THE BALIAN–LOW THEOREM FOR SYMPLECTIC LATTICES IN HIGHER DIMENSIONS

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Abstract. The Balian–Low theorem expresses the fact that time-frequency concentration is incompatible with non-redundancy for Gabor systems that form orthonormal or Riesz bases for $L^2(\mathbb{R})$. We extend the Balian–Low theorem for Riesz bases to higher dimensions, obtaining a weak form valid for all sets of time-frequency shifts which form a lattice in $\mathbb{R}^{2d}$, and a strong form valid for symplectic lattices in $\mathbb{R}^{2d}$. For the orthonormal basis case, we obtain a strong form valid for general non-lattice sets which are symmetric with respect to the origin.

1. Introduction

The Balian–Low theorem (BLT) is a key result in time-frequency analysis. It expresses the fact that time-frequency concentration and non-redundancy are incompatible properties for Gabor systems. Specifically, if for some $\alpha > 0$ and $g \in L^2(\mathbb{R})$ the set $\{e^{2\pi i \ell x/\alpha} g(x - k\alpha)\}_{(k,\ell)\in\mathbb{Z}^2}$ is an orthonormal basis for $L^2(\mathbb{R})$, then

$$\left(\int_{\mathbb{R}} |xg(x)|^2\, dt\right)\left(\int_{\mathbb{R}} |\omega\hat{g}(\omega)|^2\, d\omega\right) = \infty. \tag{1}$$

In other words, the window function g maximizes the uncertainty principle in some sense. This result was originally stated by Balian [2], and independently by Low [17]. It is only one of many examples of the fact that stability (in the form of basis properties) and good time-frequency localization cannot be simultaneously achieved.

The proofs given by Balian [2] and Low [17] each contained a gap, which was later filled by Coifman, Daubechies, and Semmes [7], who also extended the BLT to the case of Riesz bases. Battle [3] provided an elegant and entirely new proof based on the canonical commutation relations of quantum mechanics and thus demonstrated the intimate connection of the BLT to the classical uncertainty principle. Battle's proof was adapted by Daubechies and Janssen [8] to provide another proof of the BLT for Riesz bases. For historical comments and variations on the BLT we refer to [5]. Some more recent developments not reported there include the following. Zeevi and Zibulski [20] proved that BLT phenomena also appear in the multi-window setting. Balan [1] extended the BLT to the case of "superframes." A BLT variation for symplectic lattices in $\mathbb{R}^d$ (distinct from our results, and quoted as Theorem 12 below) was proved in [10]. Remarks on the BLT on locally compact abelian groups appear in [12].

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Recent results related to the optimality of the BLT were obtained by Benedetto, Czaja, Gadziński, and Powell [4].

In this note we extend the BLT (1) to higher dimensions and to more general sets of time-frequency shifts, especially lattices in time-frequency space. While the existing proofs of the BLT extend easily to the case of "rectangular" lattices of time-frequency shifts of the form $\alpha\mathbb{Z}^d \times \frac{1}{\alpha}\mathbb{Z}^d$, those proofs do not directly generalize to more general sets of time-frequency shifts. As Gabor systems using non-rectangular lattices are now being used in applications such as wireless coding, e.g., [19], it is important to understand whether and how the BLT extends to this setting. We obtain in this paper a weak form of the BLT for Gabor Riesz bases that is valid for all sets of time-frequency shifts which form a lattice in $\mathbb{R}^{2d}$, and a strong form valid for symplectic lattices in $\mathbb{R}^{2d}$. In particular, every lattice in $\mathbb{R}^2$ is a symplectic lattice (this is not the case when $d > 1$). Additionally, for the orthonormal basis case we extend the BLT to include even non-lattice sets of time-frequency shifts, requiring only that the set be symmetric with respect to the origin.

It follows from the Wexler–Raz theorem that there do exist windows g which are well-localized in time and frequency and which generate Gabor systems that are Riesz bases for their closed spans within $L^2(\mathbb{R})$ (but not for all of $L^2(\mathbb{R})$). For example, this is the case for the Gaussian window on the lattice $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ with $\alpha\beta > 1$. We show that the weak BLT for general lattices has an extension to such subspace Gabor systems. While not a *localization* restriction as such, this does shed some light on the nature of such systems.

A direct proof of the BLT for symplectic lattices would be awkward and difficult. The key ingredient in our approach is the observation that equation (1) expresses the fact that the window g does not belong to a certain modulation space. These spaces are the appropriate spaces for time-frequency analysis and appear in many different contexts, see [13] for examples. Our reformulation is yet another instance where the modulation spaces are crucial to the formulation and verification of time-frequency properties. We combine the machinery of the metaplectic representation with the invariance properties of the modulation spaces to obtain a simple and elegant approach to proving the BLT in these contexts.

Our paper is organized as follows. In Section 2 we introduce some useful concepts of time-frequency analysis, such as the metaplectic representation and the modulation spaces. In Section 3 we first generalize the weak version of the BLT introduced in [5] to lattices in higher dimensions. We then show how this result can be improved when the set of time-frequency shifts is a symplectic lattice in higher dimensions. Finally, we observe that for the orthonormal case, even the lattice requirement can be relaxed, and close with some open questions.

2. Some Concepts of Time-Frequency Analysis

In discussing lattices, Gabor systems, Gabor frames, and the metaplectic representation, we follow the definitions and notation of [13]. In particular, we write $x^2 = x \cdot x = \sum_{j=1}^d x_j^2$ for $x \in \mathbb{R}^d$ and $|x| = (x \cdot x)^{1/2}$ for the Euclidean norm on $\mathbb{R}^d$, and use the Fourier transform $\hat{f}(\omega) = \int f(x)\, e^{-2\pi i \omega \cdot x}\, dx$.

2.1. **Lattices.** A lattice Λ of $\mathbb{R}^d$ is a discrete subgroup with compact quotient. Equivalently, there exists a matrix $A \in GL(d, \mathbb{R})$ such that $\Lambda = A\mathbb{Z}^d$. The *volume* of such a lattice is

$\mathrm{vol}(\Lambda) = |\det(A)|$. The *dual lattice* of $K = A\mathbb{Z}^d$ is $K^\perp = \{x \in \mathbb{R}^d : e^{2\pi i x\cdot k} = 1\,\forall\, k \in K\} = (A^{-1})^*\mathbb{Z}^d$.

In the context of the BLT, we will deal with *time-frequency* lattices, which are lattices in $\mathbb{R}^{2d}$. Frequently only *separable lattices* of the form $A\mathbb{Z}^d \times B\mathbb{Z}^d \subseteq \mathbb{R}^{2d}$, where $A, B \in GL(d, \mathbb{R})$, will be considered. Among these, *product lattices* of the form $K \times K^\perp$ are often important.

2.2. **Time-Frequency Shifts and Gabor Systems.** For $x, \omega \in \mathbb{R}^d$, we define $T_x f(t) = f(t-x)$ and $M_\omega f(t) = e^{2\pi i \omega\cdot t} f(t)$ to be the unitary operators of translation and modulation. Writing $z = (x, \omega) \in \mathbb{R}^{2d}$ for a point in the time-frequency plane $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$, we denote the corresponding time-frequency shift by

$$\pi(z)f(t) \;=\; M_\omega T_x f(t) \;=\; e^{2\pi i \omega\cdot t} f(t-x).$$

Given a function $g \in L^2(\mathbb{R}^d)$, called a *window function*, and a lattice Λ in the time-frequency plane $\mathbb{R}^{2d}$, the corresponding *Gabor system* is

$$\mathcal{G}(g, \Lambda) \;=\; \{\pi(\lambda)g\}_{\lambda\in\Lambda}.$$

If $\mathcal{G}(g, \Lambda)$ is a *frame* for its closed span $H = \overline{\mathrm{span}}\{\pi(\lambda)g)\}_{\lambda\in\Lambda}$ in $L^2(\mathbb{R}^d)$, i.e., there exist A, $B > 0$ such that

$$\forall\, f \in H, \quad A\,\|f\|_2^2 \le \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \le B\,\|f\|_2^2,$$

then the associated *Gabor frame operator* is

$$S_{g,\Lambda} f \;=\; \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle\, \pi(\lambda)g.$$

This is a positive, invertible operator of H onto itself. The canonical *dual window* is $\gamma = S_{g,\Lambda}^{-1} g \in H$, and the canonical *dual frame* is the Gabor system $\mathcal{G}(\gamma, \Lambda)$. We have the *frame expansions*

$$\forall\, f \in H, \quad f \;=\; \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle\, \pi(\lambda)g \;=\; \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle\, \pi(\lambda)\gamma. \tag{2}$$

We recall the density theorem for Gabor frames in this setting. The following proposition is a consequence of the more general results proved in [18, 6].

Proposition 1. *If $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$, then* $\mathrm{vol}(\Lambda) \le 1$. *If $\mathcal{G}(g, \Lambda)$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then* $\mathrm{vol}(\Lambda) = 1$.

See [5, 6] for complete historical discussions of Proposition 1. In the lattice setting of this paper, Proposition 1 can be improved to say that $\mathcal{G}(g, \Lambda)$ cannot even be complete when $\mathrm{vol}(\Lambda) > 1$ [14]. It is not difficult to construct Gabor frames or Riesz bases $\mathcal{G}(g, \Lambda)$ such that Λ is not a lattice or a translate of a lattice in $\mathbb{R}^{2d}$, and generalizations of Proposition 1 can be formulated for these "irregular" Gabor frames. However, the frame hypothesis cannot be relaxed to a completeness hypothesis when Λ is not a lattice (see [5, Thm. 2.6] for a counterexample). It is shown in [16] that there even exist *orthonormal* bases $\mathcal{G}(g, \Lambda)$ such that Λ is not a translate of a lattice. In Theorem 8 below, we formulate a version of the BLT that applies to irregular Gabor orthonormal bases.

2.3. **Symplectic Lattices and Operators.** In time-frequency analysis, compositions of the symmetric time-frequency shifts $M_{\omega/2}T_xM_{\omega/2}$ often occur, and the symplectic form $[\cdot,\cdot]$ defined by

$$[(x_1,\omega_1),\,(x_2,\omega_2)] \;=\; x_2\cdot\omega_1 - x_1\cdot\omega_2, \qquad (x_1,\omega_1),(x_2,\omega_2)\in\mathbb{R}^{2d},$$

then plays an important role, cf. [13, Ch. 9]. The *symplectic group* $\mathrm{Sp}(d)$ is the group of all matrices $M \in GL(2d,\mathbb{R})$ that leave the symplectic form $[\cdot,\cdot]$ invariant, i.e., $M \in \mathrm{Sp}(d)$ satisfies

$$[Mx, My] \;=\; [x,y] \quad \text{for all } x,y\in\mathbb{R}^{2d}.$$

As a consequence of the Stone–von Neumann theorem, a symplectic transformation $M \in \mathrm{Sp}(d)$ corresponds to a unitary *symplectic operator* $\mu(M)$ on $L^2(\mathbb{R}^d)$ which satisfies

$$\pi(Mz) \;=\; \mu(M)\pi(z)\mu(M)^{-1} \qquad \text{for all } z\in\mathbb{R}^{2d}.$$

We refer to [11] and [13] for details about the construction of this metaplectic representation. In the context of time-frequency analysis, the following lattices play an important role.

Definition 2. A lattice $\Lambda \subseteq \mathbb{R}^{2d}$ is a *symplectic lattice* if

$$\Lambda \;=\; \alpha M\mathbb{Z}^{2d} \quad \text{for some } \alpha\in\mathbb{R}\backslash\{0\} \text{ and } M\in\mathrm{Sp}(d).$$

Note that if M is symplectic, then $|\det(M)| = 1$, so $\mathrm{vol}(\alpha M\mathbb{Z}^{2d}) = |\alpha|$.

Since $\mathrm{Sp}(1) = SL(2,\mathbb{R})$, every lattice in $\mathbb{R}^2$ is a symplectic lattice. However, this is not the case when $d > 1$. All product lattices are symplectic. If a symplectic lattice $\alpha M\mathbb{Z}^{2d}$ is separable, then $M\mathbb{Z}^{2d}$ is a product lattice.

The next proposition, taken from [13, Prop. 9.4.4], shows how statements for Gabor systems on rectangular lattices may be transferred to general symplectic lattices.

Proposition 3. *Let $\Lambda = \alpha M\mathbb{Z}^{2d}$ be a symplectic lattice, and let $\mathcal{G}(g,\Lambda)$ be a Gabor system such that the Gabor frame operator $S_{g,\Lambda}$ is bounded on $L^2(\mathbb{R}^d)$. Then the Gabor system and Gabor frame operator on $\alpha\mathbb{Z}^{2d}$ and on Λ are related by*

$$\mathcal{G}(g,\Lambda) \;=\; \mu(M)\;\mathcal{G}(\mu(M)^{-1}g,\,\alpha\mathbb{Z}^{2d})$$

and

$$S_{g,\Lambda} \;=\; \mu(M)\;S_{\mu(M)^{-1}g,\alpha\mathbb{Z}^{2d}}\;\mu(M)^{-1}.$$

2.4. **Modulation Spaces.** The modulation spaces quantify the time-frequency content of a function or distribution. They are defined by means of the short-time Fourier transform (or a similar time-frequency representation). Let $g \in \mathcal{S}(\mathbb{R}^d)$ be a non-zero Schwartz function. Then the *short-time Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)'$* with respect to the (fixed) window g is

$$V_gf(x,\omega) \;=\; \int_{\mathbb{R}^d} f(t)\,\overline{g(t-x)}\,e^{-2\pi i\omega\cdot t}\,dt \;=\; \langle f, M_\omega T_x g\rangle.$$

For our purpose the following special cases of the modulation spaces will be sufficient.

Definition 4. Let $v(z) \geq 1$ be a submultiplicative weight function on $\mathbb{R}^{2d}$ with at most polynomial growth. Then the modulation space M_v^p, where $1 \leq p \leq \infty$, is defined as the subspace of all $f \in \mathcal{S}(\mathbb{R}^d)'$ such that the norm

$$\|f\|_{M_v^p} := \left(\int_{\mathbb{R}^{2d}} |V_g f(z)|^p \, v(z)^p \, dz \right)^{1/p}$$

is finite, with the usual adjustment if $p = \infty$. If $v \equiv 1$, we write M^p for M_v^p.

It can be shown that M_v^p is a Banach space, and that different window functions $g \in \mathcal{S}(\mathbb{R}^d)$ yield equivalent norms for M_v^p.

For the BLT, the following identifications with standard function spaces are especially relevant, cf. [13, Prop. 11.3.1]. L_s^2 denotes the weighted L^2-space with norm $\int |f(t)|^2 \, (1 + |t|^2)^s \, dt$, and H^s denotes the Bessel potential space with norm $\int |\hat{f}(\omega)|^2 \, (1 + |\omega|^2)^s \, d\omega$.

Lemma 5. (a) *If $v(x, \omega) = (1 + |x|^2)^{s/2}$, then $M_v^2 = L_s^2$.*

(b) *If $v(x, \omega) = (1 + |\omega|^2)^{s/2}$, then $M_v^2 = H^s$.*

The weights that we shall use are

$$m(x, \omega) = (1 + |x|^2 + |\omega|^2)^{1/2} \quad \text{and} \quad m_j(x, \omega) = (1 + |x_j|^2 + |\omega_j|^2)^{1/2}, \quad j = 1, \dots, d.$$

Lemma 5 implies that $M_m^2 = L_1^2 \cap H^1$.

We will need the following special case of the invariance properties of the modulation spaces [9, Thm. 29], cf. also [13, Prop. 12.1.3] for the case $p = 1$.

Proposition 6. *If $M \in Sp(d)$, then the symplectic operator $\mu(M)$ is an isomorphism of M_m^p onto itself for each $1 \leq p \leq \infty$.*

Using Lemma 5 we can reformulate the one-dimensional BLT (1) in terms of a modulation space.

Theorem 7. *If the Gabor system $\mathcal{G}(g, \alpha\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z})$ is an orthonormal basis for $L^2(\mathbb{R})$, then $g \notin M_m^2$.*

Proof. Since we are given that g, $\hat{g} \in L^2(\mathbb{R})$, it follows that $g \in M_m^2$ if and only if (1) fails. $\square$

3. The Balian-Low Theorem

In the following we let

$$X_j f(x) \;=\; x_j f(x) \qquad \text{and} \qquad P_j f \;=\; \frac{1}{2\pi i} \frac{\partial f}{\partial x_j} \;=\; (X_j \hat{f})^\vee$$

for $j = 1, \dots, d$ denote the usual position and momentum operators. Note that

$$\|X_j f\|_2^2 \;=\; \int_{\mathbb{R}^d} |x_j f(x)|^2 \, dx \qquad \text{and} \qquad \|P_j f\|_2^2 \;=\; \int_{\mathbb{R}^d} |\omega_j \hat{f}(\omega)|^2 \, d\omega.$$

Consequently, if $f \in L^2(\mathbb{R}^d)$, then

$$f \in M_{m_j}^2 \quad \Longleftrightarrow \quad \|X_j f\|_2 \, \|Q_j f\|_2 < \infty$$

and

$$f \in M^2_m \quad \iff \quad \left(\int_{\mathbb{R}^d} \bigl(|x|\,|g(x)|\bigr)^2\,dx\right)\left(\int_{\mathbb{R}^d} \bigl(|\omega|\,|\hat{g}(\omega)|\bigr)^2\,d\omega\right) < \infty.$$

3.1. **The Weak Subspace BLT for Arbitrary Lattices.** In this section we formulate a weak version of the BLT. This result is valid for any lattice Λ in $\mathbb{R}^{2d}$ and also applies to Gabor systems which are only Riesz bases for their closed spans in $L^2(\mathbb{R}^d)$.

Theorem 8. *Let Λ be a lattice in $\mathbb{R}^{2d}$. If $g \in L^2(\mathbb{R}^d)$ is such that $\mathcal{G}(g,\Lambda)$ is a Riesz basis for its closed span $H = \overline{\operatorname{span}}\{\pi(\lambda)g\}_{\lambda\in\Lambda}$ in $L^2(\mathbb{R}^d)$ and the dual window is $\gamma = S^{-1}_{g,\Lambda}g$, then for each $j = 1,\dots,d$, one of X_jg, P_jg, $X_j\gamma$, or $P_j\gamma$ cannot lie in H.*

In particular, if $\mathcal{G}(g,\Lambda)$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then

(a) *for each $j = 1,\dots,d$, either $g \notin M^2_{m_j}$ or $\gamma \notin M^2_{m_j}$, and*

(b) *either $g \notin M^2_m$ or $\gamma \notin M^2_m$.*

Proof. The proof is an extension of Battle's argument, so we only sketch the details. Assume that X_jg, P_jg, $X_j\gamma$, $P_j\gamma \in H$. We can compute that for any $(p,q) \in \mathbb{R}^d$ we have

$$\langle X_jg, M_qT_p\gamma\rangle = \langle T_{-p}M_{-q}g, X_j\gamma\rangle \qquad \text{and} \qquad \langle M_qT_pg, P_j\gamma\rangle = \langle P_jg, T_{-p}M_{-q}\gamma\rangle.$$

Then, using the frame expansions (2), we have that

$$\begin{aligned}\langle X_jg, P_j\gamma\rangle &= \Bigl\langle \sum_{(p,q)\in\Lambda} \langle X_jg, M_qT_p\gamma\rangle\, M_qT_pg,\ P_j\gamma \Bigr\rangle \\ &= \sum_{(p,q)\in\Lambda} \langle T_{-p}M_{-q}g, X_j\gamma\rangle\, \langle P_jg, T_{-p}M_{-q}\gamma\rangle \\ &= \Bigl\langle P_jg,\ \sum_{(p,q)\in\Lambda} \langle X_j\gamma, M_qT_pg\rangle\, M_qT_p\gamma \Bigr\rangle = \langle P_jg, X_j\gamma\rangle.\end{aligned}$$

However, the canonical commutation relation $[X_j, P_j] = -\frac{1}{2\pi i}I$ leads to the contradiction

$$1 \;=\; \langle g,\gamma\rangle \;=\; 2\pi i\bigl(\langle P_jg, X_j\gamma\rangle - \langle X_jg, P_j\gamma\rangle\bigr) \;=\; 0. \quad \square$$

3.2. **The BLT for Symplectic Lattices.** In this subsection we will obtain a strong BLT for symplectic lattices. For the proof we combine the machinery of the metaplectic representation with Theorem 9.

First, we observe that the result for product lattices follows directly from the weak BLT.

Theorem 9. *Let $\Lambda = K \times K^\perp$ be a product lattice in $L^2(\mathbb{R}^{2d})$. If $g \in L^2(\mathbb{R}^d)$ is such that $\mathcal{G}(g,\Lambda)$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then*

(a) *$g \notin M^2_{m_j}$ for EACH $j = 1,\dots,d$, and*

(b) *$g \notin M^2_m$.*

Proof. We can use the same arguments as in [5, Sec. 7.3] to show that for each $j = 1,\dots,d$,

$$P_jg \in L^2(\mathbb{R}^d) \iff P_jS^{-1}_{g,\Lambda}g \in L^2(\mathbb{R}^d) \quad \text{and} \quad X_jg \in L^2(\mathbb{R}^d) \iff X_jS^{-1}_{g,\Lambda}g \in L^2(\mathbb{R}^d). \quad \square$$

Now we can extend to the case of symplectic lattices.

Theorem 10. *Let* Λ *be a symplectic lattice in* $\mathbb{R}^{2d}$. *If* $g \in L^2(\mathbb{R}^d)$ *is such that* $\mathcal{G}(g,\Lambda)$ *is a Riesz basis for* $L^2(\mathbb{R}^d)$, *then*
(a) $g \notin M^2_{m_j}$ *for SOME* $j = 1, \dots, d$, *and*
(b) $g \notin M^2_m$.

Proof. Since Λ is a symplectic lattice, there exists some $\alpha \in \mathbb{R}\backslash\{0\}$ and some $M \in \mathrm{Sp}(d)$ such that $\Lambda = \alpha M \mathbb{Z}^{2d}$. By the density theorem (Proposition 1), we must have $\alpha = 1$.
Set $\tilde{g} = \mu(M^{-1})g \in L^2(\mathbb{R}^d)$. Then Proposition 3 implies that

$$\mathcal{G}(g,\Lambda) \;=\; \mu(M)\mathcal{G}(\tilde{g},\mathbb{Z}^{2d}).$$

Since $\mu(M)$ is unitary, the Gabor system $\mathcal{G}(\tilde{g},\mathbb{Z}^{2d})$ is also a Riesz basis for $L^2(\mathbb{R}^d)$. The BLT for product lattices (Theorem 9) therefore implies that $\tilde{g} \notin M^2_m$. By the invariance property of the modulation spaces (Proposition 6), we conclude that $g = \mu(M)\tilde{g} \notin M^2_m$, and thus statement (b) holds. Finally, statement (a) follows from the fact that $M^2_m = \cap_{j=1}^d M^2_{m_j}$. □

3.3. **The BLT on Non-Lattices.** The assumption of lattice structure is not essential to the definition of a Gabor frame. In particular, if Λ is any countable sequence of points in $\mathbb{R}^{2d}$, then $\mathcal{G}(g,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ if $\sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2$ is an equivalent norm for $L^2(\mathbb{R})$. Unfortunately, if Λ is not a lattice then although a dual frame $\{h_\lambda\}_{\lambda\in\Lambda}$ will exist, it need not be a Gabor frame of the form $\mathcal{G}(\gamma,\Lambda)$. However, for the case of a so-called normalized tight frame, including orthonormal bases in particular, the dual frame coincides with the frame. In this case, we can observe that the proof of Theorem 8 requires no structural assumptions on Λ except that it be symmetric about the origin. Hence we obtain the following.

Theorem 11. *Let* Λ *be a countable sequence in* $\mathbb{R}^{2d}$ *such that* $\Lambda = -\Lambda$. *If* $g \in L^2(\mathbb{R}^d)$ *is such that* $\mathcal{G}(g,\Lambda)$ *is an orthonormal basis for* $L^2(\mathbb{R}^d)$, *then*
(a) $g \notin M^2_{m_j}$ *for each* $j = 1, \dots, d$, *and*
(b) $g \notin M^2_m$.

3.4. **Remarks and Open Questions.** It is instructive to compare Theorem 10 to the following BLT variation obtained in [10].

Theorem 12. *Let* Λ *be a symplectic lattice in* $\mathbb{R}^{2d}$. *If* $g \in L^2(\mathbb{R}^d)$ *is such that* $\mathcal{G}(g,\Lambda)$ *is a Riesz basis for* $L^2(\mathbb{R}^d)$, *then* $g \notin M^1$.

Theorems 10 and 12 are distinct (neither implies the other), because M^1 is not embedded into M^2_m, nor conversely.
Let (C_0, ℓ^1) denote the Wiener amalgam space

$$(C_0,\ell^1) \;=\; \Big\{ f : f \text{ is continuous and } \sum_{k\in\mathbb{Z}^d} \|f\cdot\chi_{Q+k}\|_\infty < \infty \Big\},$$

where $Q = [0,1)^d$. Because M^1 is embedded into (C_0,ℓ^1), we have for the case $\Lambda = \alpha\mathbb{Z}^d \times \frac{1}{\alpha}\mathbb{Z}^d$ that Theorem 12 is implied by the following result known as the *Amalgam BLT* [15].

Theorem 13. *If* $g \in L^2(\mathbb{R}^d)$ *is such that* $\mathcal{G}(g, \alpha\mathbb{Z}^d \times \frac{1}{\alpha}\mathbb{Z}^d)$ *is a Riesz basis for* $L^2(\mathbb{R}^d)$, *then* $g, \hat{g} \notin (C_0,\ell^1)$.

Since M^2_m is not embedded into (C_0, ℓ^1) nor conversely, Theorem 10 (for the case $\Lambda = \alpha\mathbb{Z}^d \times \frac{1}{\alpha}\mathbb{Z}^d$) is distinct from Theorem 13.

The proof of Theorem 12 relies on the fact that M^1 is invariant under symplectic operators. It is unknown whether (C_0, ℓ^1) is invariant under such operators, and it is an open question whether the Amalgam BLT extends to more general lattices than $\alpha\mathbb{Z}^d \times \frac{1}{\alpha}\mathbb{Z}^d$.

Finally, we observe that some of the most natural lattices in $\mathbb{R}^{2d}$ are the separable lattices. If a separable lattice with unit volume is symplectic, then it is a product lattice. Every lattice in $\mathbb{R}^2$ is symplectic, but this is not the case in $\mathbb{R}^{2d}$ when $d > 1$. It is an open question as to whether the BLT extends to the case of separable, non-product lattices in higher dimensions.

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