BOOK REVIEW

SIAM REVIEW
Vol. 35, December 1993, pp. 666–669.

Ten Lectures on Wavelets. By Ingrid Daubechies. SIAM/CBMS-NSF Regional Conference Series in Applied Mathematics, xix + 354 pp. $37.50. ISBN 0-89871-274-2.

Wavelet theory is an attempt to address the pervasive problem of describing the frequency content of a function locally in time. The wavelet approach is to analyze a function using an appropriate family of dilates and translates of one or more *wavelets*. Although this term is relatively new, wavelet-like techniques have been independently invented over the past 30 years in harmonic analysis, quantum mechanics, signal processing, etc. Some of wavelet theory has therefore been the rediscovery of those ideas, but the introduction of a single logical framework has allowed new insights and the formulation of new problems not possible before. Moreover, the applicability of wavelets to such diverse fields has led to an unusual—and highly productive—feedback between mathematicians, physicists, and engineers.

This book is a greatly expanded and updated version of ten lectures given by Daubechies at a CBMS conference at the University of Lowell in June, 1990. It is an introductory-level mathematics text, intended to introduce mathematicians and other scientists to wavelets. In this respect it is superficially similar to Chui's book [C1]; however, whereas Chui is exclusively mathematical and concentrates on only a few aspects of wavelet theory, Daubechies effectively surveys most of the current theory and specifically discusses the links with other areas, especially signal processing. To be sure, the topics which are discussed in the greatest detail are the ones in which Daubechies has herself worked—but this is not much of a limitation since she has been involved in most of the major advances, from the first "mathematical" wavelet paper [DGM] onward. All this is packaged into a well-written, organized format which should be accessible to first-year graduate students who have some knowledge of Fourier transforms and basic real analysis (a short chapter reviewing the essentials of Fourier analysis, integration theory, and Hilbert space techniques is included, but serves more as a quick-reference guide than a primer). Proofs and calculations are given in considerable detail, yet are carefully organized so that the essential ideas are clear. When details are omitted the reader is directed to the appropriate literature. There are a large number of minor typographical errors; most of these (hopefully all) will be corrected in the second printing.

The ten lectures begin with a short chapter which neatly surveys the two major incarnations of the wavelet transform, and compares these to analogous forms of a standard time-frequency transform known as the windowed Fourier transform. This chapter would be an excellent source for instructors looking for a summary of wavelets on which to base a few survey lectures. As is the case throughout most of the text, wavelets and wavelet transforms are considered here in one dimension only in order to maintain simplicity of exposition. Generalizations to higher dimensions are for the most part straightforward, and are briefly discussed in the last chapter.

The two wavelet transforms differ in the family of dilates and translates used. The continuous wavelet transform involves all possible dilates and translates $\psi^{a,b}(x) = |a|^{-1/2}\psi(a^{-1}(x-b))$, $a \neq 0$, $b \in \mathbf{R}$ of the wavelet ψ. In this case the choice of ψ can be almost arbitrary, and the

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resulting transform gives a rich—but highly redundant—time-frequency picture of a function f. There is a stable inversion formula closely related to the Calderon-Zygmund reproducing formula. Depending on the application, the extreme redundancy of the collection $\{\psi^{a,b}\}$ may be undesirable; the discrete wavelet transform "reduces" the redundancy by using only a countable collection of dilates and translates. In particular, an a_0 and b_0 are fixed and only the collection $\psi_{m,n}(x) = |a_0|^{-m/2}\psi(a_0^{-m}(x - nba_0^m))$, m, $n \in \mathbf{Z}$ is employed in the analysis. In this case ψ is considerably more constrained by the requirements that the wavelet transform both characterize f and allow a stable reconstruction. However, there is still freedom to choose the "amount" of redundancy left in the collection $\{\psi_{m,n}\}$. With redundancy eliminated—for example if $\{\psi_{m,n}\}$ is an orthonormal basis for $L^2(\mathbf{R})$—the choice of ψ is very limited, the time-frequency picture of f is sparse, and the reconstruction formula is simple. The opposite is true if redundancy is retained. Infinite variations of these two transforms are conceivable, and several are discussed at various places in the text.

The mathematics begins in the second chapter with a detailed analysis of the continuous wavelet transform. In addition to definitions and reconstruction formulas, the relationship to reproducing kernel Hilbert spaces and the parallel with the continuous windowed Fourier transform is explored. As is typical throughout the book, advanced topics such as the abstract view of both the continuous wavelet and continuous windowed Fourier transforms in terms of group representations are not discussed, although clear directions to the literature for further information are given. The latter sections illustrate the usefulness of this transform: one section builds and analyzes operators, while another takes advantage of the "zoom" property of dilations to analyze the local behavior of functions.

The majority of the remainder of the book deals with discrete families of wavelets. Chapter 3 focuses on wavelet "frames," i.e., possibly redundant families $\{\psi_{m,n}\}$ for which there exist A, $B > 0$ such that $A\,\|f\|_2^2 \leq \sum |\langle f, \psi_{m,n}\rangle|^2 \leq B\,\|f\|_2^2$ for all $f \in L^2(\mathbf{R})$. After investigating the general properties of abstract frames in Hilbert spaces, the construction and basic properties of wavelet frames and their windowed Fourier transform analogues is discussed, and applications to time-frequency localization are investigated in detail. This theme is continued in Chapter 4, which includes a proof of the important Balian–Low theorem: if an orthonormal basis is constructed for $L^2(\mathbf{R})$ using discrete windowed Fourier transform techniques then the basis elements must have poor time-frequency concentration. The remarkable clarity of this book is evidenced by the fact that this proof is the only place where I felt that Daubechies may have committed an oversight: at one point a partial derivative is taken of a function which need not be continuous. While everything can be worked out in terms of distributional derivatives, this is not a topic the reader is expected to be familiar with.

The discussion next turns to wavelet orthonormal bases $\{\psi_{m,n}\}$. Classically, the only known constructions were the Haar wavelet $\psi = \chi_{[0,1/2)} - \chi_{[1/2,0)}$ and the Littlewood-Paley wavelet $\hat{\psi} = (2\pi)^{-1/2}\chi_{[\pi,2\pi)}$, both of which have poor time-frequency localization. However, while all bases constructed via the windowed Fourier transform suffer the Balian–Low phenomena, Meyer produced in 1985 a surprising construction of an orthonormal basis $\{\psi_{m,n}\}$ with a Schwartz-class ψ. Although Meyer's original proof depended on "miraculous" cancellations, the need for miracles was eliminated when Mallat and Meyer developed multiresolution analysis. A multiresolution analysis consists of a nested family of subspaces $0 \leftarrow \cdots \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots \rightarrow L^2(\mathbf{R})$ such that $f(x) \in V_j$ if and only if $f(2^j x) \in V_0$ and for which there exists a "scaling function" φ such that $\{\varphi(x-n)\}_{n\in\mathbf{Z}}$ forms an orthonormal basis for V_0. These structures are the topic of Chapter 5; each multiresolution analysis leads to a wavelet ψ which generates an orthonormal basis for $L^2(\mathbf{R})$. For example, with φ as the box function $\chi_{[0,1)}$ we obtain the Haar wavelet ψ. All wavelet orthonormal bases of current practical interest, including the Meyer wavelet, are associated with multiresolution analyses (there do exist pathological constructions which are not).

Note that $\varphi \in V_0 \subset V_{-1}$; therefore φ must satisfy a two-scale difference equation $\varphi(x) = \sqrt{2}\sum h_n\,\varphi(2x-n)$ for some coefficients $\{h_n\}$. The 2π-periodic function $m_0(\xi) = (1/\sqrt{2})\sum h_n\, e^{-in\xi}$ plays a key role in the study of wavelet orthonormal bases. In particular, the orthonormality requirement leads to the condition $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$ a.e. This in turn leads to a connection with digital signal processing: m_0 is one of a quadrature mirror filter pair for a subband coding scheme with exact reconstruction. Daubechies clearly explains this relationship between wavelets and signal processing, without presuming any engineering expertise on the part of the reader. Although each multiresolution analysis determines one of these fast (order n) algorithms, many standard subband coding schemes are not constructed in this manner. Daubechies discusses some of the advantages or tradeoffs involved in using schemes derived from multiresolution analyses rather than other methods.

For many applications it is desirable to have a wavelet orthonormal basis such that ψ is both smooth and compactly supported and possesses several vanishing moments. The Haar wavelet is discontinuous and the Littlewood-Paley, Meyer, and related spline-wavelet constructions are infinitely supported. One of Daubechies' greatest contributions was the construction of smooth compactly supported ψ which generate wavelet bases; this is the topic of Chapters 6 and 7. Although the compact support requirement limits the smoothness to C^n, Daubechies' first (and now standard) construction was a family of scaling functions and wavelets ${}_N\phi$, ${}_N\psi$ which generate orthonormal wavelet bases, have compact support, and increase linearly in regularity and number of vanishing moments as N increases (Daubechies modestly ignores the increasingly popular notations D_{2N}, W_{2N} for this family). The price for additional smoothness is increased support size: ${}_N\psi$ is supported in an interval of length $2N-1$. These wavelets, and the other variants described, cannot be given in closed form and typically exhibit a "fractal" structure, e.g., the last continuous derivative is Hölder continuous with exponent strictly less than one. The exact regularity of these wavelets, both in a local and global sense, is an interesting problem which is explored in detail. Despite the lack of closed-form formulas, these wavelets are easily generated on a computer to any desired degree of accuracy. The methods for doing so are closely related to subdivision or refinement schemes used in computer-aided graphics, another connection that Daubechies clearly illustrates.

Chapter 8 deals with variations in the construction of wavelet bases, motivated by the fact that real-valued compactly supported wavelets ψ which generate orthonormal bases cannot be symmetric or anti-symmetric (with the exception of the Haar system). The same is true of the scaling function φ and the coefficients h_n. This is a major drawback for signal processing algorithms. Thus one topic of discussion is how to design m_0 so that φ is "almost" symmetric yet still will generate a multiresolution analysis. Alternatively, the orthonormality requirement can be relaxed: the subband coding scheme is not significantly complicated if $\{\psi_{m,n}\}$ is only a Riesz basis for $L^2(\mathbf{R})$ rather than an orthonormal basis. In this case symmetry can be achieved by a construction involving two interlocking multiresolution analyses: one corresponding to $\{\psi_{m,n}\}$ and another generating a dual, biorthogonal Riesz basis $\{\tilde{\psi}_{m,n}\}$. The analysis is now considerably more complicated, and the two wavelets ψ and $\tilde{\psi}$ may have very different regularity properties. Other tradeoffs include the distribution of vanishing moments between φ and ψ in the orthonormal case, or between ψ and $\tilde{\psi}$ in the biorthogonal case.

Chapter 9 is the last chapter of mathematics, and it is here that we leave the $L^2(\mathbf{R})$ Hilbert space setting for the first time. However, the focus is still on wavelet orthonormal bases, for these turn out to have valuable properties in spaces other than $L^2(\mathbf{R})$. In particular, this chapter discusses Meyer's proofs that these wavelet bases are also unconditional bases for many popular function spaces, including $L^p(\mathbf{R})$ for $1 < p < \infty$, Sobolev spaces, and Hölder spaces. More details can be found in Meyer's volumes [M], which will soon be translated into English.

The final chapter sketches some generalizations and recent advances. Topics include one-dimensional wavelet bases with other integer or non-integer dilation factors, multidimensional wavelet bases, wavelets on intervals, and wavelet packet constructions with increased frequency resolution. Only the outlines are discussed here, but at this point the reader can move on to the books [BF], [C2], [M], [R] or to the literature for more details or for applications of wavelets to specific problems. Also, [C1] contains an introductory treatment of some material only briefly mentioned by Daubechies, including spline-wavelets and wavelet packets.

In summary, this is a clearly written introduction to the mathematics of wavelets which provides solid background material on most of the major aspects of the current theory. Especially appealing is the way in which the relationships between wavelets and other areas are pointed out. This text would be well-suited for an introductory graduate-level course on wavelets; with some careful thought on the part of the instructor it could also be used at the advanced undergraduate level. I feel certain that this will be the major introductory text on wavelets for some time to come. It will definitely be a welcome addition to the library of anyone interested in learning the basics of wavelets.

References

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