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DENSITY OF WEIGHTED WAVELET FRAMES

BY *CHRISTOPHER HEIL AND GITTA KUTYNIOK*

ABSTRACT. *If $\psi \in L^2(\mathbf{R})$, Λ is a discrete subset of the affine group $\mathbf{A} = \mathbf{R}^+ \times \mathbf{R}$, and $w\colon \Lambda \to \mathbf{R}^+$ is a weight function, then the weighted wavelet system generated by ψ, Λ, and w is $\mathcal{W}(\psi, \Lambda, w) = \{w(a,b)^{1/2}\, a^{-1/2}\psi\big(\frac{x}{a} - b\big) : (a,b) \in \Lambda\}$. In this article we define lower and upper weighted densities $\mathcal{D}_w^-(\Lambda)$ and $\mathcal{D}_w^+(\Lambda)$ of Λ with respect to the geometry of the affine group, and prove that there exist necessary conditions on a weighted wavelet system in order that it possesses frame bounds. Specifically, we prove that if $\mathcal{W}(\psi, \Lambda, w)$ possesses an upper frame bound, then the upper weighted density is finite. Further, for the unweighted case $w = 1$, we prove that if $\mathcal{W}(\psi, \Lambda, 1)$ possesses a lower frame bound and $\mathcal{D}_w^+(\Lambda^{-1}) < \infty$, then the lower density is strictly positive. We apply these results to oversampled affine systems (which include the classical affine and the quasi-affine systems as special cases), to co-affine wavelet systems, and to systems consisting only of dilations, obtaining some new results relating density to the frame properties of these systems.*

1. **Introduction**

A common belief is that it is not possible to define a notion of density for the affine group that is analogous to the phase-space density that is defined for the Heisenberg group. While it is true that the affine group does not display a Nyquist-type phenomenon with respect to density of wavelet frames, we show in this article that a useful definition of density for the affine group does exist and that necessary conditions for the existence of wavelet frames can be formulated in terms of density conditions.

Let $\mathbf{A} = \mathbf{R}^+ \times \mathbf{R}$ denote the affine group, endowed with the multiplication

$$(a,b) \cdot (x,y) \;=\; (ax, \tfrac{b}{x} + y).$$

Let σ be the unitary representation of $\mathbf{A}$ on $L^2(\mathbf{R})$ defined by

$$\big(\sigma(a,b)f\big)(x) \;=\; a^{-1/2} f(\tfrac{x}{a} - b).$$

Given a function $\psi \in L^2(\mathbf{R})$, a subset $\Lambda \subseteq \mathbf{A}$, and a weight function $w\colon \Lambda \to \mathbf{R}$, we define the weighted wavelet system generated by ψ, Λ, and w to be the weighted collection of time-scale

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shifts of ψ given by

$$\begin{aligned}\mathcal{W}(\psi,\Lambda,w) &= \big\{w(a,b)^{1/2}\,\sigma(a,b)\psi : (a,b)\in\Lambda\big\}\\ &= \big\{w(a,b)^{1/2}\,a^{-1/2}\psi(\tfrac{x}{a}-b) : (a,b)\in\Lambda\big\}.\end{aligned} \tag{1.1}$$

When $w=1$ in this or other definitions, we will simply omit writing it. This definition of weighted wavelet systems includes as special cases the usual affine wavelet systems, the quasi-affine wavelet systems, and the co-affine wavelet systems (defined below). In particular, it is important to allow the case of nonconstant weights in (1.1) in order to obtain the quasi-affine systems. We say that $\mathcal{W}(\psi,\Lambda,w)$ is a *frame* for $L^2(\mathbf{R})$ if there exist A, $B>0$ (the *frame bounds*) such that

$$\forall\, f\in L^2(\mathbf{R}),\quad A\,\|f\|_2^2 \;\le\; \sum_{(a,b)\in\Lambda} \big|\big\langle f,\, w(a,b)^{1/2}\,\sigma(a,b)\psi\big\rangle\big|^2 \;\le\; B\,\|f\|_2^2. \tag{1.2}$$

We may also consider the upper and lower inequalities in (1.2) independently of each other.

In order to put our results into perspective, let us review the analogous density results that exist for the case of Gabor frames and the Heisenberg group, restricting our discussion to the one-dimensional case for simplicity. We refer to [6] for additional background and references.

Given a function $g\in L^2(\mathbf{R})$ and a subset $\Lambda\subseteq\mathbf{R}^2$, the Gabor system determined by g and Λ is the collection of time-frequency shifts of g defined by

$$\mathcal{G}(g,\Lambda) \;=\; \{e^{2\pi ibx}g(x-a) : (a,b)\in\Lambda\}.$$

Classical results are mostly concerned with the case of rectangular lattices of the form $\Lambda = a\mathbf{Z}\times b\mathbf{Z}$. Rieffel proved in [19], as a corollary of deep results on von Neumann algebras, that if $\mathcal{G}(g,a\mathbf{Z}\times b\mathbf{Z})$ is a complete subset of $L^2(\mathbf{R})$ then necessarily $ab\le 1$. After a number of developments, including [8], [17], [16], Ramanathan and Steger proved in [18] that all Gabor frames $\mathcal{G}(g,\Lambda)$, without restrictions on g or Λ, satisfy a certain *Homogeneous Approximation Property* (HAP), and deduced from this that if $\mathcal{G}(g,\Lambda)$ is a frame then the lower Beurling density of Λ satisfies $D^-(\Lambda)\ge 1$ (note that $D^-(a\mathbf{Z}\times b\mathbf{Z}) = 1/ab$), and that if this frame is a Riesz basis then $D^-(\Lambda) = D^+(\Lambda) = 1$. For the special case $\Lambda = a\mathbf{Z}\times b\mathbf{Z}$, they were also able to recover by this technique the completeness result of Rieffel. Some corrections and extensions were obtained in [6], compare also [12]. It was shown in [4] that the Rieffel result does not extend to non-lattices: there exist complete (but non-frame) Gabor systems with upper Beurling density ε. Recently, it has been shown in [2] that there is a fundamental connection between density properties and the so-called excess of a Gabor frame, and this connection is a manifestation of deeper implications of the HAP and related properties of localized frames [3].

In brief, in terms of necessary conditions for Gabor frames there is a critical or Nyquist density for Λ separating frames from non-frames, and furthermore the Riesz bases sit exactly at this critical density. It is natural to ask whether wavelet systems share similar properties, and the immediate answer is that there is clearly no exact analogue of the Nyquist density for wavelets. In particular, consider the case of the classical affine systems with dilation factor $a>1$ and translation parameter $b>0$, i.e.,

$$\mathcal{W}(\psi,\Lambda) \;=\; \{a^{-j/2}\psi(a^{-j}x-bk) : j,k\in\mathbf{Z}\},\qquad \Lambda = \{(a^j,bk) : j,k\in\mathbf{Z}\}. \tag{1.3}$$

It can be shown that for *each* $a>1$ and $b>0$ there exists a wavelet ψ such that $\mathcal{W}(\psi,\Lambda)$ is a frame or even an orthonormal basis for $L^2(\mathbf{R})$. In fact, the wavelet set construction of Dai, Larson, and Speegle [7] shows that this is true even in higher dimensions: wavelet orthonormal bases in the classical affine form exist for any expansive dilation matrix. For additional demonstrations of the impossibility of a Nyquist density, even given constraints on the norm or on the admissibility condition of the wavelet, see the example of Daubechies in [8, Theorem 2.10] and the more extensive analysis of Balan in [1]

However, the more general question remains: for what sets Λ and what weights w is it possible to construct wavelet frames $\mathcal{W}(\psi, \Lambda, w)$? Two important examples of wavelet systems other than classical affine systems are the quasi-affine and co-affine systems. Quasi-affine systems, introduced by Ron and Shen [20], are obtained by replacing the set Λ used in the definition of the affine system (1.3) by

$$\Lambda \,=\, \{(a^j, bk) : j < 0, k \in \mathbf{Z}\} \,\cup\, \{(a^j, a^{-j}bk) : j \geq 0, k \in \mathbf{Z}\}, \tag{1.4}$$

and using the weight

$$\begin{aligned} w(a^j, bk) &= 1, & j > 0,\ k \in \mathbf{Z}, \\ w(a^j, a^{-j}bk) &= a^{-j}, & j \geq 0,\ k \in \mathbf{Z}. \end{aligned} \tag{1.5}$$

In other words, "extra" elements are added to an affine system, and additionally the norms of the extra elements are adjusted. Ron and Shen proved that if a is integer and $b = 1$ then an affine system is a frame if and only if the quasi-affine system is a frame. The utility of the quasi-affine system is that it is integer translation-invariant, unlike the original affine system. More general quasi-affine systems with rational dilation factors were studied by Bownik in [5].

Co-affine wavelet system were studied recently by Gressman, Labate, Weiss, and Wilson [11]. If we write an affine system as $\{D_{a^j} T_k \psi\}_{j,k \in \mathbf{Z}}$, where D_{a^j} and T_k are the appropriate dilation and translation operators, then a co-affine system is $\{T_k D_{a^j} \psi\}_{j,k \in \mathbf{Z}}$. This amounts, in the terminology of this article, to taking

$$\Lambda \,=\, \{(a^j, a^{-j}bk) : j, k \in \mathbf{Z}\}, \tag{1.6}$$

and $w = 1$. It was shown in [11] that such a system can *never* form a frame for $L^2(\mathbf{R})$, and, moreover, this impossibility remains even allowing weights of the form $w(a^j, a^{-j}bk) = w(a^j)$.

In this article we will show that there is a natural density interpretation of the difference between affine/quasi-affine and co-affine systems. Moreover, we obtain this as a corollary of a more general result relating the density of arbitrary subsets Λ of the affine group to frame properties of $\mathcal{W}(\psi, \Lambda, w)$. We begin in Section 2 by defining upper and lower weighted Beurling densities $\mathcal{D}_w^+(\Lambda)$, $\mathcal{D}_w^-(\Lambda)$ of Λ that are appropriate to the geometry of the affine group, and deriving some basic properties related to these densities. Then, in Sections 3.2 and 3.4 we prove the following result, which places restrictions on when $\mathcal{W}(\psi, \Lambda, w)$ can form a frame for $L^2(\mathbf{R})$.

Theorem 1.1. Given a nonzero $\psi \in L^2(\mathbf{R})$, a subset Λ of $\mathbf{A}$, and a weight function $w\colon \Lambda \to \mathbf{R}^+$, the following statements hold.

(a) If $\mathcal{W}(\psi, \Lambda, w)$ possesses an upper frame bound for $L^2(\mathbf{R})$, then $\mathcal{D}_w^+(\Lambda) < \infty$.

(b) Let $w = 1$, and suppose that $\mathcal{D}^+(\Lambda^{-1}) < \infty$. If $\mathcal{W}(\psi, \Lambda)$ possesses a lower frame bound for $L^2(\mathbf{R})$, then $\mathcal{D}^-(\Lambda) > 0$.

Qualitatively, the statement $\mathcal{D}_w^-(\Lambda) > 0$ or $\mathcal{D}^-(\Lambda) > 0$ implies that, in a certain sense, there cannot be "gaps" in the distribution of points of Λ of arbitrarily large size, and the requirement $\mathcal{D}_w^+(\Lambda) < \infty$ or $\mathcal{D}^+(\Lambda) < \infty$ implies that there cannot be too much "crowding together" of points. In particular, we show that for the unweighted case, finite upper density is equivalent to being able to divide Λ into finitely many subsequences each of which is "separated" in a sense appropriate to the affine group.

In Section 4 we present several applications of Theorem 1.1. We consider first the "oversampled affine systems" introduced in [14], which include the affine systems, the quasi-affine systems of Ron and Shen, and the quasi-affine systems of Bownik as special cases. We show that each oversampled affine system has identical density, namely, $\mathcal{D}_w^+(\Lambda) = \mathcal{D}_w^-(\Lambda) = 1/(b \ln a)$. Next we show that the (unweighted) co-affine systems are extreme examples in regard to density: $\mathcal{D}^-(\Lambda) = 0$ while $\mathcal{D}^+(\Lambda) = \infty$. Further, $\mathcal{D}^+(\Lambda^{-1}) < \infty$, so a co-affine system can possess neither an upper nor a lower frame bound. Finally, we show that a system consisting only of dilations of a function can never form a frame for $L^2(\mathbf{R})$.

We conclude with some remarks on the hypotheses in Theorem 1.1. First, although we only obtain the result in Theorem 1.1(b) for the unweighted case, we believe that it should also be true for the weighted case. In particular, we are only able to consider the case of unweighted co-affine systems. Second, we do not know if the hypothesis $\mathcal{D}^+(\Lambda^{-1}) < \infty$ in Theorem 1.1 is necessary. Adding the assumption $\mathcal{D}^+(\Lambda) < \infty$ does not resolve this question, since, for example, if Λ is the set given in (1.3) corresponding to the affine system, then Λ^{-1} is the set corresponding to the co-affine system, so for this Λ we have $\mathcal{D}^+(\Lambda) < \infty$ yet $\mathcal{D}^+(\Lambda^{-1}) = \infty$. Further, it is not true that if both $\mathcal{D}^+(\Lambda) < \infty$ and $\mathcal{D}^+(\Lambda^{-1}) < \infty$ then necessarily $\mathcal{D}^-(\Lambda) > 0$; for example, consider the set $\Lambda = \{(a^j, bk) : j \in \mathbf{Z}, k \geq 0\}$.

A sequel article will examine the analogue of the HAP for wavelet systems, and its relation (or lack thereof) to density conditions in the wavelet case. Another interesting topic for future research is the case of higher dimensions, allowing dilation matrices in place of dilation factors.

Note added. Following submission of this article, we learned of some related references. It is well-known that density theorems for Gabor frames $\mathcal{G}(g, \Lambda)$ generated by Gaussian functions g are related to density questions in the Bargmann–Fock spaces, e.g., [21]. In [22], K. Seip introduced a notion of density for Bergman-type spaces on the unit disk, and it is possible to derive some density results for wavelet frames $\mathcal{W}(\psi, \Lambda)$ generated by certain wavelets ψ from those results. We have also learned that W. Sun and X. Zhou have simultaneously derived some results on the density of irregular wavelet frames that are related to ours. The article [23] is restricted to wavelet frames $\mathcal{W}(\psi, \Lambda)$ where Λ has the form $\Lambda = S \times T$. Additionally, the article [24] introduces a density notion for the affine group that is similar to ours, and derives a number of interesting related results. However, the results are distinct from ours, and in particular the weighted case is not considered there.

2. **Weighted affine Beurling density**

In $\mathbf{R}^n$, Beurling density is a measure of the "average" number of points of a set that lie inside a unit cube. We will define a Beurling density that is suited to the geometry of the affine group.

First we require some notation. For $h > 1$, we let Q_h denote a fixed family of neighborhoods of the identity element $e = (1, 0)$ in $\mathbf{A}$. For simplicity of computation, we will take

$$Q_h = [\tfrac{1}{h}, h) \times [-h, h).$$

For $(x, y) \in \mathbf{A}$, we let $Q_h(x, y)$ be the set Q_h left-translated via the group action so that it is "centered" at the point (x, y), i.e.,

$$Q_h(x,y) \;=\; (x,y)\cdot Q_h \;=\; \bigl\{(xa, \tfrac{y}{a}+b) : a \in [\tfrac{1}{h}, h),\, b \in [-h,h)\bigr\}.$$

Let $\mu = \frac{dx}{x}\,dy$ denote the left-invariant Haar measure on $\mathbf{A}$. Since μ is left-invariant, we have that

$$\mu(Q_h(x,y)) \;=\; \mu(Q_h) \;=\; \int_{-h}^{h}\int_{1/h}^{h} \frac{dx}{x}\,dy \;=\; 4h\ln h.$$

Next, given a subset Λ of $\mathbf{A}$ and a weight function $w\colon \Lambda \to \mathbf{R}^+$, we define the weighted number of elements of Λ lying in a subset K of $\mathbf{A}$ to be

$$\#_w(K) \;=\; \sum_{(a,b)\in K} w(a,b).$$

Then the *upper weighted affine Beurling density* of Λ is

$$\mathcal{D}_w^+(\Lambda) \;=\; \limsup_{h\to\infty} \frac{\sup_{(x,y)\in\mathbf{A}} \#_w(\Lambda\cap Q_h(x,y))}{\mu(Q_h)},$$

and the *lower weighted affine Beurling density* of Λ is

$$\mathcal{D}_w^-(\Lambda) \;=\; \liminf_{h\to\infty} \frac{\inf_{(x,y)\in\mathbf{A}} \#_w(\Lambda\cap Q_h(x,y))}{\mu(Q_h)}.$$

If $\mathcal{D}_w^-(\Lambda) = \mathcal{D}_w^+(\Lambda)$, then we say that Λ has *uniform weighted affine Beurling density* and denote this density by $\mathcal{D}_w(\Lambda)$. If $w = 1$, we omit writing it. Densities of sets associated with the affine, quasi-affine, and co-affine systems are computed in Section 4.

We will derive some equivalent ways to view the meaning of finite upper weighted density and positive lower weighted density. First, however, we will require the following technical lemma, which will be used throughout. Note that $Q_h(h^{2j},2k)\cdot(p,q) = (h^{2j},2k)\cdot Q_h\cdot(p,q)$.

Lemma 2.1. Let $h > 1$, $r \geq 1$, and $(p,q) \in \mathbf{A}$ be given.

(a) $\bigl\{Q_h(h^{2j},2k)\cdot(p,q) : j,k \in \mathbf{Z}\bigr\}$ covers $\mathbf{A}$.

(b) Any set $Q_h(x,y)$ intersects at most $N = 3(h^2p + h^2 + hp|q| + 1)$ sets of the form $Q_h(h^{2j},2k)\cdot(p,q)$.

(c) Any set $Q_{rh}(x,y)$ intersects at most $N = (\log_h r + 3)(rh^2 + h^2 + 1)$ sets of the form $Q_h(h^{2j},2k)$.

(d) If $Q_h(x,y)\cap Q_h(a,b) \neq \emptyset$, then $(x,y) \in Q_{2h^2}(a,b)$.

Proof. (a) Since $\mathbf{A}$ is invariant under right-shifts, it suffices for this part to consider the case $(p,q) = (1,0)$. Fix any $(x,y) \in \mathbf{A}$. Then $[\log_h x - 1, \log_h x + 1)$ contains a unique integer of the form $2j$, and there exists a unique $a \in [\frac{1}{h}, h)$ such that $\log_h x = 2j + \log_h a$. Further, since $\frac{2}{a} \leq 2h$, there exists at least one integer k and a number $b \in [-h,h)$ such that $y = \frac{2k}{a} + b$. Hence $(x,y) = (h^{2j}a, \frac{2k}{a} + b) = (h^{2j},2k)\cdot(a,b) \in Q_h(h^{2j},2k)$.

(b) Fix $(x,y) \in \mathbf{A}$, and suppose that $(u,v) \in Q_h(x,y)\cap Q_h(h^{2j},2k)\cdot(p,q)$. Then there exist points (a,b), $(c,d) \in Q_h$ such that

$$(u,v) \;=\; (x,y)\cdot(a,b) \;=\; (ax, \tfrac{y}{a}+b) \;\in\; Q_h(x,y)$$

and

$$(u,v) \;=\; (h^{2j},2k)\cdot(c,d)\cdot(p,q) \;=\; (h^{2j}cp, \tfrac{2k}{cp}+\tfrac{d}{p}+q) \;\in\; Q_h(h^{2j},2k)\cdot(p,q).$$

In particular, $\frac{ax}{cp}=h^{2j}$ with $\frac{1}{h}\le a,c<h$, so $\frac{x}{h^2p}\le h^{2j}\le \frac{h^2x}{p}$. Therefore

$$\frac{\log_h x}{2}-\frac{\log_h p}{2}-1 \;\le\; j \;\le\; \frac{\log_h x}{2}-\frac{\log_h p}{2}+1,$$

which is satisfied for at most 3 values of j. Further, $2k=\frac{xy}{h^{2j}}+bcp-cd-cpq$, so

$$\frac{xy}{2h^{2j}}-\frac{h^2p}{2}-\frac{h^2}{2}-\frac{hp|q|}{2} \;\le\; k \;\le\; \frac{xy}{2h^{2j}}+\frac{h^2p}{2}+\frac{h^2}{2}+\frac{hp|q|}{2}.$$

For a given value of j, this is satisfied for at most $h^2p+h^2+hp|q|+1$ values of k. Thus, $Q_h(x,y)$ can intersect at most $3(h^2p+h^2+hp|q|+1)$ sets of the form $Q_h(h^{2j},2k)$.

(c) The proof is similar to the proof of part (b).

(d) Suppose that $(c,d)\in Q_h(x,y)\cap Q_h(a,b)$. Then we would have $(c,d)=(a,b)\cdot(r,s)=(x,y)\cdot(t,u)$ for some (r,s), $(t,u)\in Q_h$. Therefore,

$$(a,b)^{-1}\cdot(x,y) \;=\; (r,s)\cdot(t,u)^{-1} \;=\; (\tfrac{r}{t}, st-tu) \;\in\; Q_{2h^2},$$

so $(x,y)\in Q_{2h^2}(a,b)$. □

Using this lemma, we can give a useful reinterpretation of finite upper density.

Proposition 2.2. If $\Lambda\subseteq \mathbf{A}$ and $w\colon \Lambda\to\mathbf{R}^+$, then the following conditions are equivalent.

(a) $\mathcal{D}^+_w(\Lambda)<\infty$.

(b) There exists $h>1$ such that $\sup_{(x,y)\in\mathbf{A}} \#_w(\Lambda\cap Q_h(x,y))<\infty$.

(c) For every $h>1$ we have $\sup_{(x,y)\in\mathbf{A}} \#_w(\Lambda\cap Q_h(x,y))<\infty$.

Proof. (a) ⇒ (b) and (c) ⇒ (b) are trivial.

(b) ⇒ (a), (c). Suppose there exists $h>1$ such that $R=\sup_{(x,y)\in\mathbf{A}} \#_w(\Lambda\cap Q_h(x,y))<\infty$.

For $1<t<h$, we have $Q_t(x,y)\subseteq Q_h(x,y)$, so $\sup_{(x,y)\in\mathbf{A}} \#_w(\Lambda\cap Q_t(x,y))<\infty$. On the other hand, if $t\ge h$ then we have $t=rh$ with $r\ge 1$. If we let $N_r=(\log_h r+3)(rh^2+h^2+1)$ be as given in Lemma 2.1(c), then each set $Q_{rh}(x,y)$ is covered by a union of at most N_r sets of the form $Q_h(h^{2j},2k)$. Consequently,

$$\sup_{(x,y)\in\mathbf{A}} \#_w(\Lambda\cap Q_{rh}(x,y)) \;\le\; N_r\cdot \sup_{j,k\in\mathbf{Z}} \#_w(\Lambda\cap Q_h(h^{2j},2k)) \;\le\; N_rR \;<\; \infty.$$

Thus statement (c) holds. Further,

$$\mathcal{D}^+_w(\Lambda) \;\le\; \limsup_{r\to\infty} \frac{N_rR}{4rh\ln rh} \;=\; \frac{Rh}{4\ln h} \;<\; \infty,$$

so statement (a) holds as well. □

A similar result holds for the case of positive lower weighted density.

Proposition 2.3. If $\Lambda \subseteq \mathbf{A}$ and $w\colon \Lambda \to \mathbf{R}^+$, then the following conditions are equivalent.

(a) $\mathcal{D}_w^-(\Lambda) > 0$.

(b) There exists some $h > 1$ such that $\inf_{(x,y)\in\mathbf{A}} \#_w(\Lambda \cap Q_h(x,y)) > 0$.

For the unweighted case, we will give a further interpretation of finite upper density in terms of the following definition.

Definition 2.4. We will say that a set $K \subseteq \mathbf{A}$ is *affinely h-separated* if

$$(a,b) \neq (c,d) \in K \implies Q_h(a,b) \cap Q_h(c,d) = \emptyset.$$

Proposition 2.5. If $\Lambda \subseteq \mathbf{A}$ and $w = 1$, then the following conditions are equivalent.

(a) $\mathcal{D}^+(\Lambda) < \infty$.

(b) There exists $h > 1$ such that Λ can be written as a finite union of subsets $\Lambda_1, \dots, \Lambda_N$, each of which is affinely h-separated.

(c) For every $h > 1$, Λ can be written as a finite union of subsets $\Lambda_1, \dots, \Lambda_N$, each of which is affinely h-separated.

Proof. (a) $\Rightarrow$ (c). Assume that $D^+(\Lambda) < \infty$, and let $h > 1$ be given. Then by Lemma 2.2, we have $M = \sup_{(x,y)\in\mathbf{A}} \#(\Lambda \cap Q_h(x,y)) < \infty$. Fix any $(a,b) \in \Lambda$. If $(c,d) \in \Lambda$ is such that $Q_h(a,b) \cap Q_h(c,d) \neq \emptyset$, then we have by Lemma 2.1(d) that $(c,d) \in Q_{2h^2}(a,b)$. Now, by Lemma 2.1(c), there exists an N, independent of (a,b), such that $Q_{2h^2}(a,b)$ is contained in a union of at most N sets of the form $Q_h(h^{2j}, 2k)$. However, each set $Q_h(h^{2j}, 2k)$ can contain at most M points of Λ. Hence $Q_{2h^2}(a,b)$ can contain at most MN points of Λ.

Thus, each $Q_h(a,b)$ with $(a,b) \in \Lambda$ can intersect at most MN sets $Q_h(c,d)$ with $(c,d) \in \Lambda$. By the disjointization principle of Feichtinger and Gröbner [10, Lemma 2.9], it follows that Λ can be divided into at most MN subsequences $\Lambda_1, \dots, \Lambda_{MN}$ such that for each fixed i, the sets $Q_h(a,b)$ with $(a,b) \in \Lambda_i$ are disjoint, or in other words, Λ_i is affinely h-separated.

(b) $\Rightarrow$ (a). Assume that $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_N$ with each Λ_i affinely h-separated. Fix δ so that $1 < 2\delta^2 < h$, and suppose that two points (a,b) and (c,d) of some Λ_i were both contained in some $Q_\delta(x,y)$. Then by Lemma 2.1(d), we would have $(x,y) \in Q_{2\delta^2}(a,b) \subseteq Q_h(a,b)$ and $(x,y) \in Q_{2\delta^2}(c,d) \subseteq Q_h(c,d)$. Hence $(a,b) = (c,d)$ since Λ_i is affinely h-separated. Thus, each $Q_\delta(x,y)$ contains at most one point of Λ_i, so $\sup_{(x,y)\in\mathbf{A}} \#(\Lambda \cap Q_\delta(x,y)) \leq N < \infty$. It therefore follows from Proposition 2.2 that $\mathcal{D}^+(\Lambda) < \infty$. □

3. Proof of Theorem 1.1

3.1. The Bergman transform.

Before beginning the proof of Theorem 1.1, we recall some basic facts. Define a function η by

$$\hat{\eta}(\xi) = \begin{cases} 2\xi\, e^{-\xi}, & \xi \geq 0, \\ 0, & \xi < 0. \end{cases}$$

We have $\|\eta\|_2 = 1$ and $\int \frac{|\hat{\eta}(\xi)|^2}{|\xi|}\, d\xi = 1 < \infty$. Therefore η is an admissible function, and consequently the continuous wavelet transform W_η defined by

$$W_\eta f(a,b) = \langle f, \sigma(a,b)\eta \rangle, \qquad (a,b) \in \mathbf{A},$$

maps $L^2(\mathbf{R})$ into $L^2(\mathbf{A}, d\mu)$. Furthermore, $W_\eta f$ is a continuous function for each $f \in L^2(\mathbf{R})$, and the roles of f and η can be interchanged by using the relation $W_f\eta(a,b) = \overline{W_\eta f((a,b)^{-1})}$.

In fact, we have chosen this particular analyzing wavelet η because $W_\eta f$ possesses a stronger property than continuity. Let $\mathbf{C}^+ = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$ denote the complex upper half-plane. The *Bergman transform* of $f \in L^2(\mathbf{R})$ is the function Gf defined on $\mathbf{C}^+$ by

$$Gf(b+ai) \;=\; \frac{1}{2\pi a^{3/2}} \left\langle f(x),\, a^{-1/2}\eta(\tfrac{x-b}{a}) \right\rangle \;=\; \frac{1}{2\pi a^{3/2}} \left\langle f,\, \sigma(a, \tfrac{b}{a})\eta \right\rangle \;=\; \frac{1}{2\pi a^{3/2}}\, W_\eta f(a, \tfrac{b}{a}).$$

For each $f \in L^2(\mathbf{R})$, Gf is an analytic function on $\mathbf{C}^+$, cf. [9, Section 2.5], [13, p. 308]. In particular, if we identify $\mathbf{A}$ with $\mathbf{C}^+$ in the obvious way, then since $\overline{Q}_h$ is a compact neighborhood of $i = 0 + 1i$ in $\mathbf{C}^+$, we have by [15, Theorem 2.2.3] that for each $h > 1$ there exists a constant K_h, independent of $f \in L^2(\mathbf{R})$, such that

$$|Gf(i)| \;=\; |Gf(0+1i)| \;\le\; K_h \iint_{Q_h} |Gf(z)|\, dz.$$

3.2. **Proof of Theorem 1.1 (a).**

We will now prove part (a) of Theorem 1.1. Assume that $\psi \in L^2(\mathbf{R}) \setminus \{0\}$, $\Lambda \subseteq \mathbf{A}$, and $w \colon \Lambda \to \mathbf{R}^+$ are given such that $\mathcal{D}_w^+(\Lambda) = \infty$. We will show that $\mathcal{W}(\psi, \Lambda, w)$ does not possess an upper frame bound.

We cannot have $\hat{\psi}$ identically zero on both $(-\infty, 0]$ and $[0, \infty)$, and we can assume without loss of generality that $\mathrm{supp}(\hat{\psi}) \cap [0, \infty) \neq \emptyset$. In this case the Bergman transform $G\psi$ of ψ is an analytic function on $\mathbf{C}^+$ that is not identically zero. Now, if $W_\psi\eta$ had a zero in every set $Q_h(c,d)$ for each $(c,d) \in \mathbf{A}$ and every $h > 1$, then, using the relation

$$x^{3/2} W_\psi\eta(x,y) \;=\; 2\pi\, \overline{G\psi(-y + \tfrac{1}{x} i)},$$

it is easy to see that the zeros of $G\psi$ would have a finite accumulation point. Since $G\psi$ is analytic, this implies $G\psi = 0$, which is a contradiction. Hence, there must be some $(c,d) \in \mathbf{A}$ and some $h > 1$ such that $W_\psi\eta$ does not vanish on $Q_h(c,d)$, and consequently,

$$\delta \;=\; \inf_{(x,y) \in Q_h(c,d)} |W_\psi\eta(x,y)| \;>\; 0.$$

Now choose any $N > 0$. Since $\mathcal{D}_w^+(\Lambda) = \infty$, it follows from Proposition 2.2 that

$$\sup_{(x,y)\in\mathbf{A}} \#_w(\Lambda \cap Q_h(x,y)) \;=\; \infty,$$

so there must exist a point $(p,q) \in \mathbf{A}$ such that $\#_w(\Lambda \cap Q_h(p,q)) \ge N$. Define

$$g \;=\; \sigma((p,q)\cdot(c,d)^{-1})\eta$$

and note that $\|g\|_2 = \|\eta\|_2 = 1$. Now,

$$(a,b) \in Q_h(p,q) = (p,q)\cdot Q_h \implies (c,d)\cdot(p,q)^{-1}\cdot(a,b) \in (c,d)\cdot Q_h = Q_h(c,d),$$

so we can compute that

$$
\begin{aligned}
&\sum_{(a,b)\in\Lambda} \left|\left\langle g,\, w(a,b)^{1/2}\,\sigma(a,b)\psi\right\rangle\right|^2 \\
&\geq \sum_{(a,b)\in\Lambda\cap Q_h(p,q)} \left|\left\langle \sigma((p,q)\cdot(c,d)^{-1})\eta,\, w(a,b)^{1/2}\sigma(a,b)\psi\right\rangle\right|^2 \\
&= \sum_{(a,b)\in\Lambda\cap Q_h(p,q)} |w(a,b)|\left|\left\langle \eta,\, \sigma((c,d)\cdot(p,q)^{-1}\cdot(a,b))\psi\right\rangle\right|^2 \\
&= \sum_{(a,b)\in\Lambda\cap Q_h(p,q)} |w(a,b)|\,|W_\psi\eta((c,d)\cdot(p,q)^{-1}\cdot(a,b))|^2 \\
&\geq \delta^2\,\#_w(\Lambda\cap Q_h(p,q)) \;\geq\; N\delta^2.
\end{aligned}
$$

Since N is arbitrary and $\|g\|_2 = 1$, we conclude that $\mathcal{W}(\psi,\Lambda,w)$ cannot possess an upper frame bound.

3.3. **Lemmas.**

In order to prove Theorem 1.1(b), we will require the following technical lemmas.

Lemma 3.1. If $\Lambda \subseteq \mathbf{A}$ satisfies $\mathcal{D}^+(\Lambda) < \infty$, then $\mathcal{D}^+(\Lambda\cdot(p,q)) < \infty$ for each $(p,q)\in\mathbf{A}$.

Proof. Since $\mathcal{D}^+(\Lambda) < \infty$, we have by Proposition 2.2 that

$$M \;=\; \sup_{(x,y)\in\mathbf{A}} \#(\Lambda\cap Q_2(x,y)) \;<\; \infty.$$

Fix any $(p,q)\in\mathbf{A}$. By Lemma 2.1, we have that $\{Q_2(2^{2j},2k)\cdot(p,q) : j,k\in\mathbf{Z}\}$ covers $\mathbf{A}$, and there exists an integer N independent of (x,y) such that each $Q_2(x,y)$ intersects at most N of the sets $Q_2(2^{2j},2k)\cdot(p,q)$. Therefore,

$$
\begin{aligned}
\#(\Lambda\cdot(p,q)\,\cap\,Q_2(x,y)) &\leq N \sup_{j,k\in\mathbf{Z}} \#(\Lambda\cdot(p,q)\,\cap\,Q_2(2^{2j},2k)\cdot(p,q)) \\
&\leq N \sup_{j,k\in\mathbf{Z}} \#(\Lambda\,\cap\,Q_2(2^{2j},2k)) \\
&\leq MN,
\end{aligned}
$$

and therefore $\mathcal{D}^+(\Lambda\cdot(p,q)) < \infty$ by Proposition 2.2. □

Lemma 3.2. Let δ, $R > 1$ be given. If $T > R(R+\delta)$, then we have for every $(p,q)\in\mathbf{A}$ that

$$(a,b)\notin Q_T(p,q) \;\Longrightarrow\; Q_R\,\cap\,Q_\delta((a,b)^{-1}\cdot(p,q)) = \emptyset.$$

Proof. Suppose that there exists a point $(x,y)\in Q_R\,\cap\,Q_\delta((a,b)^{-1}\cdot(p,q))$. Then $(x,y) = (a,b)^{-1}\cdot(p,q)\cdot(c,d)$ for some $(c,d)\in Q_\delta$. Since we also have $(x,y)\in Q_R$, we can check that

$$(p,q)^{-1}\cdot(a,b) \;=\; (c,d)\cdot(x,y)^{-1} \;=\; (\tfrac{c}{x}, dx-xy) \;\in\; Q_T.$$

Therefore $(a,b) \in (p,q)\cdot Q_T = Q_T(p,q)$. □

Lemma 3.3. For each $\delta > 1$, there exists a constant $C_\delta > 0$ such that for every (p,q), $(a,b) \in \mathbf{A}$,

$$\big|\big\langle \sigma(p,q)\eta,\, \sigma(a,b)\psi\big\rangle\big|^2 \le C_\delta \iint_{Q_\delta((a,b)^{-1}\cdot(p,q))} \big|\big\langle \psi,\, \sigma(x,y)\eta\big\rangle\big|^2 \,\frac{dx}{x}\, dy.$$

Proof. Let $(p,q) \in \mathbf{A}$ and $(a,b) \in \mathbf{A}$ be given. Set $f = \sigma((p,q)^{-1}\cdot(a,b))\psi$ and let $h = \delta^{1/2}$. Then we compute as follows:

$$\begin{aligned}
\big|\big\langle \sigma(p,q)\eta,\, \sigma(a,b)\psi\big\rangle\big|^2 &= \big|\big\langle \eta,\, \sigma((p,q)^{-1}\cdot(a,b))\psi\big\rangle\big|^2 \\
&= \big|\big\langle f,\, \sigma(1,0)\eta\big\rangle\big|^2 \\
&= (2\pi)^2\,|Gf(i)|^2 \\
&\le 4\pi^2 \left(K_h \iint_{Q_h} |Gf(z)|\,dz\right)^2 \\
&= 4\pi^2 K_h^2 \left(\iint_{Q_h} \frac{1}{2\pi x^{3/2}}\,\big|\big\langle f,\, \sigma(x,\tfrac{y}{x})\eta\big\rangle\big|\,dy\,dx\right)^2 \\
&= K_h^2 \left(\int_{1/h}^{h}\int_{-h}^{h} \frac{1}{x^{3/2}}\,\big|\big\langle f,\, \sigma(x,\tfrac{y}{x})\eta\big\rangle\big|\,dy\,dx\right)^2 \\
&= K_h^2 \left(\int_{1/h}^{h}\int_{-h/x}^{h/x} \frac{1}{x^{3/2}}\,\big|\big\langle f,\, \sigma(x,t)\eta\big\rangle\big|\,x\,dt\,dx\right)^2 \\
&\le K_h^2 \left(\int_{1/h^2}^{h^2}\int_{-h^2}^{h^2} \frac{1}{x^{1/2}}\,\big|\big\langle f,\, \sigma(x,t)\eta\big\rangle\big|\,dt\,dx\right)^2 \\
&\le K_h^2 \left(\int_{1/h^2}^{h^2}\int_{-h^2}^{h^2} dt\,dx\right)\left(\iint_{Q_{h^2}} \big|\big\langle f,\, \sigma(x,t)\eta\big\rangle\big|^2\,dt\,\frac{dx}{x}\right) \\
&= C_\delta \iint_{Q_\delta} \big|\big\langle f,\, \sigma(x,t)\eta\big\rangle\big|^2\,\frac{dx}{x}\,dt \\
&= C_\delta \iint_{Q_\delta} \big|\big\langle \sigma((p,q)^{-1}\cdot(a,b))\psi,\, \sigma(x,t)\eta\big\rangle\big|^2\,\frac{dx}{x}\,dt \\
&= C_\delta \iint_{Q_\delta} \big|\big\langle \psi,\, \sigma((a,b)^{-1}\cdot(p,q)\cdot(x,t))\eta\big\rangle\big|^2\,\frac{dx}{x}\,dt \\
&= C_\delta \iint_{Q_\delta((a,b)^{-1}\cdot(p,q))} \big|\big\langle \psi,\, \sigma(x,t)\eta\big\rangle\big|^2\,\frac{dx}{x}\,dt.
\end{aligned}$$

□

3.4. **Proof of Theorem 1.1 (b).**

Now we give the proof of part (b) of Theorem 1.1. Assume that $\psi \in L^2(\mathbf{R})$ and $\Lambda \subseteq \mathbf{A}$ are given such that $\mathcal{D}^+(\Lambda^{-1}) < \infty$ and $\mathcal{D}^-(\Lambda) = 0$. Our goal is to show that $\mathcal{W}(\psi,\Lambda)$ does not possess a lower frame bound.

Fix any $\varepsilon > 0$. We have $W_\eta\psi \in L^2(\mathbf{A}, d\mu)$, so, since the sets Q_h form a nested, increasing, exhaustive sequence of subsets of $\mathbf{A}$ there must exist some $R > 1$ such that

$$\iint_{\mathbf{A}\setminus Q_R} \left|\langle \psi,\, \sigma(x,y)\eta\rangle\right|^2 \frac{dx}{x}\, dy \;<\; \varepsilon.$$

Fix $T > R(R+1)$. Then, since $\mathcal{D}^-(\Lambda) = 0$, by Proposition 2.3 we can find a point $(p,q) \in \mathbf{A}$ such that

$$\Lambda \cap Q_T(p,q) \;=\; \emptyset.$$

Let $\delta > 1$ be chosen so that $R(R+\delta) < T$. Then by Lemma 3.2, we have that

$$\bigcup_{(a,b)\in\Lambda} Q_\delta((a,b)^{-1}\cdot(p,q)) \;\subseteq\; \mathbf{A}\setminus Q_R.$$

Now, since $\mathcal{D}^+(\Lambda^{-1}) < \infty$, we have by Lemma 3.1 that $\mathcal{D}^+(\Lambda^{-1}\cdot(p,q)) < \infty$ as well. Therefore, by Proposition 2.5 we can write Λ as a finite union $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N$ in such a way that $\Lambda_i^{-1}\cdot(p,q)$ is affinely δ-separated for each $i = 1,\ldots,N$. Consequently, for each i we have

$$\bigcup_{(a,b)\in\Lambda_i} Q_\delta((a,b)^{-1}\cdot(p,q)) \;\subseteq\; \mathbf{A}\setminus Q_R, \tag{3.1}$$

with the union being disjoint.

Finally, applying Lemma 3.3 and (3.1), we compute that

$$\begin{aligned}
\sum_{(a,b)\in\Lambda} \left|\langle \sigma(p,q)\eta,\, \sigma(a,b)\psi\rangle\right|^2 &\le C_\delta \sum_{i=1}^{N} \sum_{(a,b)\in\Lambda_i} \iint_{Q_\delta((a,b)^{-1}\cdot(p,q))} \left|\langle \psi,\, \sigma(x,y)\eta\rangle\right|^2 \frac{dx}{x}\, dy \\
&\le C_\delta \sum_{i=1}^{N} \iint_{\mathbf{A}\setminus Q_R} \left|\langle \psi,\, \sigma(x,y)\eta\rangle\right|^2 \frac{dx}{x}\, dy \\
&\le NC_\delta\, \varepsilon.
\end{aligned}$$

Since $\|\sigma(p,q)\eta\|_2 = \|\eta\|_2 = 1$, it follows that $\mathcal{W}(\psi,\Lambda)$ cannot possess a lower frame bound, which completes the proof.

4. **Examples and applications**

In this section we will apply our results to several types of weighted wavelet systems.

4.1. **Oversampled affine systems.**

A general notion of oversampled affine systems was introduced in [14], which includes the affine and quasi-affine systems as special cases. We will show that each such oversampled system has the same density.

Definition 4.1. Given $a > 1$, $b > 0$, and r_j, an oversampled affine system is a weighted wavelet system of the form $\mathcal{W}(\psi,\Lambda,w)$ with

$$\Lambda \;=\; \left\{(a^j, \tfrac{bk}{r_j}) : j,k \in \mathbf{Z}\right\} \qquad \text{and} \qquad w(a^j, \tfrac{bk}{r_j}) = \frac{1}{r_j}.$$

Example 4.2. The following are special cases of oversampled affine systems.

a. The classical affine systems are obtained by setting $r_j \equiv 1$.

b. The quasi-affine systems of Ron and Shen [20] are obtained when a is an integer, $b = 1$, and

$$r_j = \begin{cases} 1, & j < 0, \\ a^j, & j \geq 0. \end{cases}$$

c. The quasi-affine systems of Bownik [5] are obtained when $a = p/q$ is rational, $b = 1$, and

$$r_j = \begin{cases} q^{-j}, & j < 0, \\ p^j, & j \geq 0. \end{cases}$$

Proposition 4.3. If $\mathcal{W}(\psi, \Lambda, w)$ is an oversampled affine system, then Λ has uniform weighted affine Beurling density

$$\mathcal{D}_w(\Lambda) = \frac{1}{b \ln a}.$$

Proof. Fix $(x, y) \in \mathbf{A}$. If $(a^j, \frac{bk}{r_j}) \in Q_h(x, y)$, then

$$\big(\tfrac{a^j}{x}, \tfrac{bk}{r_j} - \tfrac{xy}{a^j}\big) = (x, y)^{-1} \cdot (a^j, \tfrac{bk}{r_j}) \in Q_h.$$

In particular, $\frac{a^j}{x} \in [\frac{1}{h}, h)$. There are at least $2 \log_a h$ and at most $2 \log_a h + 1$ integers j satisfying this condition. Additionally, we have $\frac{xyr_j}{a^j b} - \frac{hr_j}{b} \leq k < \frac{xyr_j}{a^j b} + \frac{hr_j}{b}$. For a given j, there are at least $\frac{2hr_j}{b}$ and at most $\frac{2hr_j}{b} + 1$ integers k satisfying this condition. Taking the weight into account, we conclude that

$$2 \log_a h \cdot \frac{1}{r_j} \cdot \frac{2hr_j}{b} \leq \#_w(\Lambda \cap Q_h(x, y)) \leq \big(2 \log_a h + 1\big) \cdot \frac{1}{r_j} \cdot \left(\frac{2hr_j}{b} + 1\right).$$

Thus

$$\mathcal{D}_w^+(\Lambda) \leq \limsup_{h \to \infty} \frac{\big(2 \log_a h + 1\big) \cdot \frac{1}{r_j} \cdot \left(\frac{2hr_j}{b} + 1\right)}{4h \ln h} = \frac{1}{b \ln a},$$

and similarly $\mathcal{D}_w^-(\Lambda) \geq \frac{1}{b \ln a}$. □

We remark that for Gabor systems $\mathcal{G}(g, a\mathbf{Z} \times b\mathbf{Z})$, the Beurling density $\frac{1}{ab}$ of the lattice $a\mathbf{Z} \times b\mathbf{Z}$ is a ubiquitous constant in a variety of formulas. For example, if $\mathcal{G}(g, a\mathbf{Z} \times b\mathbf{Z})$ is a tight frame for $L^2(\mathbf{R})$ and $\|g\|_2 = 1$, then the frame bounds are exactly $\frac{1}{ab}$. Many of these formulas have analogues for classical affine systems, with the number $\frac{1}{b \ln a}$ playing the role that $\frac{1}{ab}$ plays for Gabor systems. For this reason, Daubechies already suggested in [9, Section 4.1], that $\frac{1}{b \ln a}$ might play the role of a density for affine systems, but she also demonstrated that affine systems cannot possess an analogue of the Nyquist density that Gabor systems possess.

4.2. **Co-affine systems.**

Next we will consider the co-affine systems studied in [11]. We show that unweighted co-affine systems can possess neither an upper nor a lower frame bound.

Proposition 4.4. Fix $a > 1$ and $b > 0$, and set $\Lambda = \{(a^j, a^{-j}bk) : j, k \in \mathbf{Z}\}$. Then

$$\mathcal{D}^-(\Lambda) = 0 \qquad \text{and} \qquad \mathcal{D}^+(\Lambda) = \infty.$$

Consequently, a co-affine system $\mathcal{W}(\psi, \Lambda)$ cannot possess an upper or a lower frame bound.

Proof. Fix $(x, y) \in \mathbf{A}$. If $(a^j, a^{-j}bk) \in Q_h(x, y)$, then

$$\left(\tfrac{a^j}{x}, a^{-j}bk - \tfrac{xy}{a^j}\right) = (x, y)^{-1} \cdot (a^j, a^{-j}bk) \in Q_h.$$

This requires

$$\log_a x - \log_a h \le j < \log_a x + \log_a h$$

and

$$\frac{xy - a^j h}{b} \le k < \frac{xy + a^j h}{b}.$$

As in the proof of Proposition 4.3, terms ± 1 are not significant in the limit, so it suffices to observe that

$$\#_w(\Lambda \cap Q_h(x, y)) \approx \sum_{j=\lceil \log_a x - \log_a h\rceil}^{\lfloor \log_a x + \log_a h\rfloor} \frac{2ha^j}{b}.$$

By changing x, we can make this quantity arbitrarily large or small, which yields the conclusion $\mathcal{D}^-(\Lambda) = 0$ and $\mathcal{D}^+(\Lambda) = \infty$. Finally, since Λ^{-1} is the set corresponding to the affine system, we have $\mathcal{D}^+(\Lambda^{-1}) = 1/(b \ln a) < \infty$. The nonexistence of frame bounds therefore follows from Theorem 1.1. □

4.3. **Systems of translates and dilations.**

Finally, we examine systems which consist only of translates or of dilations of a given function.

Proposition 4.5. Let $\psi \in L^2(\mathbf{R})$, $\Lambda_1 \subseteq \mathbf{R}$, and $\Lambda_2 \subseteq \mathbf{R}^+$ be given.

(a) $\mathcal{T}(\psi, \Lambda_1) = \{\psi(x - a)\}_{a \in \Lambda_1}$ is not a frame for $L^2(\mathbf{R})$.

(b) $\mathcal{D}(\psi, \Lambda_2) = \{a^{-1/2}\psi(\frac{x}{a})\}_{a \in \Lambda_2}$ is not a frame for $L^2(\mathbf{R})$.

Proof. Note that both of these systems are special cases of weighted wavelet systems, namely,

$$\mathcal{T}(\psi, \Lambda_1) = \mathcal{W}(\psi, \{1\} \times \Lambda_1) \qquad \text{and} \qquad \mathcal{D}(\psi, \Lambda_2) = \mathcal{W}(\psi, \Lambda_2 \times \{0\}).$$

Consider first the case of pure dilations. Note that $\mathcal{D}^-(\Lambda_2 \times \{0\}) = 0$. If $\mathcal{D}^+(\Lambda_2 \times \{0\}) = \infty$, then $\mathcal{D}(\psi, \Lambda_2)$ cannot possess an upper frame bound by Theorem 1.1(a).

Suppose on the other hand that $\mathcal{D}^+(\Lambda_2 \times \{0\}) < \infty$. If $(c, 0) \in (\Lambda_2 \times \{0\})^{-1} \cap Q_h(x, y)$, then $(\frac{c}{x}, -\frac{xy}{c}) = (x, y)^{-1} \cdot (c, 0) \in Q_h$. Hence $\frac{1}{h} \le \frac{c}{x} < h$, so $-h^3 < -\frac{cy}{x} = -\frac{xy}{c}\frac{c}{x}\frac{c}{x} < h^3$. Therefore $(\frac{1}{x}, y)^{-1} \cdot (\frac{1}{c}, 0) \in Q_{h^3}$, so $(\frac{1}{c}, 0) \in Q_{h^3}(\frac{1}{x}, y)$. Thus

$$\sup_{(x,y)\in\mathbf{A}} \#((\Lambda_2 \times \{0\})^{-1} \cap Q_h(x, y)) \le \sup_{(x,y)\in\mathbf{A}} \#((\Lambda_2 \times \{0\}) \cap Q_{h^3}(x, y)) < \infty,$$

so $\mathcal{D}^+((\Lambda_2 \times \{0\})^{-1}) < \infty$ by Proposition 2.2. Consequently, Theorem 1.1(b) implies that $\mathcal{D}(\psi, \Lambda_2)$ cannot possess a lower frame bound in this case.

The proof for $\mathcal{T}(\psi, \Lambda_1)$ is similar, and was also obtained in [6] by using the fact that a system of pure translations is a Gabor system of the form $\mathcal{G}(g, \Lambda_1 \times \{0\})$. □

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References

[1] Balan, R. Stability theorems for Fourier frames and wavelet Riesz bases, *J. Fourier Anal. Appl.*, **3**(5), 499–504, (1997).

[2] Balan, R., Casazza, P.G., Heil, C., and Landau, Z. Excesses of Gabor frames, *Appl. Comput. Harmon. Anal.*, **14**(2), 87–106, (2003).

[3] Balan, R., Casazza, P.G., Heil, C., and Landau, Z. Density, excesses, and localization of frames, preprint.

[4] Benedetto, J.J., Heil, C., and Walnut, D. Differentiation and the Balian–Low theorem, *J. Fourier Anal. Appl.*, **1**(4), 355–402, (1995).

[5] Bownik, M. Quasi-affine systems and the Calderón condition, in Proc. Conf. Harmonic Analysis (Mt. Holyoke, MA, 2001), Contemp. Math., American Mathematical Society, Providence, RI, to appear.

[6] Christensen, O., Deng, B., and Heil, C. Density of Gabor frames, *Appl. Comput. Harmon. Anal.*, **7**(3), 292–304, (1999).

[7] Dai, X., Larson, D.R., and Speegle, D. M. Wavelet sets in $\mathbf{R}^n$, *J. Fourier Anal. Appl.*, **3**, 451–456, (1997).

[8] Daubechies, I. The wavelet transform, time-frequency localization and signal analysis, *IEEE Trans. Inform. Theory*, **39**, 961–1005, (1990).

[9] Daubechies, I. *Ten Lectures on Wavelets*, SIAM, Philadelphia, (1992).

[10] Feichtinger, H.G., and Gröbner, P. Banach spaces of distributions defined by decomposition methods, I, *Math. Nachr.*, **123**, 97–120, (1985).

[11] Gressman, P., Labate, D., Weiss, G., and Wilson, E.N. Affine, quasi-affine and co-affine wavelets, in *Beyond Wavelets*, J. Stöckler and G. Welland, Eds., Academic Press, San Diego, to appear.

[12] Gröchenig, K., and Razafinjatovo, H., On Landau's necessary density conditions for sampling and interpolation of band-limited functions, *J. London Math. Soc. (2)*, **54**(3), 557–565, (1996).

[13] Grossmann, A., Morlet, J., and Paul, T. Transforms associated to square integrable group representations II: examples, *Ann. Inst. H. Poincaré Phys. Théor.*, **45**, 293–309, (1986).

[14] Hernandez, E., Labate, D., Weiss, G., and Wilson, E.N. Oversampling of reproducing systems, preprint.

[15] Hörmander, L. *An Introduction to Complex Analysis in Several Variables*, North-Holland, New York, (1973).

[16] Janssen, A.J.E.M. Signal analytic proofs of two basic results on lattice expansions, *Appl. Comput. Harmon. Anal.*, **1**(4), 350–354, (1994).

[17] Landau, H. On the density of phase-space expansions, *IEEE Trans. Inform. Theory*, **39**, 1152–1156, (1993).

[18] Ramanathan, J., and Steger, T. Incompleteness of sparse coherent states, *Appl. Comput. Harmon. Anal.*, **2**(2), 148–153, (1995).

[19] Rieffel, M. Von Neumann algebras associated with pairs of lattices in Lie groups, *Math. Ann.*, **257**, 403–418, (1981).

[20] Ron, A., and Shen, Z. Affine systems in $L^2(\mathbf{R}^d)$: the analysis of the analysis operator, *J. Funct. Anal.*, **3**, 617–637, (1997).

[21] Seip, K. Sampling and interpolation in the Bargmann–Fock space, I, *J. Reine Angew. Math.*, **429**, 91–106, (1992).

[22] Seip, K. Beurling type density theorems in the unit disk, *Invent. Math.*, **113**, 21–39, (1993).

[23] Sun, W., and Zhou, X. Density of irregular wavelet frames, preprint.

[24] Sun, W., and Zhou, X. Density and stability of wavelet frames, preprint.

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School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332-0160

E-mail address: heil@math.gatech.edu

Department of Mathematics and Computer Science, University of Paderborn, 33095 Paderborn, Germany

E-mail address: gittak@uni-paderborn.de