The Homogeneous Approximation Property for Wavelet Frames

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Abstract

An irregular wavelet frame has the form $\mathcal{W}(\psi, \Lambda) = \{a^{-1/2}\psi(\frac{x}{a} - b)\}_{(a,b)\in\Lambda}$, where $\psi \in L^2(\mathbb{R})$ and Λ is an arbitrary sequence of points in the affine group $\mathbb{A} = \mathbb{R}^+ \times \mathbb{R}$. Such irregular wavelet frames are poorly understood, yet they arise naturally, e.g., from sampling theory or the inevitability of perturbations. This paper proves that irregular wavelet frames satisfy a Homogeneous Approximation Property, which essentially states that the rate of approximation of a wavelet frame expansion of a function f is invariant under time-scale shifts of f, even though Λ is not required to have any structure—it is only required that the wavelet ψ have a modest amount of time-scale concentration. It is shown that the Homogeneous Approximation Property has several implications on the geometry of Λ, and in particular a relationship between the affine Beurling density of the frame and the affine Beurling density of any other Riesz basis of wavelets is derived. This further yields necessary conditions for the existence of wavelet frames, and insight into the fundamental question of why there is no Nyquist density phenomenon for wavelet frames, as there is for Gabor frames that are generated from time-frequency shifts.

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1. Introduction

Frames have become an essential tool for many emerging applications, since they provide robust and stable—but usually nonunique—representations of vectors. For example,

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frames are used to provide stability in noisy environments and to mitigate the effect of losses in packet-based communication systems [3,4]. Wavelet frames are among the most important examples of frames, and frame properties of wavelet systems have been the focus of a number of recent studies [7,10,23,24]. Most of these results are concerned with classical wavelet systems of the form $\{a^{-j/2}\psi(a^{-j}x - bk) : j, k \in \mathbb{Z}\}$. However, due to questions arising from sampling theory and the inevitability of perturbations, the necessity of studying wavelet frames with arbitrary sets of time-scale indices has become clear. Such irregular wavelet frames are currently poorly understood.

In this paper we study special approximation properties of irregular wavelet frames

$$\mathcal{W}(\psi, \Lambda) \;=\; \{a^{-1/2}\psi(\tfrac{x}{a} - b)\}_{(a,b)\in\Lambda} \;=\; \{D_a T_b \psi\}_{(a,b)\in\Lambda},$$

where $\psi \in L^2(\mathbb{R})$ and Λ is a sequence of points in the set $\mathbb{R}^+ \times \mathbb{R}$, which is the affine group $\mathbb{A}$ when endowed with the natural multiplication corresponding to compositions of time-scale shift operators. We will focus on the approximation of L^2-functions by irregular wavelet expansions. If the approximation rate is invariant under time-scale shifts, we will say that our frame satisfies the *Homogeneous Approximation Property*, or *HAP* (discussed more below and defined precisely in Definition 4.1).

In the remainder of this introduction we will first outline three of our motivations for studying this approximation property for wavelets and then briefly outline our main results.

1.1. *Approximation properties*

Every frame yields frame expansions. Specifically, if $\mathcal{W}(\psi, \Lambda)$ is a frame then there is a dual frame $\{\tilde{\psi}_{a,b}\}_{(a,b)\in\Lambda}$ such that

$$f \;=\; \sum_{(a,b)\in\Lambda} \langle f, \tilde{\psi}_{a,b}\rangle \, D_a T_b \psi \;=\; \sum_{(a,b)\in\Lambda} \langle f, D_a T_b \psi\rangle \, \tilde{\psi}_{a,b}.$$

The very nature of the affine group leads us to ask whether there is invariance of these approximations under time-scale shifts $D_p T_q f(x) = p^{-1/2} f(\frac{a}{p} - q)$ of f. But since no structure is assumed on the index set Λ, there cannot be such invariance in any literal sense—the frame coefficients $\{\langle D_p T_q f, D_a T_b \psi\rangle\}_{(a,b)\in\Lambda}$ will not be directly related to the frame coefficients $\{\langle f, D_a T_b \psi\rangle\}_{(a,b)\in\Lambda}$. Yet we will show that the quality of approximation provided by the frame expansions of f and $D_p T_q f$ are indeed related. We will show that if we fix a nested sequence of "boxes" $\{Q_h\}_{h>0}$ which exhaust the space $\mathbb{R}^+ \times \mathbb{R}$, and consider the partial sums $\sum_{(a,b)\in\Lambda\cap Q_h} \langle f, D_a T_b \psi\rangle \, \tilde{\psi}_{a,b}$ of the frame expansion of f corresponding to these boxes, then simply shifting these boxes by the same amount yields a partial sum that approximates $D_p T_q f$ to within the same precision—even though there need be no relation between the locations or even the number of points in the original and shifted boxes. We will show that this *Homogeneous Approximation Property* holds true for all wavelet frames whose generator is chosen from a natural class that we call $\mathcal{B}_0$. This class is nearly the largest space of functions that can reasonably be used as wavelet frame generators, consisting of those admissible functions which have a moderate amount of time-scale concentration.

1.2. *Density, redundancy, and localization*

The recent paper [2] developed a powerful machinery relating the fundamental notions of density, redundancy, and localization of frames. However, irregular and even classical wavelet systems do not fit into the framework of that paper. Whether a similar relation can be derived in the wavelet setting is a question of basic importance—can the redundancy of a wavelet system be linked to its density and localization properties? A notion of density for irregular wavelet systems, called *affine density*, was introduced independently in [16] and [25], and some results linking the geometry of Λ with frame properties of the wavelet system were obtained in [19,20,25,26]. In this paper we make a first step towards linking affine density, redundancy, and localization of irregular wavelet frames, by showing that irregular wavelet frames do possess the Homogeneous Approximation Property, which is a weak version of the types of localization conditions introduced in [2]. The fact that the HAP holds provides a basic first step for a future development of the implications of redundancy and localization for wavelet systems.

1.3. *Nyquist density*

A third motivation for studying the Homogeneous Approximation Property for wavelets comes from the well-known question posed by Daubechies in [9, Sec. 4.1], namely, to explain why wavelet frames do not exhibit a Nyquist density phenomenon analogous to the one satisfied by Gabor frames. In short, for Gabor systems $\{e^{2\pi ibx}g(x-a)\}_{(a,b)\in\Lambda}$ there is a critical or Nyquist density for the set of indices Λ which separates frames from non-frames, and furthermore the Riesz bases sit exactly at this critical density (see [21,22,6]). But for wavelets there is no analogue of the Nyquist density, even given constraints on the norm or on the admissibility constant of the wavelet (see the example of Daubechies [8, Theorem 2.10] and the more extensive analysis of Balan in [1]). Some insight into this lack of a critical density was already revealed in [19], but this fundamental question is far from being satisfyingly answered. In this paper we approach this question by asking whether an analog of the key tool for proving the existence of a Nyquist density for Gabor systems is satisfied for wavelet systems, and, if so, how the implications of that tool differs between the two cases. That tool is the Homogeneous Approximation Property, and we show that wavelets do possess this property. The surprise then is that while both Gabor and wavelet frames possess the HAP, and while in the Gabor case the HAP implies the Nyquist phenomenon, in the wavelet case the implications of the HAP are quite different. Namely, we show that there is indeed a relationship between the density of any given wavelet frame compared to the density of any given wavelet Riesz basis, but this relationship depends on the rate of approximation in the HAP rather than being absolute as it is in the Gabor case. Further study of this relationship may reveal a more complete understanding of the essential differences between wavelet and Gabor systems.

1.4. *Main results*

In this paper we will show that, with a natural assumption on the time-scale concentration of the generator ψ, wavelet frames do satisfy the HAP. In particular, the approximation rates of a function f described in Section 1.1 are invariant under time-scale shifts. This gives hope that a relation between affine density, redundancy, and localization of irregular wavelet frames, as suggested in Section 1.2, might indeed be established. Further, as mentioned in Section 1.3, we prove that, as a consequence of the HAP, there exists a fundamental comparison between the affine densities of different wavelet frames. This leads to affine density conditions where the density is strongly tied to the generator of the frame, which is quite different than the Nyquist phenomenon exhibited by Gabor frames.

In addition to these main results, our techniques introduce some new tools for the study of wavelet systems. In particular, we show that any generator of a wavelet frame which satisfies our regularity assumption provides continuous wavelet transforms that lie in a particular Wiener amalgam space on the affine group.

All our results generalize with minor changes to the case of wavelet systems with multiple generators, or to wavelet systems in the spaces $\mathcal{H}^2_+(\mathbb{R})$ or $\mathcal{H}^2_-(\mathbb{R})$ consisting of functions in $L^2(\mathbb{R})$ whose Fourier transforms are supported in $[0,\infty)$ or $(-\infty,0]$, respectively.

Our paper is organized as follows. In Section 2 we present some background and notation, and review the definition of density on the affine group. In Section 3 we investigate the amalgam space properties of the continuous wavelet transform. Finally, in Section 4 we derive the HAP for wavelets and use it to obtain necessary conditions for the existence of wavelet frames.

2. Notation and Preliminary Results

2.1. *General notation*

Let $\mathbb{A} = \mathbb{R}^+ \times \mathbb{R}$ denote the *affine group*, endowed with the multiplication

$$(a,b)(x,y) \;=\; \bigl(ax, \tfrac{b}{x} + y\bigr).$$

The identity element of $\mathbb{A}$ is $e = (1,0)$, and inverses are given by $(a,b)^{-1} = \bigl(\frac{1}{a}, -ab\bigr)$.

The left-invariant Haar measure on $\mathbb{A}$ is $\mu = \frac{dx}{x}\,dy$. We denote the norm and inner product on $L^2(\mathbb{A})$ with respect to this Haar measure by $\|\cdot\|_{L^2(\mathbb{A})}$ and $\langle\cdot,\cdot\rangle_{L^2(\mathbb{A})}$, respectively, whereas the norm and inner product on $L^2(\mathbb{R})$ will be denoted by $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle$.

Let σ be the unitary representation of $\mathbb{A}$ on $L^2(\mathbb{R})$ defined by

$$\bigl(\sigma(a,b)f\bigr)(x) \;=\; D_a T_b f(x) \;=\; a^{-1/2} f(\tfrac{x}{a} - b),$$

where $D_a f(x) = a^{-1/2} f(\frac{x}{a})$ is the dilation operator and $T_b f(x) = f(x-b)$ is translation.

Given $\psi \in L^2(\mathbb{R})$ and a sequence Λ contained in $\mathbb{A}$, the *wavelet system* generated by ψ and Λ is

$$\mathcal{W}(\psi,\Lambda) \;=\; \bigl\{D_a T_b \psi\bigr\}_{(a,b)\in\Lambda} \;=\; \bigl\{\sigma(a,b)\psi\bigr\}_{(a,b)\in\Lambda} \;=\; \bigl\{a^{-1/2}\psi(\tfrac{x}{a} - b)\bigr\}_{(a,b)\in\Lambda}.$$

Although Λ will always denote a sequence of points in $\mathbb{A}$ and not merely a subset, for simplicity we will write $\Lambda \subseteq \mathbb{A}$.

The distance from $f \in L^2(\mathbb{R})$ to a closed subspace $V \subseteq L^2(\mathbb{R})$ is $\operatorname{dist}(f,V) = \inf\{\|f - v\| : v \in V\} = \|f - P_V f\|$, where P_V is the orthogonal projection of $L^2(\mathbb{R})$ onto V.

We normalize the Fourier transform on $L^1(\mathbb{R})$ as $\hat{f}(\xi) = \int f(x)\, e^{-2\pi i \xi x}\, dx$.

2.2. *The continuous wavelet transform*

Given $\psi \in L^2(\mathbb{R})$, called an *analyzing wavelet*, the *continuous wavelet transform* (CWT) of $f \in L^2(\mathbb{R})$ with respect to ψ is

$$W_\psi f(a,b) \;=\; \langle f,\, \sigma(a,b)\psi\rangle \;=\; \langle f,\, D_a T_b \psi\rangle \;=\; a^{-1/2} \int_{-\infty}^{\infty} f(x)\, \overline{\psi(\tfrac{x}{a} - b)}\, dx$$

for $(a,b) \in \mathbb{A}$. We have that $W_\psi f$ is a continuous function on $\mathbb{A}$.

We say that $\psi \in L^2(\mathbb{R})$ is *admissible* if

$$C_\psi \;=\; \int_{-\infty}^{\infty} |\hat{\psi}(\xi)|^2 \, \frac{d\xi}{|\xi|} \;<\; \infty,$$

and we set $C_\psi^+ = \int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|}$ and $C_\psi^- = \int_{-\infty}^0 |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|}$. We define

$$L^2_A(\mathbb{R}) \;=\; \{\psi \in L^2(\mathbb{R}) : \psi \text{ is admissible}\}.$$

Note that if $\psi \in L^1(\mathbb{R}) \cap L^2_A(\mathbb{R})$, then we must have $\hat{\psi}(0) = 0$, since $\hat{\psi}$ is continuous.

If ψ is admissible, then W_ψ maps $L^2(\mathbb{R})$ into $L^2(\mathbb{A})$, cf. [14, Theorem 10.1]. Precisely, we have that if $\psi \in L^2_A(\mathbb{R})$ and $f \in L^2(\mathbb{R})$, then

$$\|W_\psi f\|^2_{L^2(\mathbb{A})} \;=\; C_\psi^+ \int_0^\infty |\hat{f}(\xi)|^2 \, d\xi + C_\psi^- \int_{-\infty}^0 |\hat{f}(\xi)|^2 \, d\xi \;\le\; C_\psi \, \|f\|_2^2.$$

2.3. *Affine Beurling density*

Let $\{Q_h\}_{h>0}$ denote a fixed family of increasing, exhaustive neighborhoods of the identity in $\mathbb{A}$. For simplicity of computation, we will take

$$Q_h \;=\; [e^{-h}, e^h) \times [-h, h).$$

For $(x,y) \in \mathbb{A}$, let $(x,y)Q_h$ be the set Q_h left-translated by (x,y) via the group action, i.e.,

$$(x,y)Q_h \;=\; \bigl\{(xa, \tfrac{y}{a} + b) : a \in [e^{-h}, e^h),\, b \in [-h,h)\bigr\}.$$

Although Q_h is a rectangle, $(x,y)Q_h$ will not be rectangular in general, but we will still refer to it as a "box" and call (x,y) its "center." The Haar measure of the box $(x,y)Q_h$ is

$$\mu((x,y)Q_h) \;=\; \mu(Q_h) \;=\; \int_{-h}^{h} \int_{e^{-h}}^{e^h} \frac{dx}{x}\, dy \;=\; 4h^2.$$

The following definition was introduced in [16] (although a different set of neighborhoods Q_h was used there); a similar definition was introduced in [25]. A comparison among these and related definitions was made in [18].

Definition 2.1 *The* upper and lower affine Beurling densities *of a sequence* $\Lambda \subseteq \mathbb{A}$ *are, respectively,*

$$\mathcal{D}^+(\Lambda) \;=\; \limsup_{h \to \infty} \frac{\sup_{(x,y) \in \mathbb{A}} \#(\Lambda \cap (x,y)Q_h)}{\mu(Q_h)},$$

$$\mathcal{D}^-(\Lambda) \;=\; \liminf_{h \to \infty} \frac{\inf_{(x,y) \in \mathbb{A}} \#(\Lambda \cap (x,y)Q_h)}{\mu(Q_h)}.$$

If $\mathcal{D}^-(\Lambda) = \mathcal{D}^+(\Lambda)$ *then we say that* Λ *has* uniform affine Beurling density.

Some basic properties of affine and weighted affine Beurling densities were derived in [16]. Also, examples computing the density of classical affine systems, quasi-affine systems, co-affine systems, and oversampled affine systems are given there.

The following technical lemma will be needed later.

Lemma 2.2 *Let* $\delta > 0$ *be given. Given* $R' > 1$ *define*

$$R = R'e^{2\delta} + \delta e^{2\delta} + \delta e^{4\delta}.$$

Then for every $(p, q) \in \mathbb{A}$ *we have*

$$(p, q)Q_\delta \setminus Q_R \neq \emptyset \implies (p, q)Q_\delta \cap Q_{R'} = \emptyset.$$

PROOF. Suppose that $(p, q) \in \mathbb{A}$ and there exists $(a, b) \in (p, q)Q_\delta \setminus Q_R$. We must show that if $(c, d) \in Q_\delta$ then

$$(pc, \tfrac{q}{c} + d) = (p, q)(c, d) \notin Q_{R'}.$$

We proceed through cases, based on the facts that

$$(a, b) \notin Q_R = [e^{-R}, e^R) \times [-R, R), \tag{1}$$

$$(\tfrac{a}{p}, -\tfrac{pq}{a} + b) = (p, q)^{-1}(a, b) \in Q_\delta = [e^{-\delta}, e^\delta) \times [-\delta, \delta), \tag{2}$$

$$(c, d) \in Q_\delta = [e^{-\delta}, e^\delta) \times [-\delta, \delta). \tag{3}$$

Suppose that $a \geq e^R$. Then, using (1)–(3),

$$pc = \frac{p}{a}\, a\, c \geq e^{-\delta}\, e^R\, e^{-\delta} = e^{R-2\delta} \geq e^{R'},$$

the last inequality following from the fact that $R = R'e^{2\delta} + \delta e^{2\delta} + \delta e^{4\delta} \geq R' + \delta + \delta$. Similarly, if $a < e^{-R}$ then $pc < e^{-R'}$. In either case we conclude that $(p, q)(c, d) \notin Q_{R'}$.

Now consider the case $b \geq R$. We have

$$\frac{q}{c} + d = -\frac{a}{p}\,\frac{1}{c}\Big(-\frac{pq}{a} + b\Big) + b\,\frac{a}{p}\,\frac{1}{c} + d \geq -e^\delta e^\delta \delta + Re^{-\delta}e^{-\delta} - \delta = R'.$$

Similarly, if $b < -R$ then $\frac{q}{c} + d < -R'$. In either case we conclude that $(p, q)(c, d) \notin Q_{R'}$. □

2.4. *Frames*

We briefly recall the definition and basic properties of frames in Hilbert spaces. For more information we refer to [9,27,5]. A sequence $\{f_i\}_{i \in I}$ is a *frame* for a Hilbert space H if there exist constants A, $B > 0$ such that

$$\forall\, f \in H, \quad A\,\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i\rangle|^2 \leq B\,\|f\|^2. \tag{4}$$

The constants A and B are called *lower* and *upper frame bounds*, respectively. The *frame operator* $Sf = \sum_{i \in I} \langle f, f_i\rangle\, f_i$ is a bounded, positive, and invertible mapping of H onto itself. The *canonical dual frame* is $\{\tilde{f}_i\}_{i\in I}$ where $\tilde{f}_i = S^{-1}f_i$. For each $f \in H$ we have the *frame expansions*

$$f = \sum_{i\in I}\langle f, f_i\rangle\, \tilde{f}_i = \sum_{i\in I}\langle f, \tilde{f}_i\rangle\, f_i.$$

A sequence which satisfies the upper frame bound estimate in (4), but not necessarily the lower estimate, is called a *Bessel sequence* and B is a *Bessel bound*. In this case,

$$\Big\|\sum_{i\in I} c_i f_i\Big\|^2 \le B \sum_{i\in I} |c_i|^2 \qquad \text{for any } (c_i)_{i\in I} \in \ell^2(I). \tag{5}$$

In particular, $\|f_i\|^2 \le B$ for every $i \in I$.

A frame is a basis if and only if it is a Riesz basis, i.e., the image of an orthonormal basis for H under a continuous, invertible mapping. In this case the frame and its canonical dual frame are biorthogonal, i.e., $\langle f_i, \tilde{f}_j\rangle = \delta_{ij}$.

3. Amalgam Spaces and the CWT

3.1. *Amalgam spaces on the affine group*

An amalgam space combines a local criterion for membership with a global criterion. The first amalgam spaces were introduced by Wiener in his study of generalized harmonic analysis. A comprehensive general theory of amalgam spaces on locally compact groups was introduced and extensively studied by Feichtinger, e.g., [11–13]. For an expository introduction to Wiener amalgams on $\mathbb{R}$ with extensive references to the original literature, we refer to [15]. For our purposes, we will need the following particular amalgam spaces on the affine group.

Definition 3.1 *Given $1 \le p < \infty$, the amalgam space $W_{\mathbb{A}}(L^\infty, L^p)$ on the affine group consists of all functions $F\colon \mathbb{A} \to \mathbb{C}$ such that*

$$|||F|||_{W_{\mathbb{A}}(L^\infty,L^p)} = \left(\iint_{\mathbb{A}} \operatorname*{ess\,sup}_{(a,b)\in\mathbb{A}} |F(a,b)\,\Phi((x,y)^{-1}(a,b))|^p \,\frac{dx}{x}\,dy\right)^{1/p} < \infty,$$

where Φ is a fixed continuous function with compact support satisfying $0 \le \Phi(x,y) \le 1$ for all $(x,y) \in \mathbb{A}$ and $\Phi(x,y) = 1$ on some compact neighborhood of the identity. The amalgam space $W_{\mathbb{A}}(\mathcal{C}, L^p)$ is the closed subspace of $W_{\mathbb{A}}(L^\infty, L^p)$ consisting of the continuous functions in $W_{\mathbb{A}}(L^\infty, L^p)$.

$W_{\mathbb{A}}(L^\infty, L^p)$ is a Banach space, and its definition is independent of the choice of Φ, in the sense that each choice of Φ yields the same space under an equivalent norm. For proofs and more details, see [11,12].

Our next goal is to derive an equivalent discrete-type norm for $W_{\mathbb{A}}(L^\infty, L^p)$. First we need the following notation and lemma.

Given $h > 0$, for j, $k \in \mathbb{Z}$ define the following translates of Q_h and Q_{2h}:

$$B_{jk} = B_{jk}(h) = (e^{2jh}, 2khe^{-h})Q_h, \tag{6}$$

$$\tilde{B}_{jk} = \tilde{B}_{jk}(h) = (e^{2jh}, 2khe^{-h})Q_{2h}. \tag{7}$$

Lemma 3.2 *If $h > 0$, then*

(a) $\bigcup_{j,k\in\mathbb{Z}} B_{jk} = \mathbb{A}$, *and*

(b) *given* $m, n \in \mathbb{Z}$, *the box* $\tilde{B}_{mn}$ *can intersect at most* $N = 5(2e^{3h}+1)$ *boxes* $\tilde{B}_{jk}$ *with* $j, k \in \mathbb{Z}$.

PROOF. (a) Given $(x,y) \in \mathbb{A}$, there is a unique $j \in \mathbb{Z}$ and $a \in [e^{-h}, e^h)$ such that $x = e^{2jh}a$, i.e., $\ln x = 2jh + \ln a$. Since $\frac{2h}{ae^h} \le 2h$, there exists at least one integer $k \in \mathbb{Z}$ and number $b \in [-h, h)$ such that $y = \frac{2kh}{ae^h} + b$. Consequently,

$$(x,y) = (e^{2jh}a, \tfrac{2kh}{ae^h} + b) = (e^{2jh}, 2khe^{-h})(a,b) \in (e^{2jh}, 2khe^{-h})Q_h = B_{jk}.$$

(b) Let $(x,y) = (e^{2mh}, 2nhe^{-h})$. We must show that $\tilde{B}_{mn} = (x,y)Q_{2h}$ can intersect at most $N = 5(2e^{3h}+1)$ sets $\tilde{B}_{jk}$ with $j, k \in \mathbb{Z}$.

Suppose that $(u,v) \in (x,y)Q_{2h} \cap \tilde{B}_{jk}$. Then there exist points $(a,b) \in Q_{2h}$ and $(c,d) \in Q_{2h}$ such that both

$$(u,v) = (ax, \tfrac{y}{a} + b) = (x,y)(a,b) \in (x,y)Q_{2h}$$

and

$$(u,v) = (e^{2jh}c, \tfrac{2kh}{e^h c} + d) = (e^{2jh}, 2khe^{-h})(c,d) \in (e^{2jh}, 2khe^{-h})Q_{2h} = \tilde{B}_{jk}.$$

We have $ax = e^{2jh}c$, so

$$e^{-4h}x = \frac{e^{-2h}x}{e^{2h}} \le \frac{ax}{c} = e^{2jh} = \frac{ax}{c} \le \frac{e^{2h}x}{e^{-2h}} = e^{4h}x,$$

and hence $-4h + \ln x \le 2jh \le 4h + \ln x$. This inequality is satisfied for at most 5 values of j. Further, $\frac{y}{a} + b = \frac{2kh}{e^h c} + d$, so

$$k = \frac{ye^h c}{a2h} + \frac{be^h c}{2h} - \frac{de^h c}{2h} \le \frac{ye^h}{a2h}\,\frac{ax}{e^{2jh}} + \frac{he^h e^{2h}}{2h} + \frac{he^h e^{2h}}{2h} = \frac{xy}{2he^{(2j-1)h}} + e^{3h},$$

and similarly $k \ge \frac{xy}{2he^{(2j-1)h}} - e^{3h}$. For a given value of j, there are at most $2e^{3h}+1$ values of k for which these inequalities are satisfied. Hence, $(x,y)Q_{2h}$ can intersect at most $N = 5(2e^{3h}+1)$ sets $\tilde{B}_{jk}$. □

Proposition 3.3 *If* $1 \le p < \infty$ *and* $h > 0$, *then the following is an equivalent norm for* $W_{\mathbb{A}}(L^\infty, L^p)$:

$$\|F\|_{W_{\mathbb{A}}(L^\infty,L^p)} = \left(\sum_{j,k\in\mathbb{Z}} \|F \cdot \chi_{B_{jk}}\|_\infty^p\right)^{1/p}. \tag{8}$$

PROOF. Define $X = \{(e^{2jh}, 2khe^{-h})\}_{j,k\in\mathbb{Z}}$. In the language of [12], Lemma 3.2(a) says that X is Q_h-dense, and Lemma 3.2(b) says that X is relatively separated. In fact, if we set $N = 5(2e^{3h}+1)$, then

$$1 \le \sum_{m,n\in\mathbb{Z}} \chi_{B_{mn}} \le \sum_{m,n\in\mathbb{Z}} \chi_{\tilde{B}_{mn}} \le N. \tag{9}$$

Let $\phi_{jk} : \mathbb{A} \to \mathbb{R}$ be continuous functions such that

(i) $0 \le \phi_{jk} \le 1$,

(ii) $\mathrm{supp}(\phi_{jk}) \subseteq \tilde{B}_{jk}$,
(iii) $\phi_{jk}(x,y) = 1$ for $(x,y) \in B_{jk}$,

and define

$$\theta_{jk} = \frac{\phi_{jk}}{\sum_{m,n\in\mathbb{Z}} \phi_{mn}}.$$

Then $\{\theta_{jk}\}_{j,k\in\mathbb{Z}}$ is a bounded uniform partition of unity (BUPU) in the terminology of [12]. Therefore, [12, Prop. 3.7] implies that $\|F\| = \bigl(\sum_{j,k}\|F\cdot\theta_{jk}\|_\infty^p\bigr)^{1/p}$ defines an equivalent norm for $W_{\mathbb{A}}(L^\infty, L^p)$. Since $\chi_{B_{jk}} \le N\,\theta_{jk} \le N\,\chi_{\tilde{B}_{jk}}$, and since the sets B_{jk} cover $\mathbb{A}$ with no B_{mn} intersecting more than N sets $\tilde{B}_{jk}$, we have

$$\frac{1}{N^p}\sum_{j,k\in\mathbb{Z}}\bigl\|F\cdot\chi_{B_{jk}}\bigr\|_\infty^p \le \sum_{j,k\in\mathbb{Z}}\bigl\|F\cdot\theta_{jk}\bigr\|_\infty^p \le \sum_{j,k\in\mathbb{Z}}\bigl\|F\cdot\chi_{\tilde{B}_{jk}}\bigr\|_\infty^p \le N^p\sum_{j,k\in\mathbb{Z}}\bigl\|F\cdot\chi_{B_{jk}}\bigr\|_\infty^p.$$

Hence (8) defines an equivalent norm for $W_{\mathbb{A}}(L^\infty, L^p)$. □

Corollary 3.4 *If $1 \le p \le q < \infty$, then $W_{\mathbb{A}}(L^\infty, L^p) \subseteq W_{\mathbb{A}}(L^\infty, L^q)$.*

3.2. *A basic class of analyzing wavelets*

The basic class $\mathcal{B}_0$ of analyzing wavelets that our results will apply to is defined as follows.

Definition 3.5 *The space $\mathcal{B}_0$ consists of all functions ψ on $\mathbb{R}$ which satisfy:*

(a) $|\psi(x)| \le C(1+|x|)^{-\alpha}$ *for some $C > 0$ and $\alpha > 2$,*

(b) $\psi \in C^1(\mathbb{R})$, *i.e., ψ is differentiable, and ψ' is continuous and bounded, and*

(c) $\hat{\psi}(0) = 0$.

The most important property of the class $\mathcal{B}_0$ is that its elements possess some time-scale concentration. This concentration is naturally measured by the amalgam space properties of the continuous wavelet transform, as given in the following theorem. The proof of this result is given in Appendix A.

Theorem 3.6 (a) $\mathcal{B}_0 \subseteq L^2_A(\mathbb{R})$. *In particular, every element of $\mathcal{B}_0$ is admissible.*

(b) *If $f,\ \psi \in \mathcal{B}_0$, then $W_\psi f \in W_{\mathbb{A}}(\mathcal{C}, L^1)$.*

(c) *If $\psi \in \mathcal{B}_0$, then $W_\psi \psi \in W_{\mathbb{A}}(\mathcal{C}, L^1)$.*

While it is possible to construct wavelet frames for $L^2(\mathbb{R})$ using generators whose continuous wavelet transforms are not concentrated in time and scale, in practice such frames will have limited applicability. For example, in order that frame expansions converge in a range of function spaces rather than just L^2, or in order that the frame coefficients encode more properties of functions than just L^2-norm, requires analyzing wavelets with some regularity.

Note that if we let $\mathcal{S}_A = \{\psi \in \mathcal{S}(\mathbb{R}) : \hat{\psi}(0) = 0\}$ be the set of admissible Schwartz-class functions, then $\mathcal{S}_A \subseteq \mathcal{B}_0 \subseteq L^2_A(\mathbb{R})$. Thus the basic class $\mathcal{B}_0$ is dense in the set of admissible wavelets, which is itself dense in $L^2(\mathbb{R})$.

In Gabor analysis, the basic space of windows that can reasonably be used as generators of Gabor frames is the Feichtinger algebra S_0, which is also known as the modulation space M^1 (we refer to [14] for details on the modulation spaces). This space is defined by the time-frequency concentration of its elements, which corresponds to concentration of the Short-Time Fourier Transform (STFT) instead of the CWT. In particular, S_0 consists of those functions g whose $STFT$ $V_g g$ belongs to $W(\mathcal{C}, L^1)$, the amalgam space on the time-frequency plane $\mathbb{R}^2$. The natural analog of S_0 for wavelet analysis would be the space $\mathcal{B}$ consisting of all functions ψ such that $W_\psi \psi \in W_{\mathbb{A}}(\mathcal{C}, L^1)$. Our space $\mathcal{B}_0$ is slightly smaller, and we expect that our results should actually hold for all $\psi \in \mathcal{B}$, although we cannot yet prove this.

4. The HAP for Wavelet Frames

4.1. *The HAP*

In this section we define two versions of the HAP and prove that wavelet frames with generators from our basic class $\mathcal{B}_0$ satisfy the Strong version of the HAP. Recall that if $\mathcal{W}(\psi, \Lambda)$ is a frame for $L^2(\mathbb{R})$ then a canonical dual frame exists in $L^2(\mathbb{R})$, but that dual frame need not itself be a wavelet frame.

Definition 4.1 *Let $\psi \in L^2(\mathbb{R})$ and $\Lambda \subseteq \mathbb{A}$ be such that $\mathcal{W}(\psi, \Lambda) = \{\sigma(a,b)\psi\}_{(a,b)\in\Lambda}$ is a wavelet frame for $L^2(\mathbb{R})$, and let $\{\tilde{\psi}_{a,b}\}_{(a,b)\in\Lambda}$ denote its canonical dual frame.*

For each $h > 0$ and $(p,q) \in \mathbb{A}$, define a space

$$W(h,p,q) \;=\; \operatorname{span}\bigl\{\tilde{\psi}_{a,b} : (a,b) \in (p,q)Q_h \cap \Lambda\bigr\}. \tag{10}$$

By [16, Theorem 1.1], we have $\mathcal{D}^+(\Lambda) < \infty$, so there are only finitely many points of Λ in each box $(p,q)Q_h$, and hence $W(h,p,q)$ is finite-dimensional.

(a) *We say that $\mathcal{W}(\psi, \Lambda)$ possesses the* Weak Homogeneous Approximation Property (Weak HAP) *if for each $f \in L^2(\mathbb{R})$,*

$$\begin{aligned} &\forall\, \varepsilon > 0, \quad \exists\, R = R(f,\varepsilon) > 0 \quad \textit{such that} \\ &\forall\, (p,q) \in \mathbb{A}, \quad \operatorname{dist}\Bigl(\sigma(p,q)f,\ W(R,p,q)\Bigr) \;<\; \varepsilon. \end{aligned} \tag{11}$$

(b) *We say that $\mathcal{W}(\psi, \Lambda)$ possesses the* Strong Homogeneous Approximation Property (Strong HAP) *if given any $f \in L^2(\mathbb{R})$*

$$\begin{aligned} &\forall\, \varepsilon > 0, \quad \exists\, R = R(f,\varepsilon) > 0 \quad \textit{such that} \quad \forall\, (p,q) \in \mathbb{A}, \\ &\Bigl\| \sigma(p,q)f \;-\; \sum_{(a,b)\in(p,q)Q_R\cap\Lambda} \bigl\langle \sigma(p,q)f,\, \sigma(a,b)\psi \bigr\rangle\, \tilde{\psi}_{a,b} \Bigr\|_2 < \varepsilon. \end{aligned} \tag{12}$$

In either case we call $R(f,\varepsilon)$ an associated radius function.

Note that since the function $\sum_{(a,b)\in(p,q)Q_R\cap\Lambda} \bigl\langle \sigma(p,q)f,\, \sigma(a,b)\psi \bigr\rangle\, \tilde{\psi}_{a,b}$ is one element of the space $W(R,p,q)$, the Strong HAP implies the Weak HAP.

Theorem 4.2 *Let $\psi \in \mathcal{B}_0$ and $\Lambda \subseteq \mathbb{A}$ be such that $\mathcal{W}(\psi, \Lambda)$ is a frame for $L^2(\mathbb{R})$. Then $\mathcal{W}(\psi, \Lambda)$ satisfies the Strong HAP.*

PROOF. Let A, B be frame bounds for $\mathcal{W}(\psi,\Lambda)$. In this case $\frac{1}{B}$, $\frac{1}{A}$ are frame bounds for the canonical dual frame $\{\tilde{\psi}_{a,b}\}_{(a,b)\in\Lambda}$.

First we will show that the conditions of the Strong HAP, i.e., Eq. (12), are satisfied for functions in $\mathcal{B}_0$, and then extend by density to all of $L^2(\mathbb{R})$. Choose $g \in \mathcal{B}_0$ and fix $\varepsilon > 0$. Choose any $\delta > 0$. By [16, Theorem 1.1], we have $\mathcal{D}^+(\Lambda) < \infty$, so

$$M \;=\; \sup_{(x,y)\in\mathbb{A}} \#(\Lambda \cap (x,y)Q_\delta) \;<\; \infty.$$

Then for any $(p,q) \in \mathbb{A}$ we also have

$$\sup_{(x,y)\in\mathbb{A}} \#((p,q)^{-1}\Lambda \cap (x,y)Q_\delta) \;=\; \sup_{(x,y)\in\mathbb{A}} \#(\Lambda \cap ((p,q)(x,y))Q_\delta) \;=\; M \;<\; \infty.$$

Since g, $\psi \in \mathcal{B}_0$, it follows from Theorem 3.6 that $W_\psi g \in W_{\mathbb{A}}(\mathcal{C}, L^1) \subseteq W_{\mathbb{A}}(\mathcal{C}, L^2)$. By Lemma 3.2, the sets $B_{jk} = B_{jk}(\delta)$ defined by (6) cover $\mathbb{A}$, with no element of this family intersecting more than $N = 3(2e^{2\delta}+1)$ of the others. Considering the discrete-type norm for $W_{\mathbb{A}}(\mathcal{C}, L^2)$ given in (8), we conclude that if R' is large enough and we set

$$J \;=\; \bigl\{(j,k) \in \mathbb{Z}^2 : B_{jk} \cap Q_{R'} = \emptyset\bigr\}, \tag{13}$$

then

$$\sum_{(j,k)\in J} \bigl\|W_\psi g \cdot \chi_{B_{jk}}\bigr\|_\infty^2 < \frac{A\varepsilon^2}{M}.$$

Set $R = R(g,\varepsilon) = R'e^{2\delta} + \delta e^{2\delta} + \delta e^{4\delta}$.

Consider now any point $(p,q) \in \mathbb{A}$. The function $\sigma(p,q)g$ has the frame expansion

$$\sigma(p,q)g \;=\; \sum_{(a,b)\in\Lambda} \bigl\langle \sigma(p,q)g,\, \sigma(a,b)\psi \bigr\rangle\, \tilde{\psi}_{a,b}.$$

By applying Eq. (5) we have

$$\begin{aligned}
\Bigl\|\sigma(p,q)g \;-& \sum_{(a,b)\in(p,q)Q_R\cap\Lambda} \bigl\langle \sigma(p,q)g,\, \sigma(a,b)\psi \bigr\rangle\, \tilde{\psi}_{a,b}\Bigr\|_2^2 \\
&= \Bigl\| \sum_{(a,b)\in\Lambda\setminus(p,q)Q_R} \bigl\langle \sigma(p,q)g,\, \sigma(a,b)\psi \bigr\rangle\, \tilde{\psi}_{a,b}\Bigr\|_2^2 \\
&\le \frac{1}{A} \sum_{(a,b)\in\Lambda\setminus(p,q)Q_R} \bigl|\bigl\langle g,\, \sigma((p,q)^{-1}(a,b))\psi \bigr\rangle\bigr|^2 \\
&= \frac{1}{A} \sum_{(a,b)\in\Lambda\setminus(p,q)Q_R} \bigl|W_\psi g((p,q)^{-1}(a,b))\bigr|^2 \\
&= \frac{1}{A} \sum_{(c,d)\in(p,q)^{-1}\Lambda\setminus Q_R} \bigl|W_\psi g(c,d)\bigr|^2.
\end{aligned} \tag{14}$$

Now, each point $(c,d) \in (p,q)^{-1}\Lambda \setminus Q_R$ must lie in some set B_{jk}, and furthermore by Lemma 2.2 can only do so when $B_{jk} \cap Q_{R'} = \emptyset$, i.e., when $(j,k) \in J$. Moreover, each set B_{jk} can contain at most M elements of $(p,q)^{-1}\Lambda$. Hence we can continue (14) as follows:

$$\frac{1}{A} \sum_{(c,d)\in(p,q)^{-1}\Lambda\setminus Q_R} \bigl|W_\psi g(c,d)\bigr|^2 \;\le\; \frac{M}{A} \sum_{(j,k)\in J} \bigl\|W_\psi g \cdot \chi_{B_{jk}}\bigr\|_\infty^2 < \varepsilon^2.$$

Thus (12) is satisfied for the function g.

Now suppose that f is any function in $L^2(\mathbb{R})$, and choose any $\varepsilon > 0$. Since $\mathcal{B}_0$ is dense in $L^2(\mathbb{R})$, there exists $g \in \mathcal{B}_0$ such that

$$\|f-g\|_2 \;<\; \frac{\varepsilon A^{1/2}}{3B^{1/2}}.$$

Set $R(f,\varepsilon) = R(g,\frac{\varepsilon}{3})$, and denote this quantity by R. Then for any $(p,q) \in \mathbb{A}$, we have

$$\begin{aligned}
&\Big\|\sigma(p,q)f \;-\sum_{(a,b)\in(p,q)Q_R\cap\Lambda} \big\langle \sigma(p,q)f,\, \sigma(a,b)\psi\big\rangle\, \tilde{\psi}_{a,b}\Big\|_2 \\
&\qquad \le\; \big\|\sigma(p,q)f - \sigma(p,q)g\big\|_2 \\
&\qquad\qquad +\; \Big\|\sigma(p,q)g \;-\sum_{(a,b)\in(p,q)Q_R\cap\Lambda} \big\langle \sigma(p,q)g,\, \sigma(a,b)\psi\big\rangle\, \tilde{\psi}_{a,b}\Big\|_2 \\
&\qquad\qquad +\; \Big\|\sum_{(a,b)\in(p,q)Q_R\cap\Lambda} \big\langle \sigma(p,q)g - \sigma(p,q)f,\, \sigma(a,b)\psi\big\rangle\, \tilde{\psi}_{a,b}\Big\|_2 \\
&\qquad <\; \frac{\varepsilon A^{1/2}}{3B^{1/2}} \;+\; \frac{\varepsilon}{3} \;+\; \bigg(\frac{1}{A}\sum_{(a,b)\in(p,q)Q_R\cap\Lambda} \big|\big\langle \sigma(p,q)g - \sigma(p,q)f,\, \sigma(a,b)\psi\big\rangle\big|^2\bigg)^{1/2} \\
&\qquad \le\; \frac{\varepsilon}{3} \;+\; \frac{\varepsilon}{3} \;+\; \bigg(\frac{B}{A}\,\big\|\sigma(p,q)g - \sigma(p,q)f\big\|_2^2\bigg)^{1/2} \;<\; \varepsilon.
\end{aligned}$$

In the above calculation, the second inequality uses the fact that g satisfies the Strong HAP and that $\{\tilde{\psi}_{a,b}\}_{(a,b)\in\Lambda}$ has an upper frame bound of $\frac{1}{A}$. The third inequality follows from the fact that $\{\sigma(a,b)\psi\}_{(a,b)\in\Lambda}$ has an upper frame bound of B. Thus (12) is satisfied for the function f, so $\mathcal{W}(\psi,\Lambda)$ satisfies the Strong HAP. □

4.2. *The comparison theorem*

We saw in Theorem 4.2 that all wavelet frames $\mathcal{W}(\psi,\Lambda)$ with generators $\psi \in \mathcal{B}_0$ satisfy the Strong HAP. In this section we will show that all such wavelet frames must fulfill necessary density conditions with respect to any other wavelet Riesz bases. In fact, we will prove more generally that these density conditions apply to any frame that satisfies just the *Weak* HAP, even if the generator does not lie in $\mathcal{B}_0$.

Note that in the following result, the reference Riesz basis $\mathcal{W}(\phi,\Delta)$ is not required to satisfy the HAP, so any Riesz basis can be used, including the classical orthonormal wavelet bases. However, there is a very important difference between this result and the analogous Comparison Theorem for Gabor systems, namely that the density estimate depends on the value of the radius function associated to the frame $\mathcal{W}(\psi,\Lambda)$, whereas in the Gabor case it is independent of this value.

Theorem 4.3 (Comparison Theorem) *Assume that*

(a) $\psi \in L^2(\mathbb{R})$ *and* $\Lambda \subseteq \mathbb{A}$ *are such that* $\mathcal{W}(\psi,\Lambda)$ *is a frame for* $L^2(\mathbb{R})$ *that satisfies the* Weak HAP*, and*

(b) $\phi \in L^2(\mathbb{R})$ *and* $\Delta \subseteq \mathbb{A}$ *are such that* $\mathcal{W}(\phi,\Delta)$ *is a Riesz basis for* $L^2(\mathbb{R})$*.*

Let $\{\tilde{\phi}_{a,b}\}_{(a,b)\in\Delta}$ *denote the canonical dual frame of* $\mathcal{W}(\phi,\Delta)$*, and set*

$$C = \sup_{(p,q)\in\Delta} \|\tilde{\phi}_{p,q}\|_2. \tag{15}$$

Then for each $\varepsilon > 0$ *we have*

$$\frac{1-C\varepsilon}{e^{2R(\phi,\varepsilon)}}\,\mathcal{D}^-(\Delta) \,\le\, \mathcal{D}^-(\Lambda) \quad \textit{and} \quad \frac{1-C\varepsilon}{e^{2R(\phi,\varepsilon)}}\,\mathcal{D}^+(\Delta) \,\le\, \mathcal{D}^+(\Lambda).$$

PROOF. Note that the elements of any frame are uniformly bounded in norm, so the value C defined in (15) is indeed finite. Let $\{\tilde{\psi}_{a,b}\}_{(a,b)\in\Lambda}$ denote the canonical dual frame of $\mathcal{W}(\psi,\Lambda)$.

For each $h > 0$ and $(p,q) \in \mathbb{A}$, define

$$W(h,p,q) \,=\, \mathrm{span}\big\{\tilde{\psi}_{a,b} : (a,b) \in (p,q)Q_h \cap \Lambda\big\},$$
$$V(h,p,q) \,=\, \mathrm{span}\big\{\sigma(a,b)\phi : (a,b) \in (p,q)Q_h \cap \Delta\big\}.$$

These spaces are finite-dimensional.

Fix any $\varepsilon > 0$, and let $R = R(\phi,\varepsilon)$ be the value such that (11) holds for the function $f = \phi$. Let $(p,q) \in \mathbb{A}$, $h > 0$, and $(a,b) \in (p,q)Q_h$ be given. If $(x,y) \in (a,b)Q_R \cap \Lambda$, then since $Q_hQ_R \subseteq Q_{e^Rh+R}$, we have

$$(x,y) \,\in\, (a,b)Q_R \,\subseteq\, (p,q)Q_hQ_R \,\subseteq\, (p,q)Q_{e^Rh+R}.$$

Thus $(x,y) \in (p,q)Q_{e^Rh+R} \cap \Lambda$, which in turn implies

$$W(R,a,b) \subseteq W(e^Rh + R,p,q).$$

Combining this with the definition of the Weak HAP, we see that

$$\mathrm{dist}\big(\sigma(a,b)\phi,\ W(e^Rh+R,p,q)\big) \,\le\, \mathrm{dist}\big(\sigma(a,b)\phi,\ W(R,a,b)\big) \,<\, \varepsilon, \tag{16}$$

and this is valid for all $(p,q) \in \mathbb{A}$, $h > 1$, and $(a,b) \in (p,q)Q_h$.

Now let $h > 0$ and $(p,q) \in \mathbb{A}$ be fixed. Denote the orthogonal projections of $L^2(\mathbb{R})$ onto $V(h,p,q)$ and $W(e^Rh+R,p,q)$ by P_V and P_W, respectively. Additionally, define the map $T : V(h,p,q) \to V(h,p,q)$ by $T = P_VP_W$. Since the domain of T is $V(h,p,q)$, we have $T = P_VP_WP_V$, and hence T is self-adjoint.

By definition, $\big\{\sigma(a,b)\phi : (a,b) \in (p,q)Q_h \cap \Delta\big\}$ is a basis for $V(h,p,q)$. Although the elements $\tilde{\phi}_{a,b}$ corresponding to the same indices need not lie in $V(h,p,q)$, their orthogonal projections are in that space, and we have for (a,b) and (c,d) in $(p,q)Q_h \cap \Delta$ that

$$\big\langle \sigma(a,b)\phi,\, P_V(\tilde{\phi}_{c,d})\big\rangle \,=\, \big\langle P_V(\sigma(a,b)\phi),\, \tilde{\phi}_{c,d}\big\rangle \,=\, \big\langle \sigma(a,b)\phi,\, \tilde{\phi}_{c,d}\big\rangle \,=\, \delta_{a,c}\,\delta_{b,d}. \tag{17}$$

Since $V(h,p,q)$ is finite-dimensional, this implies that $\big\{P_V(\tilde{\phi}_{a,b}) : (a,b) \in (p,q)Q_h \cap \Delta\big\}$ is the dual basis to $\big\{\sigma(a,b)\phi : (a,b) \in (p,q)Q_h \cap \Delta\big\}$ in $V(h,p,q)$. Consequently, the trace of T is

$$\begin{aligned}
\mathrm{tr}(T) \,&=\, \sum_{(a,b)\in(p,q)Q_h\cap\Delta} \big\langle T(\sigma(a,b)\phi),\, P_V(\tilde{\phi}_{a,b})\big\rangle \\
&=\, \sum_{(a,b)\in(p,q)Q_h\cap\Delta} \big\langle P_VT(\sigma(a,b)\phi),\, \tilde{\phi}_{a,b}\big\rangle \\
&=\, \sum_{(a,b)\in(p,q)Q_h\cap\Delta} \big\langle T(\sigma(a,b)\phi),\, \tilde{\phi}_{a,b}\big\rangle.
\end{aligned} \tag{18}$$

Now, for $(a,b) \in (p,q)Q_h \cap \Delta$, we have

$$\begin{aligned}\big\langle T(\sigma(a,b)\phi),\, \tilde{\phi}_{a,b}\big\rangle &= \big\langle P_V P_W(\sigma(a,b)\phi),\, \tilde{\phi}_{a,b}\big\rangle \\ &= \big\langle P_W(\sigma(a,b)\phi),\, P_V(\tilde{\phi}_{a,b})\big\rangle \\ &= \big\langle \sigma(a,b)\phi,\, P_V(\tilde{\phi}_{a,b})\big\rangle + \big\langle (P_W - I)(\sigma(a,b)\phi),\, P_V(\tilde{\phi}_{a,b})\big\rangle. \qquad (19)\end{aligned}$$

By (17), the first term in (19) is

$$\big\langle P_V(\sigma(a,b)\phi),\, \tilde{\phi}_{a,b}\big\rangle \;=\; \big\langle \sigma(a,b)\phi,\, P_V(\tilde{\phi}_{a,b})\big\rangle \;=\; 1.$$

By the Cauchy–Schwarz inequality and Eqs. (15) and (16), the second term in (19) is bounded by

$$\big|\big\langle (P_W - I)(\sigma(a,b)\phi),\, P_V(\tilde{\phi}_{a,b})\big\rangle\big| \;\le\; \|(P_W - I)(\sigma(a,b)\phi)\|_2\, \|P_V(\tilde{\phi}_{a,b})\|_2 \;\le\; \varepsilon\, C.$$

This yields a lower bound for the trace of T:

$$\operatorname{tr}(T) \;\ge \sum_{(a,b)\in(p,q)Q_h\cap\Delta} (1 - C\varepsilon) \;=\; (1 - C\varepsilon)\, \#((p,q)Q_h \cap \Delta).$$

On the other hand, the operator norm of T satisfies $\|T\| \le \|P_V\|\,\|P_W\| \le 1$, so all eigenvalues of T must satisfy $|\lambda| \le \|T\| \le 1$. This in turn provides us with an upper bound for the trace of T, because the trace is the sum of the nonzero eigenvalues, so

$$\operatorname{tr}(T) \;\le\; \operatorname{rank}(T) \;\le\; \dim\big(W(e^R h + R, p, q)\big) \;\le\; \#((p,q)Q_{e^R h+R} \cap \Lambda).$$

Combining these two estimates, we see that for each $h > 1$ and all $(p,q) \in \mathbb{A}$, we have

$$(1 - C\varepsilon)\, \#((p,q)Q_h \cap \Delta) \;\le\; \#((p,q)Q_{e^R h+R} \cap \Lambda).$$

Therefore,

$$(1 - C\varepsilon)\, \frac{\#((p,q)Q_h \cap \Delta)}{4h^2} \;\le\; \frac{\#((p,q)Q_{e^R h+R} \cap \Lambda)}{4(e^R h + R)^2}\, \frac{4(e^R h + R)^2}{4h^2}.$$

Taking the infimum over all points $(p,q) \in \mathbb{A}$ and then the liminf as $h \to \infty$, or the supremum over all points $(p,q) \in \mathbb{A}$ and then the limsup as $h \to \infty$, therefore yields the estimates

$$(1 - C\varepsilon) D^-(\Delta) \;\le\; e^{2R}\, D^-(\Lambda) \quad \text{and} \quad (1 - C\varepsilon) D^+(\Delta) \;\le\; e^{2R}\, D^+(\Lambda).$$

Since we took $R = R(\phi, \varepsilon)$, this completes the proof. □

As a corollary, we obtain the following necessary density condition. This density condition was also obtained in [16, Theorem 1.1(b)], but with the extremely restrictive additional hypothesis that $\mathcal{D}^+(\Lambda^{-1}) < \infty$. Moreover, Theorem 4.3 provides more information, in terms of the value of the associated radius function, than merely the fact that $\mathcal{D}^-(\Lambda)$ must be positive. In particular, the following result applies to any frame generated by a wavelet $\psi \in \mathcal{B}_0$, since by Theorem 4.2, such a frame will satisfy the Strong (and hence the Weak) HAP.

Corollary 4.4 *Let $\psi \in L^2(\mathbb{R})$ and $\Lambda \subseteq \mathbb{A}$ be such that $\mathcal{W}(\psi, \Lambda)$ is a frame for $L^2(\mathbb{R})$ that satisfies the* Weak HAP. *Then $0 < \mathcal{D}^-(\Lambda) \le \mathcal{D}^+(\Lambda) < \infty$.*

PROOF. The fact that $\mathcal{D}^+(\Lambda) < \infty$ follows from [16, Theorem 1.1(a)]. To show the lower density is positive, let $\phi = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)} \in L^2(\mathbb{R})$ be the Haar wavelet and set $\Delta = \{(2^j, k)\}_{j,k\in\mathbb{Z}}$. Then $\mathcal{W}(\phi, \Delta)$ is the classical Haar orthonormal basis for $L^2(\mathbb{R})$. A proof similar to [16, Prop. 4.3] shows that $\mathcal{D}^-(\Delta) = \frac{1}{\ln 2}$. Therefore, Theorem 4.3 applied to the frame $\mathcal{W}(\psi, \Lambda)$ and the Haar basis $\mathcal{W}(\phi, \Delta)$ implies that for any $0 < \varepsilon < 1$ we have

$$\mathcal{D}^-(\Lambda) \geq \frac{1-\varepsilon}{e^{2R(\phi,\varepsilon)} \ln 2} > 0,$$

which completes the proof. □

Appendix A. Proof of Theorem 3.6

In this appendix we will prove Theorem 3.6.

First, we require the following two results concerning decay of the CWT. The first result is similar to [9, Theorem 2.9.1].

Theorem A.1 *Assume that*

(a) $\int_{-\infty}^{\infty} (1 + |x|)\, |\psi(x)|\, dx < \infty$, *and*

(b) $f \in C^1(\mathbb{R})$, *i.e., f is differentiable and f' is continuous and bounded,*

(c) $\hat{\psi}(0) = 0$.

Then there exists $C > 0$ such that $|W_\psi f(a, b)| \leq C\, a^{3/2}$ for all $(a, b) \in \mathbb{A}$.

PROOF. By the Mean-Value Theorem, we have $|f(x)-f(y)| \leq \|f'\|_\infty\, |x-y|$. Therefore,

$$\begin{aligned}
|W_\psi f(a, b)| &= \left| a^{-1/2} \int_{-\infty}^{\infty} f(x)\, \overline{\psi(\tfrac{x}{a} - b)}\, dx - a^{-1/2} f(ab) \int_{-\infty}^{\infty} \overline{\psi(\tfrac{x}{a} - b)}\, dx \right| \\
&\leq a^{-1/2} \int_{-\infty}^{\infty} |f(x) - f(ab)|\, |\psi(\tfrac{x}{a} - b)|\, dx \\
&\leq a^{-1/2}\, \|f'\|_\infty \int_{-\infty}^{\infty} |x - ab|\, |\psi(\tfrac{x-ab}{a})|\, dx \\
&= a^{-1/2}\, \|f'\|_\infty\, a^2 \int_{-\infty}^{\infty} |x\, \psi(x)|\, dx \;=\; C\, a^{3/2}. \qquad \square
\end{aligned}$$

Theorem A.2 *Assume that the functions ψ and f satisfy*

$$|\psi(x)| \leq C(1 + |x|)^{-\alpha} \qquad \text{and} \qquad |f(x)| \leq C(1 + |x|)^{-\alpha}$$

for some $C > 0$ and $\alpha > 1$. Then there exists $C' > 0$ such that

$$|W_\psi f(a, b)| \leq C' \frac{a^{1/2}}{1+a} \left(1 + \frac{a|b|}{1+a}\right)^{-\alpha}, \qquad (a, b) \in \mathbb{A}.$$

PROOF. The wavelet transform $\tilde{W}_\psi f$ used in [17] is related to the wavelet transform of this paper by the equality $\tilde{W}_\psi f(a, b) = a^{-1/2} W_\psi f(a, b/a)$. By [17, Theorem 11.0.2], we have

$$|\tilde{W}_\psi f(a, b)| \leq C' \frac{1}{1+a} \left(1 + \frac{|b|}{1+a}\right)^{-\alpha}.$$

A change of variables therefore completes the proof. □

We can now prove Theorem 3.6.

PROOF. [Proof of Theorem 3.6] Assume that ψ, f belong to $\mathcal{B}_0$. In particular, we have that

(a) there exists $C > 0$ and $\alpha > 2$ such that $|f(x)|, |\psi(x)| \leq C(1+|x|)^{-\alpha}$,
(b) $f, \psi \in C^1(\mathbb{R})$, and
(c) $\hat{f}(0) = \hat{\psi}(0) = 0$.

Since $\alpha > 2$, we have that $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Since we also have $\hat{\psi}(0) = 0$, this implies that ψ is admissible, cf. [9, p. 24].

Furthermore,

$$\int_{-\infty}^{\infty} (1+|x|)\,|\psi(x)|\,dx \;\leq\; C \int_{-\infty}^{\infty} (1+|x|)^{1-\alpha}\,dx \;<\; \infty, \tag{A.1}$$

so Theorem A.1 implies that there exists $C_1 > 0$ such that

$$|W_\psi f(a,b)| \;\leq\; C_1\, a^{3/2}, \qquad (a,b) \in \mathbb{A}. \tag{A.2}$$

Additionally, by Theorem A.2, there exists $C_2 > 0$ such that

$$|W_\psi f(a,b)| \;\leq\; C_2\, \frac{a^{1/2}}{1+a} \left(1 + \frac{a|b|}{1+a}\right)^{-\alpha}, \qquad (a,b) \in \mathbb{A}. \tag{A.3}$$

Now set $h = 1$, and let $B_{jk} = B_{jk}(1) = (e^{2j}, \frac{2k}{e})Q_1$ as in (6). Since $\alpha > 2$, we can find γ such that $\frac{2\alpha-1}{\alpha-1} < \gamma < 3$. Set $N_j = e^{-\gamma j+2}$.

Define

$$\begin{aligned}
S_1 \;&=\; \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} \|W_\psi f \cdot \chi_{B_{jk}}\|_\infty, \\
S_2 \;&=\; \sum_{j=-\infty}^{0} \sum_{|k|\leq N_j} \|W_\psi f \cdot \chi_{B_{jk}}\|_\infty, \\
S_3 \;&=\; \sum_{j=-\infty}^{0} \sum_{|k|> N_j} \|W_\psi f \cdot \chi_{B_{jk}}\|_\infty.
\end{aligned}$$

We will show that $S_1, S_2, S_3 < \infty$. This implies by Proposition 3.3 that $W_\psi f \in W_{\mathbb{A}}(L^\infty, L^1)$, and since we already know that $W_\psi f$ is continuous, the proof will be complete.

Before doing this, however, let us make a generic observation. If we take a point $(a,b) \in B_{jk}$ for some $j, k \in \mathbb{Z}$, then

$$(a,b) \;=\; (e^{2j}, \tfrac{2k}{e})(x,y) \;=\; (e^{2j}x, \tfrac{2k}{ex} + y)$$

for some $(x,y) \in Q_1 = [\frac{1}{e}, e) \times [-1, 1)$. Therefore

$$e^{2j-1} \;\leq\; a \;\leq\; e^{2j+1} \qquad \text{and} \qquad \frac{2|k|}{e^2} - 1 \;\leq\; |b| \;\leq\; 2|k| + 1.$$

Estimate S_1. Suppose that $(a,b) \in B_{jk}$ with $j > 0$, $k \in \mathbb{Z}$. Then $a \geq 1$, so

$$1 + \frac{a|b|}{1+a} \;\geq\; 1 + \frac{|b|}{2} \;\geq\; 1 + \frac{|k|}{e^2} - \frac{1}{2} \;=\; \frac{2|k| + e^2}{2e^2}.$$

Hence, we have from (A.3) that

$$|W_\psi f(a,b)| \;\leq\; C_2 \, \frac{e^{(2j+1)/2}}{1 + e^{2j-1}} \left(\frac{2e^2}{2|k| + e^2} \right)^{\alpha} \;\leq\; C_3 \, e^{-j} \, \frac{1}{(2|k| + e^2)^{\alpha}}.$$

Since $\alpha > 1$, we therefore have

$$S_1 \;=\; \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \|W_\psi f \cdot \chi_{B_{jk}}\|_\infty \;\leq\; C_3 \sum_{j=1}^{\infty} e^{-j} \sum_{k \in \mathbb{Z}} \frac{1}{(2|k| + e^2)^{\alpha}} \;<\; \infty.$$

Estimate S_2. Suppose that $(a,b) \in B_{jk}$ with $j \leq 0$, $|k| \leq N_j = e^{-\gamma j + 2}$. By (A.2), we have

$$|W_\psi f(a,b)| \;\leq\; C_1 \, a^{3/2} \;\leq\; C_4 \, e^{3j}.$$

Therefore, since $N_j \geq 1$ for all $j \leq 0$, we have

$$\begin{aligned} S_2 \;=\; \sum_{j=-\infty}^{0} \sum_{|k| \leq N_j} \|W_\psi f \cdot \chi_{B_{jk}}\|_\infty \;&\leq\; \sum_{j=-\infty}^{0} \sum_{|k| \leq N_j} C_4 \, e^{3j} \\ &\leq\; C_4 \sum_{j=-\infty}^{0} (2N_j + 1) e^{3j} \\ &\leq\; 3e^2 C_4 \sum_{j=-\infty}^{0} e^{-\gamma j + 3j} \;<\; \infty, \end{aligned}$$

the finiteness following from the fact that $\gamma < 3$.

Estimate S_3. If $(a,b) \in B_{jk}$ with $j \leq 0$, $|k| > N_j = e^{-\gamma j + 2}$, then, since $a \leq 1$,

$$1 + \frac{a|b|}{1+a} \;\geq\; \frac{a|b|}{2} \;\geq\; \frac{e^{2j-1}}{2} \left(\frac{2|k|}{e^2} - 1 \right) \;=\; e^{2j} \left(\frac{2|k| - e^2}{2e^3} \right).$$

Therefore, by (A.3) and the fact that $|k| > N_j \geq e^2$, we have

$$|W_\psi f(a,b)| \;\leq\; C_2 \, \frac{e^{(2j+1)/2}}{1+0} \left(e^{-2j} \frac{2e^3}{2|k| - e^2} \right)^{\alpha} \;=\; C_5 \, e^{j(1-2\alpha)} \, \frac{1}{(2|k| - e^2)^{\alpha}}.$$

Now, since $N_j \geq e^2$ we have for each $j \leq 0$ that

$$\begin{aligned} \sum_{|k| > N_j} \frac{1}{(2|k| - e^2)^{\alpha}} \;\leq\; 2 \int_{N_j}^{\infty} \frac{1}{(2x - e^2)^{\alpha}} \, dx \;&=\; \frac{1}{\alpha - 1} \, \frac{1}{(2N_j - e^2)^{\alpha - 1}} \\ &\leq\; \frac{1}{\alpha - 1} \, N_j^{1-\alpha} \\ &=\; \frac{e^{2(1-\alpha)}}{\alpha - 1} \, e^{\gamma j (\alpha - 1)}. \end{aligned}$$

Hence

$$S_3 = \sum_{j=-\infty}^{0} \sum_{|k|>N_j} \|W_\psi f \cdot \chi_{B_{jk}}\|_\infty \le \sum_{j=-\infty}^{0} \sum_{|k|>N_j} C_5\, e^{j(1-2\alpha)} \frac{1}{(2|k|-e^2)^\alpha}$$

$$\le C_6 \sum_{j=-\infty}^{0} e^{j(1-2\alpha)}\, e^{\gamma j(\alpha-1)}$$

$$= C_6 \sum_{j=-\infty}^{0} e^{j(1-2\alpha+\gamma(\alpha-1))}.$$

However,

$$1-2\alpha+\gamma(\alpha-1) > 1-2\alpha+\frac{2\alpha-1}{\alpha-1}(\alpha-1) = 0,$$

so we have $S_3 < \infty$. $\square$

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