{\mathcal{F}} = \{\tilde{f}_i^n\}_{n\in\mathbf{N},i=1,\dots,n}$ are dual Riesz bases for H.

Another direct calculation shows that

$$|\langle f_i^m, e_j^n\rangle| \;=\; \begin{cases} 1, & i=j, m=n,\\ \frac{1}{2\sqrt{n}}, & m=n, j=1, i=2,\dots,n,\\ 0, & \text{otherwise.}\end{cases}$$

Now let $\{f_k\}_{k\in\mathbf{N}}$ be the natural enumeration of $\{f_i^m\}_{i=1,\dots,m,\,m\in\mathbf{N}}$ and likewise let $\{e_\ell\}_{\ell\in\mathbf{N}}$ be the natural enumeration of $\{e_j^n\}_{j=1,\dots,n,\,n\in\mathbf{N}}$.

Choose any $\varepsilon>0$, and fix $N>\varepsilon/2$. Choose any k. Then $f_k = f_i^m$ for some i, m. Consider

$$S_N(k) \;=\; [k-\tfrac{N}{2}, k+\tfrac{N}{2}] \cap \mathbf{N}.$$

Suppose $\ell \in \mathbf{N}\backslash S_N(k)$, and write $e_\ell = e_j^n$. If $m\neq n$, then we have

$$\langle f_k, e_\ell\rangle \;=\; \langle f_i^m, e_j^n\rangle \;=\; 0.$$

If $m \le N/2$ then $m\neq n$ is the only possibility. On the other hand, if $m>N/2$ and $m=n$, then we must have $j\neq i$. Hence

$$\langle f_k, e_\ell\rangle \;=\; \langle f_i^m, e_j^n\rangle \;=\; \begin{cases} \frac{1}{2\sqrt{m}}, & j=1,\\ 0, & j\neq 1.\end{cases}$$

Consequently,

$$\sum_{\ell\in\mathbf{N}\backslash S_N(k)} |\langle f_k, e_\ell\rangle|^2 \;=\; \frac{1}{4m} \;<\; \frac{1}{2N} \;<\; \varepsilon.$$

After mapping the index set of $\mathcal{E}$ and $\mathcal{F}$ onto $\mathbf{Z}$, similarly to Example 6, we conclude that $(\mathcal{F}, a, \mathcal{E})$ does have ℓ^2-row decay.

On the other hand, suppose that N is any fixed number, and choose any $\ell\in\mathbf{N}$. Then $e_\ell = e_j^n$ for some j, n. Consider the case $j=1$. If n is large enough, then there will be at least $n-N$ choices of $k\in\mathbf{N}\backslash S_N(\ell)$ for which $f_k = f_i^m$ with $m=n$. For each of these we will have

$$\langle f_k, e_\ell\rangle \;=\; \langle f_i^m, e_j^n\rangle \;\ge\; \frac{1}{2\sqrt{n}}.$$

Hence

$$\sum_{k\in\mathbf{N}\backslash S_N(\ell)} |\langle f_k, e_\ell\rangle|^2 \;\ge\; \frac{n-N}{4n} \;=\; \frac{1}{4} - \frac{N}{4n}.$$

Again after mapping the index set of $\mathcal{E}$ and $\mathcal{F}$ onto $\mathbf{Z}$, similarly to Example 6, we conclude that $(\mathcal{F}, a, \mathcal{E})$ does not have ℓ^2-column decay.