International Journal of Wavelets, Multiresolution and Information Processing

Smoothness of refinable function vectors on $\mathbb{R}^n$

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Received (Day Month Year)
Revised (Day Month Year)
Accepted (Day Month Year)
Published (Day Month Year)

Let A be a dilation matrix, an $n \times n$ expansive matrix that maps $\mathbb{Z}^n$ into itself. Let Λ be a finite subset of $\mathbb{Z}^n$, and for $k \in \Lambda$ let c_k be $r \times r$ complex matrices. The refinement equation corresponding to A, $\mathbb{Z}^n$, Λ, and $c = \{c_k\}_{k\in\Lambda}$ is $f(x) = \sum_{k\in\Lambda} c_k\, f(Ax - k)$. A solution $f\colon \mathbb{R}^n \to \mathbb{C}^r$, if one exists, is called a refinable vector function or a vector scaling function of multiplicity r. This paper characterizes the higher-order smoothness of compactly supported solutions of the refinement equation, in terms of the p-norm joint spectral radius of a finite set of finite matrices determined by the coefficients c_k.

Keywords: Cascade Algorithm; dilation matrix; joint spectral radius; multiresolution analysis; multiwavelets; nonseparable wavelets; refinement equations; refinable functions; scaling functions; self-similarity; wavelets.

AMS Subject Classification: Primary 39A10; Secondary 39B62

1. Introduction

Wavelet bases for $L^2(\mathbb{R})$, which consist of translates and dilations of a single "wavelet," possess excellent time-frequency localization properties. For this reason, wavelet bases have proved useful for numerous applications in mathematics and engineering, including signal processing, image compression, and numerical analysis (for example, see Refs. 5 or 21). In the classical setting, the construction of the wavelet basis begins with a refinable function, which is a solution to the refinement equation $f(x) = \sum_{k\in\mathbb{Z}} c_k\, f(2x - k)$. The continuity of such functions can be

characterized in terms of the joint spectral radius of a certain family of associated matrices; we refer to Ref. 4 for background and references.

The smoothness of the wavelet is an important consideration in applications such as image processing and computer graphics. For example, quoting Mallat (see Sec. 7.2 of Ref. 18), "For image coding applications, a smooth error is often less visible than an irregular error, even though they have the same energy." In one dimension, there are many constructions of smooth wavelets, starting with the classical compactly supported Daubechies wavelets. However, for image and video processing, we need to work in higher dimensions, where there is considerably less known both in terms of theory and actual constructions.

In higher dimensions the traditional approach is through tensor products of one-dimensional wavelets. This produces a variety of drawbacks, including horizontal and vertical biases and a lack of design freedom (for illustrations of application where directional sensitivity is important, see, e.g., Refs. 1, 9). One way to counteract these deficiencies is to use multiple generating functions, which leads to a multiwavelet basis consisting of translates and dilates of several functions. By combining this with dilation by a matrix rather than uniform dilation by 2 we can also address issues with directional biases. Such non-tensor product wavelets have been used in a variety of applications; for example, see Refs. 23, 24, 25 for some recent two and three dimensional applications.

In this paper we study smooth solutions to the refinement equation

$$f(x) = \sum_{k\in\Lambda} c_k\, f(Ax-k), \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where A is a dilation matrix (an $n \times n$ expansive matrix that maps $\mathbb{Z}^n$ into itself), Λ is a finite subset of $\mathbb{Z}^n$, the c_k are $r \times r$ complex matrices, and $f\colon \mathbb{R}^n \to \mathbb{C}^r$ is vector-valued. A solution f, if one exists, is called a refinable vector function of multiplicity r. The existence of continuous solutions to such vector refinement equations was studied in Ref. 4, but here we are concerned with higher-order smoothness of compactly supported solutions to the refinement equation. Some initial results in this direction were obtained in Ref. 12. Here we characterize the higher-order smoothness of refinable vector functions in terms of the p-norm joint spectral radius of a finite set of finite matrices determined by the coefficients c_k.

2. Notation

Most computations in this manuscript are independent of the choice of norm on $\mathbb{R}^n$; if not specifically stated then the norm is taken to be taken to be the Euclidean norm on $\mathbb{R}^n$.

The transpose of a matrix or vector B is B^T, while we let B^* denote the conjugate transpose of B.

We use a generalized matrix notation which allows matrices or vectors to be indexed by arbitrary countable sets. If desired, such generalized matrices can always be realized as ordinary matrices by choosing a specific ordering for the index set.

The actual ordering used is not important as long as the same ordering is used consistently. To be precise, let J and K be finite or countable index sets. Let $m_{j,k}$ be $r \times s$ matrices for $j \in J$ and $k \in K$. Then we say $M = [m_{j,k}]_{j\in J,k\in K} \in (\mathbb{C}^{r\times s})^{J\times K}$ is a $J \times K$ matrix (with $r \times s$ block entries). If $N = [n_{k,l}]_{k\in K,l\in L} \in (\mathbb{C}^{s\times t})^{K\times L}$, then the product of the $J \times K$ matrix M with the $K \times L$ matrix N is the $J \times L$ matrix formally defined by $MN = \left[\sum_{k\in K} m_{j,k}n_{k,l}\right]_{j\in J,l\in L}$.

The *Kronecker product* of an $m\times n$ matrix C and a $p\times q$ matrix D is the $mp\times nq$ matrix $C \otimes D$ defined in block form by

$$C \otimes D = [c_{ij}D]_{\substack{i=1,\dots,m\\ j=1,\dots,n}} = \begin{bmatrix} c_{11}D & \cdots & c_{1n}D \\ \vdots & \ddots & \vdots \\ c_{m1}D & \cdots & c_{mn}D \end{bmatrix}.$$

A "column vector" is a $J \times 1$ matrix, which we will denote by $v = [v_j]_{j\in J}$. The entries v_j may be scalars or $r \times s$ blocks. In particular,

$$\mathbb{C}^r = \mathbb{C}^{r\times 1} = \left\{ \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} : u_1, \dots, u_r \in \mathbb{C} \right\}$$

is the space of column vectors of length r. Analogously, a "row vector" is a $1 \times J$ matrix, which we will denote by $u = (u_j)_{j\in J}$. In particular, $\mathbb{C}^{1\times r}$ is the space of all row vectors of length r, i.e.,

$$\mathbb{C}^{1\times r} = \{u^T : u \in \mathbb{C}^r\} = \{(u_1, \dots, u_r) : u_1, \dots, u_r \in \mathbb{C}\}.$$

The support of a vector-valued function $g = (g_1, \dots, g_r)^T : \mathbb{R}^n \to \mathbb{C}^n$ is the closure of $\{x \in \mathbb{R}^n : g(x) \neq 0\}$. Integrals of g are computed componentwise. In particular, if g is integrable then we define its Fourier transform by

$$\hat{g}(w) = \int_{\mathbb{R}^n} g(x)e^{-2\pi ix\cdot w}dx = \left(\int_{\mathbb{R}^n} g_1(x)e^{-2\pi ix\cdot w}dx, \dots, \int_{\mathbb{R}^n} g_r(x)e^{-2\pi ix\cdot w}dx\right)^T,$$

where $x \cdot w$ is the usual inner product.

3. Preliminaries

3.1. *Standing Assumptions*

We assume throughout that the refinement Eq. (1.1) is fixed. We assume $A\mathbb{Z}^n \subseteq \mathbb{Z}^n$ and A is expansive, i.e., every eigenvalue λ satisfies $|\lambda| > 1$. For simplicity of notation, throughout this paper we let

$$B = A^T.$$

The lattice $\mathbb{Z}^n$ is chosen for simplicity; all of our results generalize easily to dilation matrices that map a full-rank lattice Γ in $\mathbb{R}^n$ into itself.

Since $A\mathbb{Z}^n \subseteq \mathbb{Z}^n$, the dilation matrix A necessarily has integer determinant. We set

$$m = |\det(A)|,$$

and let $D = \{d_1, d_2, \ldots, d_m\}$ be fixed full set of digits with respect to A, i.e., a full set of representatives of $\mathbb{Z}^n/A(\mathbb{Z}^n)$. Then $\mathbb{Z}^n$ is partitioned into the disjoint cosets

$$\Gamma_d = A(\mathbb{Z}^n) - d = \{Ak - d : k \in \mathbb{Z}^n\}, \qquad d \in D. \tag{3.1}$$

Without loss of generality, we may assume that $0 \in D$.

The refinement operator associated with the refinement equation is the operator S defined on vector-valued functions $g\colon \mathbb{R}^n \to \mathbb{C}^r$ by

$$Sg(x) = \sum_{k\in\Lambda} c_k g(Ax - k).$$

A refinable function is a fixed point of S. The cascade algorithm associated with the refinement equation is the iteration of the operator S given by $f^{i+1}(x) = Sf^i(x)$.

3.2. *Attractors and Tiles*

For $k \in \mathbb{Z}^n$, let $w_k : \mathbb{R}^n \to \mathbb{R}^n$ be the affine map $w_k(x) = A^{-1}(x + k)$. Given a finite subset $H \subseteq \mathbb{Z}^n$ the attractor of the iterated function system (IFS) generated by $\{w_k\}_{k\in H}$ is the unique nonempty compact set K_H that satisfies[11]

$$A^{-1}(K_H + H) = K_H.$$

K_H can also be expressed as

$$K_H = \sum_{j=1}^{\infty} A^{-j}(H). \tag{3.2}$$

In this article, the attractors K_D and K_Λ will be of particular interest. We note that any compactly supported solution f of the refinement Eq. (1.1) must satisfy[4, 13]

$$\operatorname{supp}(f) \subseteq K_\Lambda.$$

We say that Q is a tile if $Q + \mathbb{Z}^n = \mathbb{R}^n$ and Q and $Q + k$ only intersect on a set of Lebesgue measure zero, i.e., $|Q \cap (Q + k)| = 0$ for $k \in \mathbb{Z}^n \setminus \{0\}$. Equivalent conditions for Q to be a tile can be found in Ref. 6; in particular, Q is a tile if and only if $|Q| = 1$.

From now on, we denote the attractor K_D by

$$Q = K_D.$$

If $n \le 3$ or $m > n$ then there always exists a full set of digits such that $Q = K_D$ is a tile.[15–17] If $n \ge 4$ then there do exist dilation matrices A for which no choice of digit set D is $Q = K_D$ a tile.[19] However, such matrices are rare in some sense, and therefore throughout this article we will assume that $Q = K_D$ is a tile. Since K_Λ is compact and Q is a tile, there exists a finite set $\Omega \subseteq \mathbb{Z}^n$ such that $K_\Lambda \subseteq Q + \Omega$.

3.3. *A-nary Expansions*

By applying Eq. (3.2) to the attractor $Q = K_D$, we see that each point $x \in Q$ has an A-nary expansion

$$x = .\varepsilon_1\varepsilon_2\cdots = \sum_{j=1}^{\infty} A^{-j}\varepsilon_j$$

with $\varepsilon_j \in D$ for each j. That this expansion may not be unique.

It will be useful to define a function $\tau\colon Q \to Q$ that represents a left-shift of this A-nary expansion. Since Q is the attractor of the IFS generated by $\{w_k\}_{k\in D}$, we have

$$Q = A^{-1}(Q + D) = \bigcup_{d\in D} w_{d_i}(Q).$$

Therefore each point $x \in Q$ satisfies $x \in w_{d_i}(Q)$ for at least one $d_i \in D$. If $x \in w_{d_i}(Q)$ for a unique digit d_i, then we define

$$\tau x = w_{d_i}^{-1}(x) = Ax - d_i.$$

In this case we have $d_i = \varepsilon_1$ and $\tau x = .\varepsilon_2\varepsilon_3\cdots$. However, x might belong to more than one $w_{d_i}(Q)$. To eliminate the ambiguity of this definition for other points in Q, we partition Q into the disjoint subsets $Q_1, \ldots, Q_m$ given by

$$Q_1 = w_{d_1}(Q) = A^{-1}(Q + d_1),$$

$$Q_i = w_{d_i}(Q) \setminus \left(\bigcup_{j=1}^{i-1} Q_j\right), \qquad i = 2, \ldots, m.$$

Then $Q = \bigcup_{i=1}^{m} Q_i$ and each $x \in Q$ lies in a unique Q_i. We use this i to define $\tau x = w_{d_i}^{-1}(x) = Ax - d_i$. Now τ is well-defined.

3.4. *Matrix Form of the Refinement Equation*

Let $g\colon \mathbb{R}^n \to \mathbb{C}^r$ be any function such that $\operatorname{supp}(g) \subseteq K_\Lambda$. Define the *folding* of g to be the function $\Phi\colon Q \to (\mathbb{C}^{r\times 1})^{\Omega\times 1}$ given by

$$\Phi g(x) = [g(x + k)]_{k\in\Omega}, \qquad x \in Q.$$

For each $d \in D$, we define the $\Omega \times \Omega$ matrix $T_d = [c_{Aj-k+d}]_{j,k\in\Omega}$. We define an operator T acting on functions $u\colon Q \to (\mathbb{C}^{r\times 1})^{\Omega\times 1}$ by

$$Tu(x) = \sum_{i=1}^{m} \chi_{Q_i}(x)\, T_{d_i} u(Ax - d_i).$$

In other words,

$$Tu(x) = T_{d_i} u(\tau x) \quad \text{if } x \in Q_i.$$

As shown in Ref. 4 (Prop. 2.13), for any $g\colon \mathbb{R}^n \to \mathbb{C}^r$ with $\operatorname{supp}(g) \subseteq K_\Lambda$ satisfying $g(x) = 0$ for $x \in \partial K_\Lambda$, the refinement operator can be written in the equivalent matrix form

$$\Phi S g = T\Phi g.$$

3.5. *Generalized Polynomial Vectors*

We use the standard multiindex notation $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$. The degree or order of α or x^α is $\deg(x^\alpha) = |\alpha| = \alpha_1 + \cdots + \alpha_n$. If $s \geq 0$ then the number of multiindices of degree s is $d_s = \binom{s+n-1}{n-1}$. If $\beta_i \leq \alpha_i$ for each i (i.e., $\beta \leq \alpha$), then we set $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$, otherwise $\binom{\alpha}{\beta} = 0$.

Let $X_{[s]}$ denote the vector of all monomials of degree s with respect to some fixed ordering of the multiindices:

$$X_{[s]} = [x^\alpha]_{|\alpha|=s}.$$

Given an $n \times n$ matrix $M = [m_{ij}]_{i,j=1,\ldots,n}$ and given $s \geq 0$, we let $M_{[s]} = [m^s_{\alpha,\beta}]_{|\alpha|=s,|\beta|=s}$ be the $d_s \times d_s$ matrix whose entries are defined by the equation

$$\sum_{|\beta|=s} m^s_{\alpha,\beta} x^\beta = (Mx)^\alpha = \prod_{i=1}^n (m_{i,1}x_1 + \cdots + m_{i,n}x_n)^{\alpha_i}. \tag{3.3}$$

Then we have $X_{[s]}(Mx) = M_{[s]}X_{[s]}(x)$.

Given a collection $\{v_\alpha = (v_{\alpha,1}, \ldots, v_{\alpha,r}) : 0 \leq |\alpha| < \kappa\}$ of row vectors of length r, we group these vectors together by degree to form an associated $d_s \times r$ matrix $v_{[s]}$ defined by

$$v_{[s]} = [v_\alpha]_{|\alpha|=s}.$$

We also associate to each multiindex α the row vector of polynomials $y_\alpha \colon \mathbb{R}^n \to \mathbb{C}^{1\times r}$ defined by

$$y_\alpha(x) = \sum_{0\leq\beta\leq\alpha} (-1)^{|\alpha|-|\beta|} \binom{\alpha}{\beta} v_\beta \, x^{\alpha-\beta}, \tag{3.4}$$

and then define the $d_s \times r$ matrix of polynomials

$$y_{[s]}(x) = [y_\alpha(x)]_{|\alpha|=s}.$$

We also collect the matrices $y_{[s]}(x+k)$ into a row to form the function $Y_{[s]} \colon \mathbb{R}^n \to (\mathbb{C}^{d_s\times r})^{1\times\mathbb{Z}^n}$ given by

$$Y_{[s]}(x) = (y_{[s]}(x+k))_{k\in\mathbb{Z}^n}.$$

The matrix $Y_{[s]}(0)$ is the row vector formed by evaluating the matrix of polynomials $y_{[s]}(x)$ at the lattice points in $\mathbb{Z}^n$.

3.6. *The Joint Spectral Radius*

The joint spectral radius of a finite set of square matrices $\mathcal{A} = \{A_1, \ldots, A_k\}$ is

$$\hat{\rho}_\infty(\mathcal{A}) = \lim_{\ell\to\infty} \max_{\Pi\in\mathcal{P}_\ell} \|\Pi\|^{1/\ell}, \tag{3.5}$$

where $\mathcal{P}_\ell$ is the set of all matrix products of ℓ elements of $\mathcal{A}$. Specifically $\mathcal{P}_0 = \{I\}$ and $\mathcal{P}_\ell = \{\mathcal{A}_{j_1} \cdots \mathcal{A}_{j_\ell} : j_i = 1, \ldots, k\}$. This definition is independent of the choice

of the norm $\|\cdot\|$. It was conjectured by Daubechies and Lagarias[8] and later proved by Berger and Wang[2] that (3.5) can be restated as

$$\hat{\rho}_\infty(\mathcal{A}) = \lim_{\ell\to\infty} \max_{\Pi\in\mathcal{P}_\ell} \rho(\Pi)^{1/\ell} \tag{3.6}$$

where $\rho(\Pi)$ is the usual spectral radius of the matrix Π.

The joint spectral radius was first introduced by Rota and Strang.[20] It was rediscovered independently by Daubechies and Lagarias,[7] who used it to analyze refinable functions $f\colon \mathbb{R}\to\mathbb{C}$ with dilation factor 2. They developed conditions on the joint spectral radius of two specific matrices that determine the differentiability of a scaling function in this one-dimensional setting. Since then the joint spectral radius has found a wide range of applications, including coding and graph theory as well as wavelet theory. We refer to Ref. 14 for a recent survey of the joint spectral radius, including techniques for computing or approximating it and extensive references.

4. Existence of Solution to the Refinement Equation

4.1. *Accuracy of a Refinable Function*

The *accuracy* of a refinable function $f\colon \mathbb{R}^n\to\mathbb{C}^r$ is the largest integer $\kappa > 0$ such that every multivariate polynomial $q(x)$ with $\deg(q) < \kappa$ can be written

$$q(x) = \sum_{k\in\mathbb{Z}^n} a_k f(x+k) = \sum_{k\in\mathbb{Z}^n}\sum_{i=1}^{r} a_{k,i} f_i(x+k) \text{ a.e.}, \qquad x\in\mathbb{R}^n,$$

for some row vectors $a_k = (a_{k,1},\ldots,a_{k,r}) \in \mathbb{C}^{1\times r}$. If no polynomials are reproducible from translates of f then we set $\kappa = 0$. We say that translates of f along Γ are linearly independent if $\sum_{k\in\Gamma} a_k f(x+k) = 0$ implies $a_k = 0$ for each k. The following result[4] states the minimal accuracy conditions for a compactly supported solution of the refinement equation.

Theorem 4.1. *Let $f\colon \mathbb{R}^n\to\mathbb{C}^r$ be a compactly supported solution to the refinement Eq.* (1.1)*. Let $\{\Gamma_d\}_{d\in D}$ be the cosets defined in* (3.1)*. If there exists a $1\times r$ row vector u_0 such that $u_0\hat{f}(0)\neq 0$ and*

$$u_0 = \sum_{k\in\Gamma_d} u_0 c_k \quad \textit{for each } d\in D, \tag{4.1}$$

then f has accuracy $\kappa\geq 1$ and $\sum_{k\in\mathbb{Z}^n} u_0 f(x+k) = 1$ a.e.

For higher-order accuracy, we have the following characterization.[3] Here we use the notation introduced in Section 3.5.

Theorem 4.2. *Assume that $f\colon \mathbb{R}^n\to\mathbb{C}^r$ is a compactly supported solution to the refinement Eq.* (1.1)*. Let $L = [c_{Aj-k}]_{j,k\in\mathbb{Z}^n}$. If there exists a set of $1\times r$ vectors $\{v_\alpha : 0\leq|\alpha|<\kappa\}$ such that*

(i) $v_0\hat{f}(0) \neq 0$, *and*
(ii) $Y_{[s]}(0) = A_{[s]}Y_{[s]}(0)L$ *for* $0 \leq s < \kappa$,

then f *has accuracy* κ. *Furthermore, after scaling each of the vectors* v_α *by an appropriate constant,*

$$X_{[s]}(x) = \sum_{k\in\mathbb{Z}^n} y_{[s]}(k)f(x+k) = Y_{[s]}(0)F(x), \qquad 0 \leq s < \kappa, \tag{4.2}$$

where $F(x) = [f(x+k)]_{k\in\mathbb{Z}^n}$.

It follows from (4.2) that the vectors $y_\alpha(k)$ are exactly those needed to reproduce the monomials from translates of f:

$$x^\alpha = \sum_{k\in\mathbb{Z}^n} y_\alpha(k)f(x+k), \qquad 0 \leq |\alpha| < \kappa.$$

By (3.4), the coefficients $y_\alpha(k)$ have the form of polynomials $y_\alpha(x)$ evaluated at $k \in \mathbb{Z}^n$.

If higher-order accuracy conditions are satisfied, a significant structure is imposed on the matrices T_d. Assume that the conditions for accuracy κ given in Theorem 4.2 are satisfied. In particular $u_0 \neq 0$, and therefore the vector of polynomials y_α defined by (3.4) has degree $|\alpha|$. We define the finite row vectors

$$e_\alpha = (y_\alpha(k))_{k\in\Omega} \in (\mathbb{C}^{1\times r})^{1\times\Omega}, \qquad 0 \leq |\alpha| < \kappa, \tag{4.3}$$

formed by restricting the infinite row vectors $(y_\alpha(k))_{k\in\mathbb{Z}^n}$ to components whose indices lie in Ω. It is shown in Ref. 4 (see Lem. 3.16) that if a compactly supported solution to the refinement Eq. (1.1) exists, then these vectors are linearly independent.

For each $0 \leq s < \kappa$, we define U_s to be the subspace of $(\mathbb{C}^{1\times r})^{1\times\Omega}$ given by

$$U_s = \operatorname{span}\{e_\alpha : 0 \leq |\alpha| \leq s\}.$$

Since the set $\{e_\alpha : 0 \leq |\alpha| \leq s\}$ is linearly independent, it forms a basis for U_s. We define the dot product of two column vectors $u = [u_k]_{k\in\Omega}$ and $v = [v_k]_{k\in\Omega} \in (\mathbb{C}^{1\times r})^{1\times\Omega}$ by

$$u \cdot v = u^*v = \sum_{k\in\Omega} u_k^*\, v_k = \sum_{k\in\Omega}\sum_{i=1}^{r} \overline{u_{k,i}}\, v_{k,i},$$

where u^* is the conjugate transpose of u. Then for a row vector u and a column vector v, $uv = u^* \cdot v$. Define V_s to be the orthogonal complement of U_s^*:

$$V_s = \{e_\alpha^* : 0 \leq |\alpha| \leq s\}^\perp = \{v \in (\mathbb{C}^{1\times r})^{1\times\Omega} : e_\alpha v = 0 \text{ for } 0 \leq |\alpha| \leq s\}. \tag{4.4}$$

The following theorem[4] shows that the matrices $\{T_d\}_{d\in D}$ can be simultaneously block triangularized.

Theorem 4.3. *Let* $\Omega \subseteq \mathbb{Z}^n$ *be a finite set such that* $K_\Lambda \subseteq Q + \Omega$. *Assume there exist row vectors* $\{v_\alpha : 0 \leq |\alpha| \leq s\}$ *such that the conditions of Theorem 4.2 are*

satisfied. Let the vectors e_α *be given by* (4.3) *and assume that these vectors are linearly independent. Let* $V_s \subseteq (\mathbb{C}^{1\times r})^{1\times\Omega}$ *be given by* (4.4). *Let* $\{\tilde{e}_\alpha : 0 \le |\alpha| \le s\}$ *be the result of applying the Gram-Schmidt orthogonalization procedure to* $\{e_\alpha : 0 \le |\alpha| \le s\}$. *Let* $\mathcal{B}_V$ *be an orthonormal basis for* $V_{\kappa-1}$. *Then*

$$\mathcal{B} = \{\tilde{e}_\alpha : 0 \le |\alpha| \le s\} \cup \mathcal{B}_V$$

is an orthonormal basis for $(\mathbb{C}^{1\times r})^{1\times\Omega}$, *and for each* $d \in D$, *the matrix* T_d *in this basis has the block form*

$$[T_d]_{\mathcal{B}} = \begin{bmatrix} B_0 & & & 0 \\ & B_1 & & \\ & & \ddots & \\ & & & B_{\kappa-1} \\ * & & & C_d \end{bmatrix}, \tag{4.5}$$

where each B_s *is similar to* $A_{[s]}^{-1}$, *and* $C_d = [T_d|_{V_{\kappa-1}}]_{\mathcal{B}_V}$.

4.2. *Convergence of Cascade Algorithm*

This section addresses the convergence of the cascade algorithm. The following theorem[4] guarantees the convergence of the cascade algorithm, and furthermore, for a certain set of starting functions, ensures the uniform convergence of the cascade algorithm to a continuous solution.

Theorem 4.4. *Let* $\Omega \subseteq \mathbb{Z}^n$ *be a finite set such that* $K_\Lambda \subseteq Q + \Omega$. *Assume that there exists a nonzero vector* $u_0 \in \mathbb{C}^{1\times r}$ *such that* (4.1) *holds. Let* $e_0 = (u_0)_{k\in\Omega} \in (\mathbb{C}^{1\times r})^{1\times\Omega}$, *and define*

$$V_0 = \{e_0^*\}^\perp = \{v \in (\mathbb{C}^{r\times 1})^{\Omega\times 1} : e_0 v = \sum_{k\in\Omega} u_0 v_k = 0\}. \tag{4.6}$$

Set

$$I_0 = \Big\{g \in C(\mathbb{R}^n, \mathbb{C}^r) : \mathrm{supp}(g) \subseteq K_\Lambda \text{ and } \sum_{k\in\mathbb{Z}^n} u_0 g(x+k) = 1 \text{ a.e.}\Big\} \tag{4.7}$$

If $I_0 \neq \emptyset$ *and* $\hat{\rho}_\infty(\{T_d|_{V_0}\}_{d\in D}) < 1$, *then for any initial* $f^0 \in I_0$ *the cascade algorithm converges uniformly to a continuous solution* f *of the refinement equation.*

As seen above, the sufficient condition for the cascade algorithm to converge depends on the set I_0 and the computation of $\hat{\rho}_\infty(\{T_d|_{V_0}\}_{d\in D})$. Some auxiliary facts are provided below.

The set I_0 defined by (4.7) is determined by two quantities: the set Λ and the row vector u_0. Since any continuous function supported in K_Λ must be zero on the boundary of K_Λ, it is sufficient to study the question of when the set

$$I(\Lambda, u_0) = \Big\{g \in L^\infty(\mathbb{R}^n, \mathbb{C}^r) : g(x) = 0 \text{ for } x \notin K_\Lambda^\circ \text{ and } \sum_{k\in\Gamma} u_0 g(x+k) = 1\Big\}$$

contains a continuous function. The following result[4] shows that $I(\Lambda, u_0)$ contains such a function exactly when it contains any functions at all. Further, whether $I(\Lambda, u_0)$ is nonempty or not is independent of the value of u_0.

Lemma 4.1. *Let $\Lambda \subseteq \Gamma$ be finite and let $u_0 \in \mathbb{C}^{1\times r}$ be nonzero. Then the following statements are equivalent.*

(a) $I(\Lambda, u_0) \neq \emptyset$
(b) $I(\Lambda, u_0)$ *contains a continuous function.*
(c) $K_\Lambda^\circ + \Gamma = \mathbb{R}^n$, *i.e., lattice translates of* K_Λ° *cover* $\mathbb{R}^n$.

If the coefficients $\{c_k\}_{k\in\mathbb{Z}^n}$ satisfy higher-order accuracy conditions, then Theorem 4.3 can be used to reduce the computation of $\hat\rho_\infty(\{T_d|_{V_0}\}_{d\in D})$ to computing the joint spectral radius on a smaller subspace of V_0.

Corollary 4.1. *Let $\Omega \subseteq \mathbb{Z}^n$ be a finite set such that $K_\Lambda \subseteq Q+\Omega$. Assume that there exist row vectors $\{v_\alpha : 0 \leq |\alpha| \leq s\}$ such that the conditions of Theorem 4.2 are satisfied. Let the vectors e_α be given by* (4.3) *and assume these vectors are linearly independent. Let $V_s \subseteq (\mathbb{C}^{1\times r})^{1\times\Omega}$ be given by* (4.4). *Then*

$$\hat\rho_\infty(\{T_d|_{V_0}\}_{d\in D}) = \max\big(\rho(A^{-1}),\, \hat\rho_\infty(\{T_d|_{V_{\kappa-1}}\}_{d\in D})\big).$$

5. Refinement Equation for Derivatives of a Refinable Function

Assume that $f \in \mathbb{C}^\nu(\mathbb{R}^n; \mathbb{C}^r)$ is a smooth vector-valued function that satisfies the refinement Eq. (1.1). It was proved in Ref. 12, that the derivatives of f of order $0 \leq s \leq \nu$ are refinable and can be expressed in terms of the refinement coefficients of f. As the proof has not appeared outside of Ref. 12, we include it here for completeness as the next Proposition.

Given s with $s \leq \nu$, each component f_i of f has d_s partial derivatives of order s. We write

$$D_{[s]}f_i = \left[\frac{\partial^s f_i}{\partial x^\alpha}\right]_{|\alpha|=s}, \qquad 0 \leq s \leq \nu,\ 1 \leq i \leq r.$$

Then $D_{[s]}f\colon \mathbb{R}^n \to (\mathbb{C}^{d_s\times 1})^{r\times 1}$ is given by $D_{[s]}f = \begin{bmatrix} D_{[s]}f_1 \\ \vdots \\ D_{[s]}f_r \end{bmatrix}$.

Proposition 5.1. *Let $f\colon \mathbb{R}^n \to \mathbb{C}^r$ be refinable with coefficients $\{c_k\}_{k\in\mathbb{Z}^n}$, and assume $f \in \mathbb{C}^\nu(\mathbb{R}^n)$. For each $0 \leq s \leq \nu$, the function $D_{[s]}f$ satisfies the refinement equation*

$$D_{[s]}f = \sum_{k\in\Lambda} (c_k \otimes B_{[s]})\, (D_{[s]}f)(Ax - k).$$

Proof. Let $a_1, \ldots, a_n$ be the columns of A and $b_1, \ldots, b_n$ the rows of $B = A^{\mathrm{T}}$, so we have $a_j = b_j^{\mathrm{T}}$. Consider the first-order partial derivatives of $f_i(Ax-k)$:

$$\begin{aligned} D_j\big(f_i(Ax-k)\big) &= \sum_{\ell=1}^{n} (D_\ell f_i)(Ax-k)\, a_{\ell j} \\ &= a_j^{\mathrm{T}} \begin{bmatrix} (D_1 f_i)(Ax-k) \\ \vdots \\ (D_n f_i)(Ax-k) \end{bmatrix} = b_j \left(D_{[1]} f_i\right)(Ax-k), \end{aligned}$$

where $D_j = \frac{\partial}{\partial x_j}$. Since $f \in \mathbb{C}^\nu(\mathbb{R}^n)$, the partial derivative operators commute. Therefore, if $|\alpha| = s$ then

$$\begin{aligned} \frac{\partial^s f_i(Ax-k)}{\partial x^\alpha} &= D_1^{\alpha_1} \cdots D_n^{\alpha_n} \Big(f_i(Ax-k)\Big) \\ &= \Big((b_1 D_{[1]})^{\alpha_1} \cdots (b_n D_{[1]})^{\alpha_n} f_i\Big)(Ax-k) \\ &= \Big((b_{11}D_1 + \cdots + b_{1n}D_n)^{\alpha_1} \cdots (b_{n1}D_1 + \cdots + b_{nn}D_n)^{\alpha_n} f_i\Big)(Ax-k) \\ &= \Big((BD_{[1]})^\alpha f_i\Big)(Ax-k) = \sum_{|\beta|=s} b_{\alpha,\beta}^s \frac{\partial^s f_i}{\partial x^\beta}. \end{aligned}$$

where the last equality comes from (3.3). Combining all derivatives of f_i of order s gives the relationship

$$D_{[s]}\Big(f_i(Ax-k)\Big) = B_{[s]}\Big((D_{[s]} f_i)(Ax-k)\Big). \tag{5.1}$$

For each $k \in \mathbb{Z}^d$, let $c_k = \{c_k^{\ell,j}\}_{\ell\, j=1}^r$ where $c_k^{\ell,j}$ is the ℓ, j-th entry of c_k. The refinement equation then can be rewritten as $f_i(x) = \sum_{k\in\Lambda} \left[\sum_{j=1}^r c_k^{i,j} f_j(Ax-k)\right]$. Substituting Eq. (5.1) into this yields

$$D_{[s]} f_i(x) = \sum_{k\in\Lambda} \left[\sum_{j=1}^r c_k^{i,j} B_{[s]} (D_{[s]} f_j)(Ax-k)\right],$$

and therefore

$$D_{[s]} f(x) = \sum_{k\in\Lambda} (c_k \otimes B_{[s]})(D_{[s]} f)(Ax-k).$$

□

As seen above, the coefficients $\{c_k \otimes B_{[s]}\}_{k\in\Lambda}$ will play an important role in dealing the derivative of a refinable function.

6. Existence of a Differentiable Solution to the Refinement Equation

For any compact set $K \subseteq \mathbb{R}^n$, the space $C^\nu(K, \mathbb{C}^r)$ is a Banach space.[22] This leads to the following lemma which will be useful in considering the convergence of the cascade algorithm for the derivatives.

Lemma 6.1. *Let $K \subseteq \mathbb{R}^n$ be compact. Suppose that $f^i \in C^\nu(K, \mathbb{C}^r)$ and f^i converges uniformly to f. If for each $|\alpha| = \nu$ there exists a g_α such that $\frac{\partial^\nu f^i}{\partial x_\alpha} \to g_\alpha$, then $f \in C^\nu(K, \mathbb{C}^r)$ and $g_\alpha = \frac{\partial^\nu f^i}{\partial x_\alpha}$.*

Using the refinement coefficient for the derivative $D_{[s]}$, we define matrices $\{T_d^s\}_{d\in D}$ analogously to the matrices T_d:

$$T_d^s = \left[\, c_{Aj-k+d} \otimes B_{[s]} \,\right]_{j,k\in\Omega} = T_d \otimes B_{[s]}. \tag{6.1}$$

It is these matrices that will be of interest when considering the convergence of the cascade algorithm for the derivative.

The proof of Theorem 3.17 in Ref. 4 shows that for each $d \in D$ and $0 \le s \le \kappa$ we have

$$[e_\alpha T_d]_{|\alpha|=s} = A_{[s]}^{-1}[e_\alpha]_{|\alpha|=s} + A_{[s]}^{-1} \sum_{t=0}^{s-1} Q_{[s,t]}(-d)[e_\beta]_{|\beta|=s},$$

where

$$Q_{[s,t]}(y) = (-1)^{s-t} \left[\binom{\alpha}{\beta} y^{\alpha-\beta} \right]_{|\alpha|=s,|\beta|=s}.$$

This leads to the triangularization in Eq. (4.5), and is useful for obtaining a similar triangularization form for $\{T_d^s\}_{d\in D}$. This form will be useful in deriving the conditions similar to those of Theorem 4.4 for the convergence of the cascade algorithm to a ν-times continuously differentiable solution.

It is necessary to find the right class of starting functions for the cascade algorithm. If the accuracy conditions corresponding to $\kappa = \nu + 1$ are satisfied, then for $0 \le |\alpha| \le \nu$ we have $e_\alpha \Phi f(x) = x^\alpha$ for $x \in Q$. Therefore, for $x \in Q$ and $|\beta| = \nu$,

$$e_\alpha \Phi \frac{\partial^\nu f(x)}{\partial x^\beta} = \begin{cases} \alpha!, & \text{if } \beta = \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Let I_ν be the $d_\nu \times d_\nu$ identity matrix, and let u_β be the row β of I_ν. Then for each $|\alpha| = |\beta| = \nu$,

$$(e_\alpha \otimes u_\beta)\Phi D_{[\nu]} f = e_\alpha \Phi \frac{\partial^\nu f(x)}{\partial x^\beta} = \alpha!\, \delta_{\alpha,\beta}. \tag{6.2}$$

where $\delta_{\alpha,\beta}$ is the Kronecker delta. It is necessary that the starting functions for the cascade algorithm satisfy Eq. (6.2).

In order to prove our main theorem, it is useful to state a more general form of Theorem 3.6.

Theorem 6.1. *Let $\nu \ge 1$. Let $\Omega \subseteq \mathbb{Z}^n$ be a finite set such that $K_\Lambda \subseteq Q + \Omega$. Let E be a nonempty closed subset of $((\mathbb{C}^{r\times 1})^{d_\nu \times 1})^{\Omega\times 1}$ such that $T_d^\nu(E) \subseteq E$ for each $d \in D$. Let V be subspace of $((\mathbb{C}^{r\times 1})^{d_\nu \times 1})^{\Omega\times 1}$ that contains $E - E$ and is right-invariant under each T_d^ν (defined in* (6.1)*). Define*

$$I_0 = \{g \in \mathbb{C}(\mathbb{R}^n, (\mathbb{C}^{r\times 1})^{d_\nu \times 1}) : \operatorname{supp}(g) \subseteq K_\Lambda \text{ and } \Phi_g(Q) \subseteq E\}.$$

If $I_0 \neq \emptyset$ and $\hat{\rho}_\infty\big(\{T_d|_V\}_{d\in D}\big) < 1$, then for any initial $f^0 \in I_0$, the cascade algorithm

$$f^{i+1} = \sum_{k\in\Lambda} (c_k \otimes B_{[\nu]}) f^i(Ax - k)$$

converges uniformly to a continuous solution f of the refinement equation.

Now we come to the main result.

Theorem 6.2. *Let $\nu \geq 1$. Let $\Omega \subseteq \mathbb{Z}^n$ be a finite set such that $K_\Lambda \subseteq Q+\Omega$. Assume that there exist $1 \times r$ row vectors $\{v_\alpha : 0 \leq |\alpha| \leq \nu+1\}$ such that the conditions of Theorem 4.2 hold. For each $0 \leq |\alpha| \leq \nu$, let the vectors $e_\alpha = (y_\alpha(k))_{k\in\Omega}$ be given by Eq. (4.3), and let $V_0 = \{e_0^*\}^\perp$ be as in Eq. (4.6). Let I_0 consist of all functions $g \in \mathbb{C}^\nu(\mathbb{R}^n, \mathbb{C}^r)$ such that:*

(a) $\operatorname{supp}(g) \subseteq K_\Lambda$,

(b) $\displaystyle\sum_{k\in\mathbb{Z}^n} u_0 g(x+k) = 1$,

(c) $\displaystyle\sum_{k\in\mathbb{Z}^n} y_\alpha(k) \frac{\partial^\nu g(x+k)}{\partial x^\beta} = \alpha!\, \delta_{\alpha,\beta}$ *for each* $0 \leq |\alpha| \leq \nu, |\beta| = \nu$,

(d) $(e_\alpha \otimes u_\beta) T_d^\nu = (e_\alpha \otimes u_\beta)$ *for each* $0 \leq |\alpha| \leq \nu, |\beta| = \nu$.

If $I_0 \neq \emptyset$ and

$$\hat{\rho}_\infty\big(\{T_d|_{V_0}\}_{d\in D}\big) < \frac{1}{\rho(A)^\nu}, \tag{6.3}$$

then for any initial $f^0 \in I_0$, the cascade algorithm converges uniformly to a ν-times continuously differentiable solution f of the refinement equation.

Proof.

Let $f^0 \in I_0$.

Claim 1 $f^{i+1} = \sum c_k f^i(Ax-k)$ converges uniformly to a continuous function f.

To see why, recall that A is expansive, so $\rho(A) > 1$. Consequently, Eq. (6.3) implies that $\hat{\rho}_\infty\big(\{T_d|_{V_0}\}_{d\in D}\big) < \frac{1}{\rho(A)^\nu} < 1$. All of the conditions of Theorem 4.4 are therefore satisfied, so the convergence of the cascade algorithm to a continuous solution f is ensured. This establishes the claim.

Claim 2 Let $g^0 = D_\nu[f^0]$. Then $g^{i+1} = \sum (c_k \otimes B_{[\nu]})\, g^i(Ax-k)$ converges to a ν-differentiable solution g of $g(x) = \sum (c_k \otimes B_{[\nu]})\, g(Ax-k)$.

We will establish this claim by applying Theorem 6.1 using appropriate sets E and V. For each $0 \leq |\alpha| \leq \nu$, define e_α^ν by

$$e_\alpha^\nu = e_\alpha \otimes I_\nu \in ((\mathbb{C}^{d_\nu \times d_\nu})^{1\times r})^{1\times\Omega},$$

and let $e^\nu_{\alpha,\beta}$ be the collection of the d_ν row vectors e^ν_α:

$$\{e^\nu_{\alpha,\beta}\}_{|\beta|=\nu} = \{e_\alpha \otimes u_\beta\}_{|\beta|=\nu} \in ((\mathbb{C}^{1\times d_\nu})^{1\times r})^{1\times\Omega}.$$

Now set

$$\begin{aligned} V &= V^\nu_\nu \\ &= \{(e^\nu_{\alpha,\beta})^* : 0 \le |\alpha| \le \nu,\, |\beta| = \nu\}^\perp \\ &= \{v \in (((\mathbb{C}^{d_\nu\times 1})^{r\times 1})^{\Omega\times 1} : e^\nu_{\alpha,\beta} v = 0 \text{ for each } 0 \le |\alpha| \le \nu,\, |\beta| = \nu\} \end{aligned}$$

and

$$E = \{v \in (((\mathbb{C}^{1\times d_\nu})^{1\times r})^{1\times\Omega} : (e_\alpha \otimes u_\beta)v = \alpha!\,\delta_{\alpha,\beta} \text{ for each } 0 \le |\alpha| \le \nu, |\beta| = \nu\}.$$

In order to apply Theorem 6.1, we must show that the following conditions hold.

(i) $T^\nu_d(E) \subseteq E$.
(ii) V is a subspace such that $E - E \subseteq V$
(iii) V is right invariant under each T^ν_d
(iv) $\operatorname{supp}(g^0) \subseteq K_\Lambda$ and $\Phi_{g^0}(Q) \subseteq E$.
(v) $\hat\rho_\infty\big(\{T^\nu_d|_V\}_{d\in D}\big) < 1$.

Statement (ii) is clear from the definitions of V and E, while statement (iv) follows from assumptions (a), (b), and (c) and Eq. (6.2).

Proof of (i): By assumption (d), $(e_\alpha \otimes u_\beta)T^\nu_d = (e_\alpha \otimes u_\beta)$. Let $v \in E$. Then $(e_\alpha \otimes u_\beta)v = \alpha!\,\delta_{\alpha,\beta}$. Now $(e_\alpha \otimes u_\beta)T^\nu_d v = (e_\alpha \otimes u_\beta)v = \alpha!\,\delta_{\alpha,\beta}$, so $T^\nu_d(E) \subseteq E$.

Proof of (v): This is done by establishing the relationship between $\hat\rho_\infty(\{T^\nu_d|_V\}_{d\in D})$ and $\hat\rho_\infty(\{T_d|_{V_0}\}_{d\in D})$. In doing that, V_ν plays an intermediate role. Note that V_ν is given by (4.4) with $s = \nu$. We give the proof step by step.

Step 1. A useful property of Kronecker products[10] is that if $\{A_1, \dots, A_k\}$ are $n \times n$ matrices and $\{B_1, \dots, B_k\}$ are $m \times m$ matrices, then

$$(A_1 \cdots A_k) \otimes (B_1 \cdots B_k) = (A_1 \otimes B_1) \cdots (A_k \otimes B_k). \tag{6.4}$$

Using (6.4), it follows that for $v \in V_\nu$,

$$0 \otimes I_\nu = (e_\alpha v) \otimes (I_\nu I_\nu) = (e_\alpha \otimes I_\nu)(v \otimes I_\nu),$$

and therefore $V = V_\nu \otimes I_\nu$.

Step 2. Let $\mathcal{B}$ be the orthonormal basis for $(\mathbb{C}^{r\times 1})^{\Omega\times 1}$ given by Theorem 4.3. There exists a unitary matrix $\mathcal{U}$ formed by the vectors in $\mathcal{B}$ so that for each $d \in D$, U^*T_dU has the form given by Eq. (4.5). Let $\mathcal{B}^\nu = \{b \otimes I_\nu : b \in \mathcal{B}\}$, and correspondingly let $U^\nu = U \otimes I_\nu$. Then

$$\begin{aligned} (U^\nu)^* T^\nu_d U^\nu &= (U \otimes I_\nu)^*(T_d \otimes B_{[\nu]})(U \otimes I_\nu) \\ &= (U^* \otimes I_\nu)(T_d \otimes B_{[\nu]})(U \otimes I_\nu) \\ &= (U^*T_dU) \otimes (I_\nu B_{[\nu]} I_\nu) \qquad \text{by (6.4)} \\ &= [T_d]_{\mathcal{B}} \otimes B_{[\nu]}. \end{aligned}$$

Therefore, $[T_d^\nu]_{\mathcal{B}^\nu} = [T_d]_{\mathcal{B}} \otimes B_{[\nu]}$. Furthermore, $T_d^\nu|_V = T_d|_{V_\nu} \otimes B_{[\nu]}$.

Step 3: By the expression for the joint spectral radius given in (3.6),

$$\begin{aligned}
\hat{\rho}_\infty(\{T_d^\nu|_V\}_{d\in D}) &= \lim_{\ell\to\infty} \max_{\epsilon_i\in D} \rho(T_{\epsilon_1}^\nu|_V T_{\epsilon_2}^\nu|_V \cdots T_{\epsilon_\ell}^\nu|_V)^{1/\ell} \\
&= \lim_{\ell\to\infty} \max_{\epsilon_i\in D} \rho((T_{\epsilon_1}|_{V_\nu} \otimes B_{[\nu]}) \cdots (T_{\epsilon_\ell}|_{V_\nu} \otimes B_{[\nu]}))^{1/\ell} \\
&= \lim_{\ell\to\infty} \max_{\epsilon_i\in D} \rho((T_{\epsilon_1}|_{V_\nu} \cdots T_{\epsilon_\ell}|_{V_\nu}) \otimes B_{[\nu]}^\ell))^{1/\ell} \\
&= \lim_{\ell\to\infty} \max_{\epsilon_i\in D} \rho((T_{\epsilon_1}|_{V_\nu} \cdots T_{\epsilon_\ell}|_{V_\nu})\, \rho(B_{[\nu]}^\ell))^{1/\ell} \\
&= \rho(B_{[\nu]})\, \hat{\rho}_\infty(\{T_d|_{V_\nu}\}_{d\in D}).
\end{aligned}$$

If $\lambda = (\lambda_1, ..., \lambda_n)^T$ is the vector of the eigenvalues of A, and thus the eigenvalues of B, then the eigenvalues of $B_{[\nu]}$ are $\{\lambda^\alpha : |\alpha| = \nu\}$ (see Ref. 3). Therefore

$$\rho(B_{[\nu]}) = \max\{|\lambda^\alpha| : |\alpha| = \nu\} = (\rho(A))^\nu.$$

Furthermore, $\hat{\rho}_\infty(\{T_d|_{V_\nu}\}_{d\in D}) \le \hat{\rho}_\infty(\{T_d|_{V_0}\}_{d\in D})$ by Corollary 4.1. Consequently,

$$\hat{\rho}_\infty(\{T_d^\nu|_V\}_{d\in D}) \le (\rho(A))^\nu\, \hat{\rho}_\infty(\{T_d|_{V_0}\}_{d\in D}). \tag{6.5}$$

Step 4: If $\hat{\rho}_\infty(\{T_d|_{V_0}\}_{d\in D}) < \frac{1}{(\rho(A))^\nu}$, then by (6.5), $\hat{\rho}_\infty(\{T_d^\nu|_V\}_{d\in D}) < 1$. This completes the proof of Claim 2.

Claim 3 $f^{i+1} = \sum c_k f^i(Ax - k)$ converges to a ν-differentiable solution f of $f(x) = \sum c_k f(Ax - k)$.

To establish this final claim, observe that f_i converges uniformly to a continuous function f by Claim 1. Since $g^0 = D_{[\nu]} f^0$, the function g^i defined in Claim 2 satisfies $g^i = D_{[\nu]} f^i$ by Proposition 5.1. Claim 2 therefore implies that g^i converges uniformly to a continuous function g. By Lemma 6.1, it follows that $f \in C^\nu(K, \mathbb{C}^r)$ and $g = D_{[\nu]} f$. In other words, the cascade algorithm converges to a ν-times continuously differentiable solution of the refinement Eq. (1.1). □

Acknowledgements

The third author was partially supported by Simons Foundation grant #229035.

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