DENSITY, OVERCOMPLETENESS, AND LOCALIZATION OF FRAMES

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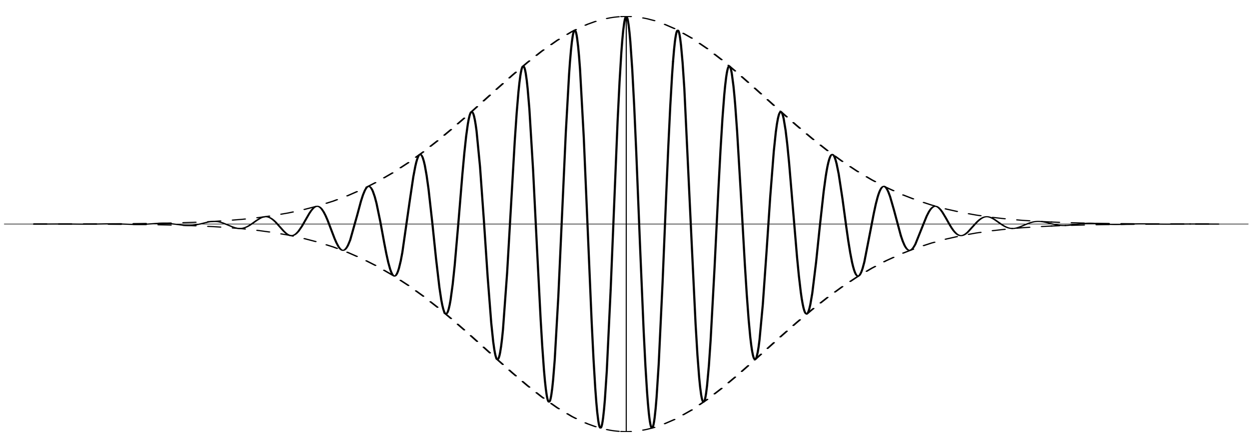
Goal

Music-like bases or frames for $L^2(\mathbf{R})$.

Model of a note at time α and frequency β:



$$e^{2\pi i\beta x} g(x-\alpha) \;=\; M_\beta T_\alpha g(x)$$

I. INTRODUCTION

Regular Gabor System

$$\mathcal{G}(g, \alpha\mathbf{Z} \times \beta\mathbf{Z}) \;=\; \left\{e^{2\pi i n\beta x} g(x - k\alpha)\right\}_{k,n\in\mathbf{Z}} \;=\; \left\{M_{n\beta} T_{k\alpha} g\right\}_{k,n\in\mathbf{Z}}$$

Density Theorem

(a) **Frame** $\Longrightarrow$ $0 < \alpha\beta \leq 1$**.**

(b) **Riesz basis** $\Longrightarrow$ $\alpha\beta = 1$**.**

(c) $\alpha\beta > 1$ $\Longrightarrow$ **incomplete.**

Techniques

Baggett, Rieffel: von Neumann algebra generated by T_α**,** M_β

Daubechies: Zak Transform

Janssen: Wexler–Raz relations,

$$\left\langle M_{\frac{n}{\alpha}} T_{\frac{k}{\beta}} g,\, M_{\frac{n'}{\alpha}} T_{\frac{k'}{\beta}} \tilde{g} \right\rangle \;=\; (\alpha\beta)\, \delta_{nn'} \delta_{kk'}$$

Irregular Gabor Systems

Let $\Lambda \subset \mathbf{R}^2$ be any countable sequence of points. Define

$$\mathcal{G}(g, \Lambda) \;=\; \left\{e^{2\pi i \beta x} g(x-\alpha)\right\}_{(\alpha,\beta)\in\Lambda} \;=\; \left\{M_\beta T_\alpha g\right\}_{(\alpha,\beta)\in\Lambda}$$

Beurling Densities of Λ

$$D^-(\Lambda) \;=\; \liminf_{r\to\infty} \inf_{z\in\mathbf{R}^2} \frac{\#(\Lambda \cap Q_r(z))}{r^2},$$

$$D^+(\Lambda) \;=\; \limsup_{r\to\infty} \sup_{z\in\mathbf{R}^2} \frac{\#(\Lambda \cap Q_r(z))}{r^2},$$

where $Q_r(z)$ is the square centered at z with side lengths r.

Examples

$$D^-(\alpha\mathbf{Z} \times \beta\mathbf{Z}) = D^+(\alpha\mathbf{Z} \times \beta\mathbf{Z}) = \frac{1}{\alpha\beta}$$

$$D^-(\alpha\mathbf{Z} \times \beta\mathbf{Z}^+) = 0, \qquad D^+(\alpha\mathbf{Z} \times \beta\mathbf{Z}^+) = \frac{1}{\alpha\beta}$$

Density for Irregular Gabor Systems (Ramanathan/Steger)

Given $g \in L^2(\mathbf{R})$ and $\Lambda \subset \mathbf{R}^2$, the Gabor system

$$\mathcal{G}(g,\Lambda) \;=\; \left\{e^{2\pi i\beta x} g(x-\alpha)\right\}_{(\alpha,\beta)\in\Lambda}$$

has the following properties:

(a) **Frame** $\implies 1 \le D^-(\Lambda) \le D^+(\Lambda) < \infty$.

(b) **Riesz basis** $\implies D^-(\Lambda) = D^+(\Lambda) = 1$.

Remarks

- **Irregular Gabor systems can be complete (but not frames) even if they are very sparse [Walnut/H.]**
- $\exists$ **(very) irregular Gabor ONB [Y. Wang]**
- **Problem: The dual frame $\{\tilde{g}_{\alpha,\beta}\}_{(\alpha,\beta)\in\Lambda}$ need not be a Gabor frame; does it have any structure?**

II. NEW RESULTS FOR IRREGULAR GABOR FRAMES

MetaTheorem

If $\mathcal{F} = \{f_i\}_{i \in I}$ is a frame sequence that is "localized" with respect to another frame sequence $\mathcal{E} = \{e_j\}_{j \in G}$, then

$$D(I) \cdot \mathcal{M}_{\mathcal{E}}(\mathcal{F}) \;=\; \mathcal{M}_{\mathcal{F}}(\mathcal{E})$$

where

$$\mathcal{M}_{\mathcal{E}}(\mathcal{F}) \;=\; \begin{cases} \textbf{Limits of averages of diagonal} \\ \textbf{elements of } \big[\langle P_{\mathcal{E}} f_i, \tilde{f}_j \rangle\big]_{i,j \in I} \end{cases}$$

and

$$\mathcal{M}_{\mathcal{F}}(\mathcal{E}) \;=\; \begin{cases} \textbf{Limits of averages of diagonal} \\ \textbf{elements of } \big[\langle P_{\mathcal{F}} e_i, \tilde{e}_j \rangle\big]_{i,j \in G} \end{cases}$$

Example

If $\mathcal{F}$ is a frame and $\mathcal{E}$ is a Riesz basis then $\mathcal{M}_{\mathcal{F}}(\mathcal{E}) = 1$.

Remark

"Limits" include Beurling-type upper and lower limits as well as ultrafilter limits.

Application 1: Necessary Density Conditions

Let $\mathcal{G}(g,\Lambda) = \{M_\beta T_\alpha g\}_{(\alpha,\beta)\in\Lambda}$ **be Gabor frame for** $L^2(\mathbf{R})$**. Then:**

(a) $D^{\pm}(\Lambda) = \dfrac{1}{\mathcal{M}^{\mp}(\mathcal{G}(g,\Lambda))}$**.**

(b) $D^{-}(\Lambda) \geq 1$**.**

(c) **Riesz basis** $\implies D^{-}(\Lambda) = D^{+}(\Lambda) = 1$**.**

Application 2: Relations between Density and Frame Bounds

Let $\mathcal{G}(g,\Lambda) = \{M_\beta T_\alpha g\}_{(\alpha,\beta)\in\Lambda}$ **be Gabor frame for** $L^2(\mathbf{R})$ **with frame bounds** A**,** B**. Then:**

(a) $A \leq \|g\|_2^2\, D^{-}(\Lambda) \leq \|g\|_2^2\, D^{+}(\Lambda) \leq B$**.**

(b) **Tight frame** $\implies D^{-}(\Lambda) = D^{+}(\Lambda)$**.**

Definition: Time-frequency concentration

The STFT of g with respect to a nice window ϕ is

$$V_\phi g(x,\omega) \;=\; \langle g, M_\beta T_\alpha \phi \rangle.$$

We have

$$\|V_\phi g\|_2 \;=\; \|g\|_2 \, \|\phi\|_2 \;<\; \infty.$$

We say that g belongs to the modulation space M^1 if

$$\|V_\phi g\|_1 \;<\; \infty.$$

Application 3: Quantifying Excess

Let $\mathcal{G}(g,\Lambda) \;=\; \{M_\beta T_\alpha g\}_{(\alpha,\beta)\in\Lambda}$ be Gabor frame for $L^2(\mathbf{R})$ with $g \in M^1$. Then:

(a) **If $D^-(\Lambda) > 1$, then there exists $J \subset \Lambda$ with $D^-(J) = D^+(J) > 0$ such that $\mathcal{G}(g, \Lambda \setminus J)$ is a frame for $L^2(\mathbf{R})$.**

(b) **$\mathcal{G}(g,\Lambda)$ can be written as a finite union of Riesz sequences.**

Application 4: Structure of the Dual Frame

Let $\mathcal{G}(g,\Lambda) = \{M_\beta T_\alpha g\}_{(\alpha,\beta)\in\Lambda}$ **be Gabor frame for** $L^2(\mathbf{R})$ **with** $g \in M^1$**. Then:**

(a) **The dual frame** $\tilde{\mathcal{G}} = \{\tilde{g}_{\alpha,\beta}\}_{(\alpha,\beta)\in\Lambda}$ **is also contained in** M^1**.**

Remark: If Λ **is a lattice, this recovers Gröchenig/Leinert.**

(b) **The dual frame** $\tilde{\mathcal{G}} = \{\tilde{g}_{\alpha,\beta}\}_{(\alpha,\beta)\in\Lambda}$ **is a set of** *Gabor molecules***, i.e.,** $\exists\, F \in L^1(\mathbf{R}^2)$ **such that**

$$|V_\phi(\tilde{g}_{\alpha,\beta})(x,\omega)| \ \le\ F(x-\alpha,\omega-\beta).$$

Compare:

$$|V_\phi(M_\beta T_\alpha g)(x,\omega)| \ =\ |V_\phi g(x-\alpha,\omega-\beta)|.$$

Remark

Applications 1–4 continue to hold (with minor changes) if the Gabor frame $\mathcal{G}(g,\Lambda)$ **is replaced by a frame of Gabor molecules** $\{g_{\alpha,\beta}\}_{(\alpha,\beta)\in\Lambda}$**.**

An Open Problem Regarding Irregular Gabor Frames

Gabor frames are (probably) finitely independent.

The following conjecture is known to hold for many special cases, but is open in the generality stated.

Conjecture (H./Ramanathan/Topiwala)

If $g \in L^2(\mathbf{R})$ is nonzero and $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^N$ are distinct points in $\mathbf{R}^2$, then $\mathcal{G}(g, \Lambda) = \{e^{2\pi i \beta_k x} g(x - \alpha_k)\}_{k=1}^N$ is linearly independent.

Even the following special case seems to be open.

Conjecture

If $g \in L^2(\mathbf{R})$ is continuous and nonzero then

$$\{g(x),\ g(x-1),\ e^{2\pi i x} g(x),\ e^{2\pi i \sqrt{2} x} g(x - \sqrt{2})\}$$

is linearly independent ($\Lambda = \{(0,0),\ (1,0),\ (0,1), (\sqrt{2}, \sqrt{2})\}$).

What makes the density theorem work?

<u>**Homogeneous Approximation Property**</u> **(Ramanathan/Steger)**
If $\mathcal{G}(g,\Lambda) = \{M_\beta T_\alpha g\}_{(\alpha,\beta)\in\Lambda}$ **is a frame and** $f \in L^2(\mathbf{R})$**, then**

$$f \;=\; \sum_{(\alpha,\beta)\in\Lambda} \langle f, M_\beta T_\alpha g\rangle \; \tilde{g}_{\alpha,\beta}$$

and therefore $\exists$ **square** $Q_r(0,0)$ **such that**

$$\Bigl\| f - \sum_{(\alpha,\beta)\in Q_r(0,0)} \langle f, M_\beta T_\alpha g\rangle \; \tilde{g}_{\alpha,\beta} \Bigr\|_2 \;<\; \varepsilon.$$

But in fact, for any $(p,q) \in \mathbf{R}^2$**,**

$$\Bigl\| M_q T_p f - \sum_{(\alpha,\beta)\in Q_r(p,q)} \langle M_q T_p f, M_\beta T_\alpha g\rangle \; \tilde{g}_{\alpha,\beta} \Bigr\|_2 \;<\; \varepsilon.$$

Note that the (α,β) **points lying in** $Q_r(p,q)$ **are not just translates of the** (α,β) **points lying in** $Q_r(0,0)$**.**

Definition: Abstract HAP

Given a frame $\mathcal{F} = \{f_i\}_{i\in I}$ with dual $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$, a sequence $\mathcal{E} = \{e_j\}_{j\in G}$, and a map $a\colon I \to G$. Set $I_N(j) = a^{-1}(S_N(j))$.

(a) $(\mathcal{F}, a, \mathcal{E})$ **has the** *weak HAP* **if** $\forall\, \varepsilon > 0$, $\exists\, N_\varepsilon > 0$ **such that**

$$\forall\, j, \quad \exists\, c_{i,j}, \quad \Bigl\| e_j - \sum_{i\in I_{N_\varepsilon}(j)} c_{i,j}\, \tilde{f}_i \Bigr\| \;<\; \varepsilon.$$

(b) $(\mathcal{F}, a, \mathcal{E})$ **has the** *strong HAP* **if** $\forall\, \varepsilon > 0$, $\exists\, N_\varepsilon > 0$ **such that**

$$\forall\, j, \quad \Bigl\| e_j - \sum_{i\in I_{N_\varepsilon}(j)} \langle e_j, f_i\rangle\, \tilde{f}_i \Bigr\| \;<\; \varepsilon.$$

Theorem: Abstract Necessary Density Condition

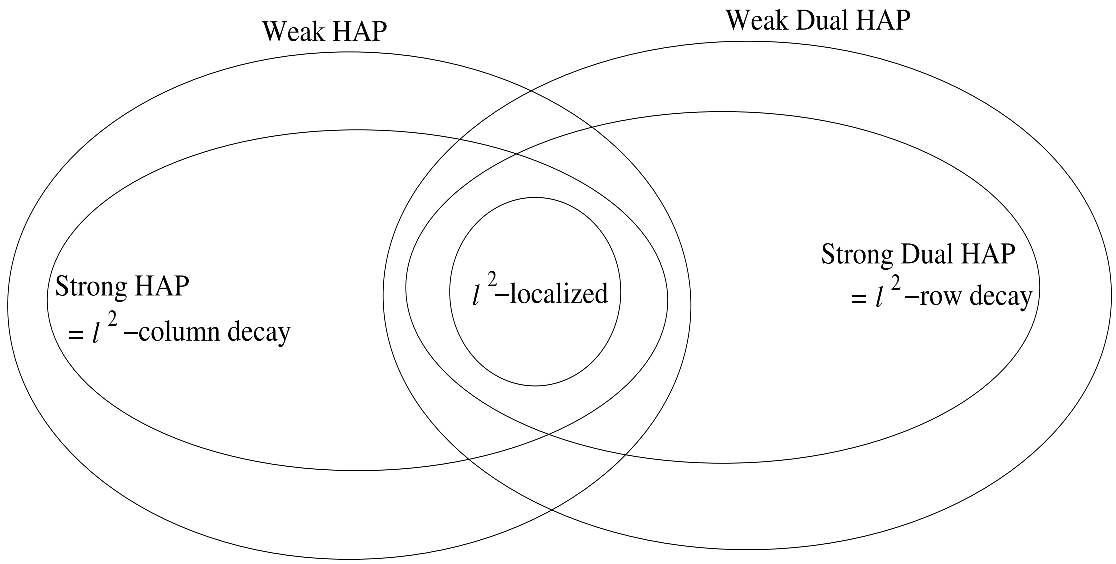
If $\mathcal{F}$ is a frame and $\mathcal{E}$ is a Riesz basis then

$$\textbf{weak HAP} \implies D^-(I) \geq 1.$$

Question: When does the HAP hold?

Main issue: Properties of $[\langle f_i, e_j\rangle]_{i\in I, j\in G}$

Relations among localization and HAP properties



Definition: Localized Frames

Given $\mathcal{F} = \{f_i\}_{i\in I}$, $\mathcal{E} = \{e_j\}_{j\in G}$, and $a\colon I \to G$.

(a) **($\mathcal{F}$, a, $\mathcal{E}$) is** *ℓ^p-localized* **if** $\exists\, r = (r_k)_{k\in G} \in \ell^p(G)$ **such that**

$$|\langle f_i, e_j\rangle| \;\leq\; r_{a(i)-j}$$

(b) $(\mathcal{F}, a, \mathcal{E})$ **has** *ℓ^p-column decay* **if** $\forall\, \varepsilon > 0$, $\exists\, N_\varepsilon > 0$ **such that**

$$\forall\, j \in G, \qquad \sum_{i\in I\setminus I_{N_\varepsilon}(j)} |\langle f_i, e_j\rangle|^p \;<\; \varepsilon$$

(c) $(\mathcal{F}, a, \mathcal{E})$ **has** *ℓ^p-row decay* **if** $\forall\, \varepsilon > 0$, $\exists\, N_\varepsilon > 0$ **such that**

$$\forall\, i \in I, \qquad \sum_{j\in G\setminus S_{N_\varepsilon}(a(i))} |\langle f_i, e_j\rangle|^p \;<\; \varepsilon$$

Example: Regular Gabor Frames

Setup:

$$\mathcal{F} \;=\; \mathcal{G}(g, \alpha\mathbf{Z} \times \beta\mathbf{Z}) \;=\; \big\{M_{\beta n} T_{\alpha k} g\big\}_{k,n \in \mathbf{Z}}$$

$$\mathcal{E} \;=\; \mathcal{G}(\phi, \alpha\mathbf{Z} \times \beta\mathbf{Z}) \;=\; \big\{M_{\beta n} T_{\alpha k} \phi\big\}_{k,n \in \mathbf{Z}}$$

$$G \;=\; \alpha\mathbf{Z} \times \beta\mathbf{Z}$$

$$a \;=\; \textbf{identity}$$

Then the Gram-like matrix

$$\begin{aligned} A \;&=\; \Big[|\langle M_{\beta n} T_{\alpha k} g, M_{\beta m} T_{\alpha j} \phi\rangle|\Big]_{(k,n),(m,j) \in \mathbf{Z}^2} \\ &=\; \Big[|\langle g, M_{\beta(m-n)} T_{\alpha(j-k)} \phi\rangle|\Big]_{(k,n),(m,j) \in \mathbf{Z}^2} \end{aligned}$$

is Toeplitz, and the diagonals satisfy

$$\sum_{(k,n) \in \mathbf{Z}^2} |\langle g, M_{\beta n} T_{\alpha k} \phi\rangle|^2 \;\le\; B\,\|g\|_2^2 \;<\; \infty$$

Thus $(\mathcal{F}, a, \mathcal{E})$ is ℓ^2-localized.

Remark

If $\phi = \tilde{g}$ (dual frame generator), then main diagonal of A is

$$|\langle M_{\beta n} T_{\alpha k} g, M_{\beta n} T_{\alpha k} \tilde{g}\rangle| \;=\; |\langle g, \tilde{g}\rangle| \;=\; \alpha\beta \;=\; \frac{1}{D^{\pm}(\alpha\mathbf{Z} \times \beta\mathbf{Z})}$$

Theorem: Localization of Irregular Gabor Frames

Let $\mathcal{E} = \mathcal{G}(\phi, \alpha\mathbf{Z} \times \beta\mathbf{Z})$ be a frame with $\phi \in M^1$.

Let $\mathcal{F} = \mathcal{G}(g, \Lambda)$ be a frame with $g \in M^p$ (note $M^2 = L^2$).

Let a round to the nearest element of $G = \alpha\mathbf{Z} \times \beta\mathbf{Z}$.

Then $(\mathcal{F}, a, \mathcal{E})$ is ℓ^p-localized.

Theorem: Density vs. Relative Measures

If $\mathcal{F}$, $\mathcal{E}$ are frame sequences and $(\mathcal{F}, a, \mathcal{E})$ has both ℓ^2-column and row decay, then (with respect to any ultrafilter limit),

$$D(I) \cdot \mathcal{M}_{\mathcal{E}}(\mathcal{F}) \;=\; \mathcal{M}_{\mathcal{F}}(\mathcal{E})$$

where

$$\mathcal{M}_{\mathcal{E}}(\mathcal{F}) \;=\; \begin{cases} \textbf{Limits of averages of diagonal} \\ \textbf{elements of } \big[\langle P_{\mathcal{E}} f_i, \tilde{f}_j\rangle\big]_{i,j\in I} \end{cases}$$

and

$$\mathcal{M}_{\mathcal{F}}(\mathcal{E}) \;=\; \begin{cases} \textbf{Limits of averages of diagonal} \\ \textbf{elements of } \big[\langle P_{\mathcal{F}} e_i, \tilde{e}_j\rangle\big]_{i,j\in G} \end{cases}$$

In particular, if $\mathcal{F}$ is a frame and $\mathcal{E}$ is a Riesz basis, then

$$D^+(I) \;=\; \frac{1}{\mathcal{M}^-(\mathcal{F})} \qquad \textbf{and} \qquad D^-(I) \;=\; \frac{1}{\mathcal{M}^+(\mathcal{F})}$$

Sample (easy) Corollary

Let $\mathcal{F}$ be a frame with frame operator S, and let $\mathcal{E}$ be a Riesz basis such that $(\mathcal{F}, a, \mathcal{E})$ has both ℓ^2-column and row decay. Then

$$\begin{aligned}
\frac{1}{D^-(\Lambda)} \;=\; \mathcal{M}^+(\mathcal{G}) \;&=\; \textbf{averages of } \langle f_i, \tilde{f}_i \rangle \\
&=\; \textbf{averages of } \langle f_i, S^{-1}(f_i) \rangle \\
&\le\; \textbf{averages of } \frac{1}{A} \langle f_i, f_i \rangle \\
&=\; \textbf{averages of } \frac{1}{A} \|f_i\|^2
\end{aligned}$$

Consequently if $\|f_i\|^2 = C$ then

$$A \;\le\; D^-(\Lambda)\, C \;\le\; D^+(\Lambda)\, C \;\le\; B$$

and therefore

$$\mathcal{F} \textbf{ is tight } (A = B) \;\Longrightarrow\; D^-(\Lambda) = D^+(\Lambda) = \frac{A}{C}$$

Remark

Versions of the other Applications made for Gabor systems hold for abstract localized frames.

Localization of the Dual Frame

This relies on the following "noncommutative Wiener's Lemma".

Sjöstrand's Lemma

If a matrix A defines an invertible bounded map $\ell^2(G) \to \ell^2(G)$ and A has ℓ^1-type decay, then A^{-1} does also.

Theorem

If $(\mathcal{F}, a, \mathcal{F})$ is ℓ^1-localized, then so are $(\mathcal{F}, a, \tilde{\mathcal{F}})$ and $(\tilde{\mathcal{F}}, a, \tilde{\mathcal{F}})$.

Theorem (Gröchenig)

a. Given a polynomially localized frame $\mathcal{F} = \{f_i\}_{i \in I}$, there exists a family of associated Banach spaces $\mathcal{H}_s^p$ for which $\mathcal{F}$ simultaneously provides frame expansions.

b. With respect to polynomial or exponential localization, the dual frame shares the localization of the original frame.

<u>Contrast: Wavelets</u>

Wavelets are not localized in the same sense, and do not possess a Nyquist density.

<u>Theorem</u> (H./Kutyniok)

A wavelet frame

$$\mathcal{W}(\psi, \Lambda) \;=\; \left\{a^{-1/2}\psi(\tfrac{x}{a} - b)\right\}_{(a,b)\in\Lambda}$$

satisfies an affine version of the HAP, but in the wavelet case this implies only

$$0 \;<\; D^{-}(\Lambda) \;\leq\; D^{+}(\Lambda) \;<\; \infty,$$

using an affine analogue of Beurling density.