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Overcomplete Reproducing Pairs

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Abstract

The Gaussian Gabor system at the critical density has the property that it is overcomplete in $L^2(\mathbb{R})$ by exactly one element, and if any single element is removed then the resulting system is complete but is not a Schauder basis. This paper characterizes systems that are overcomplete by finitely many elements. Among other results, it is shown that if such a system has a reproducing partner, then it contains a Schauder basis. While a Schauder basis provides a strong reproducing property for elements of a space, the existence of a reproducing partner only requires a weak type of representation of elements. Thus for these systems weak representations imply strong representations. The results are applied to systems of weighted exponentials and to Gabor systems at the critical density. In particular, it is shown that the Gaussian Gabor system does not possess a reproducing partner.

Keywords Frames · Gabor systems · Reproducing pairs · Weighted exponentials · Zak transform

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Dedicated to Prof. Dr. Karlheinz Gröchenig on the occasion of his 65th birthday.

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1 Introduction

Frames are generalizations of orthonormal bases and Riesz bases. They were first introduced by Duffin and Schaeffer in their study of non-harmonic Fourier series [11]. A sequence of vectors $\{f_n\}_{n\geq 0}$ is a *frame* for a separable, infinite dimensional Hilbert space $\mathbb{H}$ if there exist constants $A,\ B > 0$ such that the following norm equivalence holds:

$$A\,\|f\|^2 \leq \sum_{n=0}^{\infty} |\langle f, f_n\rangle|^2 \leq B\,\|f\|^2, \qquad \text{for every } f \in \mathbb{H}.$$

We refer to A as a *lower frame bound* and B as an *upper frame bound* for $\{f_n\}_{n\geq 0}$. A frame is similar to an unconditional basis in that for every $f \in \mathbb{H}$ we have an unconditionally convergent expansion

$$f = \sum_{n=0}^{\infty} c_n f_n \tag{1}$$

for some scalars $(c_n)_{n\geq 0}$. However, for a frame the coefficients need not be unique. If the scalars c_n are unique for every f, then $\{f_n\}_{n\geq 0}$ is a *Riesz basis* for $\mathbb{H}$.

There are a variety of other notions related to the existence of representations of elements. A (strong) *Schauder basis* is a sequence $\{f_n\}_{n\geq 0}$ such that for each $f \in \mathbb{H}$ there exist unique scalars c_n such that equation (1) holds. A Schauder basis need not have either a lower or upper frame bound. Further, the representations in equation (1) may converge conditionally, although there must be a single fixed ordering of the index set with respect to which the series converge for every f. A *weak Schauder basis* is similar, except that we only require that the series in equation (1) converge weakly for every f. However, the *Weak Basis Theorem* implies that every weak Schauder basis is a Schauder basis.

Every Schauder basis is *exact*, or both minimal and complete. Complete means that the finite linear span is dense, while minimal means that no element f_m lies in the closed span of the other elements $\{f_n\}_{n\neq m}$. A sequence can be minimal without being exact. We refer to texts such as [8] and [20] for details on frames, Riesz bases, Schauder bases, and related systems.

Given a function $g \in L^2(\mathbb{R})$ and a countable index set $\Lambda \subseteq \mathbb{R}^2$, the *Gabor system* generated by g and Λ is

$$G(g, \Lambda) = \{M_\xi T_x g\}_{(x,\xi)\in\Lambda} = \{e^{2\pi i \xi t} g(t - x)\}_{(x,\xi)\in\Lambda},$$

where T_x is the translation operator $T_x g(t) = g(t - x)$ and M_ξ is the modulation operator $M_\xi g(t) = e^{2\pi i \xi t} g(t)$. The compositions $T_x M_\xi$ and $M_\xi T_x$ are *time-frequency shift operators*.

Usually the index set Λ contains some structure. For example, it may be a lattice $A(\mathbb{Z}^2)$ for some invertible matrix A, or a rectangular lattice $\alpha\mathbb{Z} \times \beta\mathbb{Z}$. In this paper, we will focus on the case where Λ is a rectangular lattice with density 1 (the *critical* *density*), which means that $\alpha\beta = 1$. By a change of variables, this can always be reduced to the case $\alpha = \beta = 1$.

The structure of Gabor frames makes them suitable for applications involving time-dependent frequency content. Hence, it is not unexpected that Gabor theory has a long history. Gröchenig [16] and Janssen [25] mention that von Neumann [32] claimed (without proof) that, for the Gaussian atom $\varphi(t) = 2^{1/4}e^{-\pi t^2}$ and the lattice $\Lambda = \mathbb{Z}^2$, the Gabor system $G(\varphi, \mathbb{Z}^2)$ is complete in $L^2(\mathbb{R})$. Additionally, Gabor conjectured in [13] that every function $f \in L^2(\mathbb{R})$ can be represented in the form

$$f = \sum_{k,n\in\mathbb{Z}} c_{nk}(f)\, M_n T_k \varphi, \tag{2}$$

for some scalars $c_{nk}(f)$. Later, Janssen [24] proved that Gabor's conjecture is true, but with convergence of the series in equation (2) only in the sense of tempered distributions and not in the norm of L^2. Further, the coefficients $c_{nk}(f)$ may grow with k and n. Gabor's original system $G(\varphi, \mathbb{Z}^2)$ is not a frame. It is overcomplete by exactly one element (that is, if any single element of the system is removed then it is still complete, but if two elements are removed then it is incomplete). Moreover, the system with one element removed is exact, but it is not a Schauder basis, and it is not a frame.

Many generalizations of or variations on frames have been introduced. A *Bessel sequence* need only satisfy the upper frame bound. A *semi-frame* [1, 2], need only satisfy one of the two frame bounds. Moreover, an upper semi-frame is a complete sequence that only needs to satisfy the upper frame bound. A *quasibasis* or *Schauder frame* [7] is a sequence $\{f_n\}_{n\geq 0}$ for which there exists a sequence $\{g_n\}_{n\geq 0}$ such that

$$f = \sum_{n\geq 0} \langle f, g_n\rangle\, f_n, \qquad \text{for every } f \in \mathbb{H},$$

where the series converges in norm with respect to some fixed ordering of the index set.

Recently, Speckbacher and Balazs [38] introduced *reproducing pairs* (see also [3]). The general definition is related to *continuous frames* with respect to arbitrary Borel measures. However, in this paper we are entirely focused on the discrete setting. In that context, the definition takes the following form.

Definition 1.1 Let $\Psi = \{\psi_i\}_{i\in\mathcal{I}}$ and $\Phi = \{\phi_i\}_{i\in\mathcal{I}}$ be two countable families in $\mathbb{H}$. Then (Ψ, Φ) is a *reproducing pair* for $\mathbb{H}$ if the operator $S_{\Psi,\Phi} : \mathbb{H} \to \mathbb{H}$ that is weakly defined by

$$\langle S_{\Psi,\Phi} f, g\rangle = \sum_{i\in\mathcal{I}} \langle f, \psi_i\rangle\, \langle \phi_i, g\rangle, \qquad \text{for } f,\ g \in \mathbb{H}, \tag{3}$$

is bounded and boundedly invertible (that is, a topological isomorphism using the terminology of [20]). In this case, we say that Ψ is a reproducing partner for Φ, and conversely Φ is a reproducing partner for Ψ.

We allow the convergence of the infinite series in equation (3) to be conditional, in the sense that there exists some fixed ordering of the index set $\mathcal{I}$ such that the partial sums of the series converge with respect to that ordering. Since we will mostly be interested in sequences that are overcomplete by one element (or finitely many later in the paper), we will often take the index set to be $\mathcal{I} = \{0, 1, 2, \dots\}$, and in that case assume that the convergence is with respect to the natural ordering. However, we will make applications to sequences, such as Gabor systems, that are indexed by other countable sets, and in those settings we will assume that an ordering has been fixed on the index set.

Since $S^*_{\Psi,\Phi} = S_{\Phi,\Psi}$, if (Ψ, Φ) is a reproducing pair, then $(\Psi, S^{-1}_{\Psi,\Phi}\Phi)$ is also a reproducing pair. Therefore, we can assume without loss of generality that $S_{\Psi,\Phi} = I$ (see [38]). With this assumption, (Ψ, Φ) is a reproducing pair if for each $f \in \mathbb{H}$ we have that the representation $f = \sum \langle f, \psi_i \rangle \phi_i$ holds weakly, i.e.,

$$\langle f, g \rangle = \sum_{i \in \mathcal{I}} \langle f, \psi_i \rangle \langle \phi_i, g \rangle, \qquad \text{for all } f,\ g \in \mathbb{H}. \tag{4}$$

In this sense, a reproducing pair is a weak analogue of a quasibasis or Schauder frame. Certainly every quasibasis is a reproducing pair; however, it is unclear to us whether the converse implication holds in general.

In this paper we will consider reproducing pair properties of sequences that are overcomplete by finitely many elements. In Section 2 we consider a sequence Φ that, like the original Gabor system $G(\varphi, \mathbb{Z}^2)$, is overcomplete by a single element. We prove in Theorem 2.2 that if such a sequence Φ has a reproducing partner Ψ, then it must contain a Schauder basis. Specifically, the exact sequence obtained by removing that single element from Φ is a Schauder basis for $\mathbb{H}$.

In Section 3, we focus on systems of weighted exponentials $\{e^{2\pi i n t} g(t)\}_{n\in\mathbb{Z}}$ in $L^2[0, 1)$. We recover a result from [21], explicitly showing the existence of weighted exponential systems that are overcomplete by one element. We prove in Theorem 3.1 that these systems do not contain a Schauder basis. By applying Theorem 2.2 we construct families of sequences that do not possess a reproducing partner.

We consider Gabor systems in Section 4. Through the use of the Zak transform, we prove in Theorem 4.2 that there exist families of Gabor sequences at the critical density that do not possess a reproducing partner. In particular, we see that the Gaussian Gabor system belongs to this family. Balazs and Speckbacher claimed in [39] that Gaussian Gabor system $G(\varphi, \mathbb{Z}^2)$ does have a reproducing partner. However, we demonstrate that this is not possible.

Finally, in Section 5 we show how our results generalize to sequences that are overcomplete by finitely many elements. These proofs require that we address certain issues of convergence. Lastly, we give an example of a weighted exponential system that is overcomplete by n elements, but does not possess a reproducing partner.

2 Reproducing Pairs and Schauder Bases

2.1 Main Theorem.

We begin by considering a sequence Φ that is exact (both complete and minimal). A standard fact is that $\Phi = \{\phi_k\}_{k\geq 1}$ is minimal if and only if there exists a *biorthogonal sequence* $\widetilde{\Phi} = \{\widetilde{\phi}_k\}_{k\geq 1}$ that satisfies $\langle \phi_j, \widetilde{\phi}_k \rangle = \delta_{jk}$. Further, an exact sequence has a unique biorthogonal sequence. (We refer to texts such as [8] or [20] for details.)

We show first that if an exact sequence has a reproducing partner, then it is a Schauder basis.

Lemma 2.1 *If an exact sequence $\Phi = \{\phi_k\}_{k\geq 1}$ has a reproducing partner, then Φ is a Schauder basis for $\mathbb{H}$.*

Proof Suppose that a reproducing partner $\Psi = \{\psi_k\}_{k\geq 1}$ for Φ did exist. Then

$$\langle f, g\rangle = \sum_{k\geq 1} \langle f, \psi_k\rangle \langle \phi_k, g\rangle, \qquad \text{for all } f,\ g \in \mathbb{H}. \tag{5}$$

Since Φ is exact, it has a biorthogonal sequence $\widetilde{\Phi} = \{\widetilde{\phi}_k\}_{k\geq 1}$. Therefore, by equation (5) we have for every f that

$$\langle f, \widetilde{\phi}_j\rangle = \sum_{k\geq 1}\langle f, \psi_k\rangle \langle \phi_k, \widetilde{\phi}_j\rangle = \langle f, \psi_j\rangle.$$

Consequently $\psi_j = \widetilde{\phi}_j$ for every j. Therefore $f = \sum \langle f, \widetilde{\phi}_k\rangle\, \phi_k$ weakly for every f.

Now fix $f \in \mathbb{H}$, and suppose that $(c_k)_{k\geq 1}$ is a scalar sequence such that $f = \sum c_k \phi_k$ weakly. Then

$$\langle f, \widetilde{\phi}_j\rangle = \sum_{k\geq 1} c_k \langle \phi_k, \widetilde{\phi}_j\rangle = \sum_{k\geq 1} c_k\, \delta_{jk} = c_j.$$

Hence there is a unique choice of coefficients for which we have $f = \sum c_k \phi_k$ weakly. Therefore Φ is a weak Schauder basis for $\mathbb{H}$, and so, by the Weak Basis Theorem (see [20, Thm. 4.30]), Φ is a strong Schauder basis for $\mathbb{H}$. □

Now we prove that if a sequence Φ that is overcomplete by one element possesses a reproducing partner, then it must contain a Schauder basis.

Theorem 2.2 *Assume that $\Phi = \{\phi_k\}_{k\geq 0}$ satisfies the following properties.*

(a) $\Phi' = \{\phi_k\}_{k\geq 1}$ *is exact in* $\mathbb{H}$.
(b) Φ *has a reproducing partner* $\Psi = \{\psi_k\}_{k\geq 0}$.

Then Φ' is a Schauder basis for $\mathbb{H}$.

Proof Since Φ' is exact, it has a biorthogonal sequence $\widetilde{\Phi} = \{\widetilde{\phi}_k\}_{k\geq 1}$. Further, since (Ψ, Φ) is a reproducing pair, equation (4) holds with $\mathcal{I} = \{0, 1, 2, \dots\}$. If $j \geq 1$, then we have for all $f \in \mathbb{H}$ that

$$\begin{aligned}\langle f, \widetilde{\phi}_j\rangle &= \sum_{k=0}^{\infty}\langle f, \psi_k\rangle \langle \phi_k, \widetilde{\phi}_j\rangle = \langle f, \psi_0\rangle \langle \phi_0, \widetilde{\phi}_j\rangle + \sum_{k=1}^{\infty}\langle f, \psi_k\rangle \langle \phi_k, \widetilde{\phi}_j\rangle \\ &= \Big\langle f, \langle \widetilde{\phi}_j, \phi_0\rangle \psi_0\Big\rangle + \langle f, \psi_j\rangle \qquad \text{(by biorthogonality)} \\ &= \Big\langle f, \langle \widetilde{\phi}_j, \phi_0\rangle \psi_0 + \psi_j\Big\rangle.\end{aligned}$$

Therefore

$$\widetilde{\phi}_j = \langle \widetilde{\phi}_j, \phi_0\rangle \psi_0 + \psi_j, \qquad \text{for every } j \geq 1. \tag{6}$$

Now assume that $\psi_0 \neq 0$, as otherwise the result follows trivially. Since Φ' is complete, there is some $n \geq 1$ such that $\langle \phi_n, \psi_0\rangle \neq 0$. Using equation (6) to substitute for ψ_k, we compute that if $g \in \mathbb{H}$ then

$$\begin{aligned}\langle \phi_n, g\rangle &= \sum_{k=0}^{\infty} \langle \phi_n, \psi_k\rangle \langle \phi_k, g\rangle \\ &= \langle \phi_n, \psi_0\rangle \langle \phi_0, g\rangle + \sum_{k=1}^{\infty} \langle \phi_n, \psi_k\rangle\langle \phi_k, g\rangle \\ &= \langle \phi_n, \psi_0\rangle \langle \phi_0, g\rangle + \sum_{k=1}^{\infty} \Big\langle \phi_n, \widetilde{\phi}_k - \langle \widetilde{\phi}_k, \phi_0\rangle \psi_0\Big\rangle \langle \phi_k, g\rangle \qquad \text{(by equation (6))} \\ &= \langle \phi_n, \psi_0\rangle\langle \phi_0, g\rangle + \sum_{k=1}^{\infty} \Big(\langle \phi_n, \widetilde{\phi}_k\rangle - \langle \phi_0, \widetilde{\phi}_k\rangle \langle \phi_n, \psi_0\rangle\Big) \langle \phi_k, g\rangle \\ &= \langle \phi_n, \psi_0\rangle \langle \phi_0, g\rangle + \sum_{k=1}^{\infty} \Big(\delta_{kn} - \langle \phi_0, \widetilde{\phi}_k\rangle \langle \phi_n, \psi_0\rangle\Big) \langle \phi_k, g\rangle \\ &= \langle \phi_n, \psi_0\rangle \langle \phi_0, g\rangle + \langle \phi_n, g\rangle - \sum_{k=1}^{\infty}\langle \phi_0, \widetilde{\phi}_k\rangle \langle \phi_n, \psi_0\rangle\langle \phi_k, g\rangle.\end{aligned}$$

Therefore,

$$\langle \phi_n, \psi_0\rangle \langle \phi_0, g\rangle = \langle \phi_n, \psi_0\rangle \sum_{k=1}^{\infty} \langle \phi_0, \widetilde{\phi}_k\rangle \langle \phi_k, g\rangle.$$

Since $\langle \phi_n, \psi_0\rangle \neq 0$, we can cancel that factor and conclude that

$$\phi_0 = \sum_{k=1}^{\infty} \langle \phi_0, \widetilde{\phi}_k\rangle \phi_k \quad \text{weakly}. \tag{7}$$

Now we will show that Φ' is a Schauder basis. If $g, h \in \mathbb{H}$, then

$$\begin{aligned}
\langle g, h\rangle &= \langle g, \psi_0\rangle \langle \phi_0, h\rangle + \sum_{k=1}^{\infty} \langle g, \psi_k\rangle \langle \phi_k, h\rangle && \text{(reproducing property)}\\
&= \langle g, \psi_0\rangle \sum_{k=1}^{\infty} \langle \phi_0, \widetilde{\phi}_k\rangle \langle \phi_k, h\rangle + \sum_{k=1}^{\infty} \langle g, \psi_k\rangle \langle \phi_k, h\rangle && \text{(by equation (7))}\\
&= \sum_{k=1}^{\infty} \Big(\langle g, \psi_0\rangle \langle \phi_0, \widetilde{\phi}_k\rangle + \langle g, \psi_k\rangle\Big) \langle \phi_k, h\rangle\\
&= \sum_{k=1}^{\infty} \Big\langle g,\ \langle \widetilde{\phi}_k, \phi_0\rangle\, \psi_0 + \psi_k\Big\rangle \langle \phi_k, h\rangle\\
&= \sum_{k=1}^{\infty} \langle g, \widetilde{\phi}_k\rangle \langle \phi_k, h\rangle. && \text{(by equation (6))}
\end{aligned}$$

Therefore $(\widetilde{\Phi}, \Phi')$ is a reproducing pair, so Lemma 2.1 implies that Φ' is a Schauder basis for $\mathbb{H}$. □

2.2 Applications

We utilize Lemma 2.1 and Theorem 2.2 to immediately show that certain sequences do not have reproducing partners.

Let $\Lambda = \{\lambda_n\}_{n\geq 0}$ be an increasing sequence of non-negative real numbers that diverge to infinity and satisfy the *Müntz condition*

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty.$$

Let $M(\Lambda) = \{t^{\lambda_n}\}_{n\geq 0}$, and let $[M(\Lambda)]$ be the closed linear span of $M(\Lambda)$ in $L^2(\mathbb{T})$. It follows from the Müntz–Szász theorem (see [20, Thm. 5.6]) that $[M(\Lambda)]$ is a proper subspace of $L^2(\mathbb{T})$. Further, $M(\Lambda)$ is complete in $[M(\Lambda)]$ by definition, and also $M(\Lambda)$ is minimal because it satisfies the Müntz condition (see [19, Prop. 6.1.4]). Therefore $M(\Lambda)$ is exact in $[M(\Lambda)]$.

Definition 2.3 A sequence $\{\lambda_n\}_{n\geq 0}$ is *lacunary* if there exists some $q > 1$ such that $\lambda_{n+1}/\lambda_n > q$ for every $n \geq 1$.

The next result is part of the $p = 2$ case of [19, Thm. 9.2.2].

Theorem 2.4 *Λ is lacunary if and only if $M(\Lambda)$ is a Schauder basis for $[M(\Lambda)]$.*

Therefore, the next corollary follows immediately.

Corollary 2.5 *If Λ is not lacunary, then $M(\Lambda)$ does not have a reproducing partner in $[M(\Lambda)]$.*

Proof Assume that $M(\Lambda)$ has a reproducing partner in $[M(\Lambda)]$. Then since $M(\Lambda)$ is exact, Lemma 2.1 implies that it is a Schauder basis for $[M(\Lambda)]$. By Theorem 2.4, we therefore conclude that Λ is lacunary. □

In the context of Gabor systems, which will be expanded upon in Section 4, we obtain the following corollary.

Corollary 2.6 *The Gaussian Gabor system $G(\varphi, \mathbb{Z}^2)$ with $\varphi(t) = 2^{\frac{1}{4}} e^{-\pi t^2}$ does not have a reproducing partner.*

This corollary is a consequence of Theorem 2.2, because $G(\varphi, \mathbb{Z}^2)$ is overcomplete by one element, but does not contain a Schauder basis. A detailed proof will be presented in Corollary 4.3.

3 Weighted Exponentials and Reproducing Partners

3.1 Background

In this section we will apply our results to sequences of weighted exponentials in $L^2(\mathbb{T})$, where $\mathbb{T} = [0, 1)$. These have the form $E(g, \mathbb{Z}) = \{g e_n\}_{n\in\mathbb{Z}}$, where $e_n(t) = e^{2\pi i n t}$ (equivalently, we could consider the trigonometric system $\{e_n\}_{n\in\mathbb{Z}}$ in a weighted L^2 space). A characterization of when $E(g, \mathbb{Z})$ is complete, minimal, exact, a frame, an unconditional basis, or an orthonormal basis can be found in textbooks such as [8] or [20]. A much deeper classical result due to Hunt, Muckenhoupt, and Wheeden [23] is that $E(g, \mathbb{Z})$ is a Schauder basis for $L^2(\mathbb{T})$ with respect to the ordering $\mathbb{Z} = \{0, -1, 1, -2, 2, \dots\}$ if and only if $|g|^2$ is an $\mathcal{A}_2(\mathbb{T})$ *weight*.

However, those results apply when the index set is the full set of integers. We are interested in systems that are overcomplete by one or finitely many elements, and hence we will deal with subsequences $E(g, \Lambda)$ that are indexed by a proper subset Λ of $\mathbb{Z}$. Kazarian [28] characterized the functions g and index sets Λ such that $E(g, \Lambda)$ is complete or minimal in $L^p(\mathbb{T})$ for $1 \le p < \infty$. Additional related results are in [22], [30], and [29]. A characterization of functions g and finite sets $F \subseteq \mathbb{Z}$ such that $E(g, \mathbb{Z}\setminus F)$ is exact in $L^2(\mathbb{T})$ was given by Heil and Yoon [21]. Recently, Zikkos [43] proved that the closed span in $L^2(\gamma, \beta)$ of the system $E(t^k, \Lambda) = \{t^k e^{\lambda_n t} : n \in \mathbb{N},\ k = 1, 2, \dots, \mu_n - 1\}$, with $\mu_i \in \mathbb{N}$, is equal to the closed span of its unique biorthogonal sequence $r_\Lambda = \{r_{n,k} : n \in \mathbb{N},\ k = 1, 2, \dots, \mu_n - 1\}$ if some constraints on Λ and the μ_i are satisfied. Further, in this case each $f \in L^2(\gamma, \beta)$ admits a Fourier-like series representation

$$f(t) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\mu_n - 1} \langle f, r_{n,k} \rangle\, t^k \right) e^{\lambda_n t},$$

where the series converges uniformly on closed subintervals of (γ, β).

3.2 Applications

The following theorem combines a characterization from [21] with a basis result from [37]. We include the proof for completeness, and because a similar technique will be used to prove Theorem 4.2.

Theorem 3.1 *Let $g \in L^2(\mathbb{T})$ be such that $1/g \notin L^2(\mathbb{T})$. If $(t-t_0)/g(t) \in L^2(\mathbb{T})$ for some $t_0 \in \mathbb{T}$, then $E(g, \mathbb{Z}\backslash\{k\})$ is exact for every $k \in \mathbb{Z}$. However, there is no ordering of $\mathbb{Z}\backslash\{k\}$ such that $E(g, \mathbb{Z}\backslash\{k\})$ is a Schauder basis for $L^2(\mathbb{T})$.*

Proof Fix $k \in \mathbb{Z}$. We will first show that $E(g, \mathbb{Z}\backslash\{k\})$ is complete. If $f \in L^2(\mathbb{T})$ satisfies $\langle f, g e_n\rangle = 0$ for all $n \neq k$, then $\langle f\overline{g}, e_n\rangle = 0$ for every $n \neq k$. Since $f\overline{g} \in L^1(\mathbb{T})$ and functions in $L^1(\mathbb{T})$ are uniquely determined by their Fourier coefficients, it follows that $f\overline{g} = c e_k$ for some constant c, and hence $f e_{-k} = c/\overline{g}$. Since $1/g \notin L^2(\mathbb{T})$, we must have $c = 0$. Therefore $E(g, \mathbb{Z}\backslash\{k\})$ is complete.

Next, for each $n \neq k$ let $c_n = -e^{2\pi i(n-k)t_0}$, so that $e_n + c_n e_k$ vanishes at t_0. Then the function

$$\widetilde{g}_n = \frac{e_n + c_n e_k}{\overline{g}}$$

belongs to $L^2(\mathbb{T})$, and $\langle g e_m, \widetilde{g}_n\rangle = \langle e_m, e_n + c_n e_k\rangle = \delta_{mn}$ for $m, n \neq k$. Therefore $\{\widetilde{g}_n\}_{n\neq k}$ is biorthogonal to $E(g, \mathbb{Z}\backslash\{k\})$, so this sequence is minimal.

Now we will show that $E(g, \mathbb{Z}\backslash\{k\})$ is not a Schauder basis. Assume that there were some ordering of $\mathbb{Z}\backslash\{k\}$ such that $E(g, \mathbb{Z}\backslash\{k\})$ formed a Schauder basis for $L^2(\mathbb{T})$. Then there would exist unique coefficients d_n such that

$$g e_k = \sum_{n\neq k} d_n g e_n, \tag{8}$$

where this sum converges in norm with respect to the specified ordering of $\mathbb{Z}\backslash\{k\}$. Using the biorthogonality established earlier, it follows that if $m \neq k$ then

$$\langle g e_k, \widetilde{g}_m\rangle = \sum_{n\neq k} d_n \langle g e_n, \widetilde{g}_m\rangle = \sum_{n\neq k} d_n \delta_{nm} = d_m.$$

However, if $m \neq k$ then we also have that

$$\langle g e_k, \widetilde{g}_m\rangle = \left\langle g e_k, \frac{e_m + c_m e_k}{\overline{g}}\right\rangle = \langle e_k, e_m + c_m e_k\rangle = \delta_{km} + \overline{c_m} = \overline{c_m}.$$

Therefore $d_m = \overline{c_m}$ for $m \neq k$. Consequently, since the series in equation (8) converges in norm, we must have $\|\overline{c_n} g e_n\|_2 \to 0$ as $n \to \infty$. But $|c_n| = 1$, so $\|\overline{c_n} g e_n\|_2 = \|g\|_2$ for every n, which is a contradiction. □

Applying Theorem 2.2, we obtain the following corollary.

Corollary 3.2 *Let $g \in L^2(\mathbb{T})$ be such that $1/g \notin L^2(\mathbb{T})$. If there exists some point $t_0 \in \mathbb{T}$ such that $(t - t_0)/g(t) \in L^2(\mathbb{T})$, then there does not exist any sequence $\Psi \subseteq L^2(T)$ such that $\big(\Psi, E(g, \mathbb{Z})\big)$ is a reproducing pair.*

Proof By Theorem 3.1, the sequence $E(g, \mathbb{Z})$ is overcomplete by one element, and if we remove any element then the resulting sequence is not a Schauder basis. Theorem 2.2 therefore implies that there is no sequence $\Psi \subseteq L^2(\mathbb{T})$ such that $\big(\Psi, E(g, \mathbb{Z})\big)$ is a reproducing pair. □

For example, the functions $g_1(t) = t$, $g_2(t) = e^t - 1$, and $g_3(t) = \sin(t)$ satisfy the hypotheses of Corollary 3.2. Hence, for $i = 1, 2, 3$ and any $k \in \mathbb{Z}$, the systems $E(g_i, \mathbb{Z} \setminus \{k\})$ do not possess a reproducing partner. For $i = 1$, this also follows from results in [15] and [21].

4 Gabor Systems and Reproducing Pairs

4.1 Background

Von Neumann's claim that the Gaussian Gabor system at the critical density is complete was proven independently by Perelomov [33], Bargmann, Butera, Girardello, and Klauder [6], and Bacry, Grossmann, and Zak [4]. It was later conjectured by Daubechies and Grossmann that $G(\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a frame if and only if $0 < \alpha\beta < 1$ [9, 10]. This conjecture was proven in full by Lyubarskii [31] and by Seip and Wallstén [35, 36]. At the critical density, Folland [12] proved that $G(\varphi, \alpha\mathbb{Z} \times 1/\alpha\mathbb{Z})$ is not a frame, and is overcomplete by exactly one element. He further showed that if any single element is removed, the resulting system is exact but not a Schauder basis. We include an alternative proof of this result in Theorem 4.2. On the topic of overcompleteness, it was shown in [5] that every Gabor frame $G(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ with $\alpha\beta < 1$ has infinite excess (so is overcomplete by infinitely many elements). Gröchenig and Stöckler [18] showed that if $g \in L^2(\mathbb{R})$ is a totally positive function of finite type (which includes the Gaussian function φ), then the Gabor system $G(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a frame for $L^2(\mathbb{R})$ if and only if $\alpha\beta < 1$. Recently, Gröchenig proved [17] that if $g \in L^1(\mathbb{R})$ is totally positive and $\alpha\beta \in \mathbb{Q}$, then $G(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a frame for $L^2(\mathbb{R})$ if and only if $\alpha\beta < 1$.

The Zak transform is an important tool for studying Gabor systems at the critical density $\alpha\beta = 1$ (which we reduce to $\alpha = \beta = 1$ by a change of variables). Gröchenig [16] remarks that the Zak transform was first introduced by Gel'fand [14]. As with many useful tools, it has been rediscovered numerous times and goes by a variety of names. Weil [40] defined a Zak transform for locally compact abelian groups, and this transform is often called the *Weil-Brezin* map in representation theory and abstract harmonic analysis [34]. Zak rediscovered this transform, which he called the *k-q* transform, in his work on quantum mechanics [42]. The terminology "Zak transform" has become customary in applied mathematics and signal processing. We refer to texts such as [8, 16, 20] for details on the Zak transform.

Let $Q = [0,1)^2$. The *Zak transform* is the unitary map $Z : L^2(\mathbb{R}) \to L^2(Q)$ defined by

$$Zf(x,\xi) = \sum_{j\in\mathbb{Z}} f(x-j)\, e^{2\pi i j\xi}, \qquad \text{for } (x,\xi) \in Q. \tag{9}$$

The series in equation (9) converges unconditionally in the norm of $L^2(Q)$. Since Z is unitary, it preserves properties such as completeness, minimality, being a frame, being a Riesz basis, and so forth.

If $g \in L^2(\mathbb{R})$, then the Gabor system generated by g at the critical density is $G(g, \mathbb{Z}^2) = \{M_n T_k g\}_{k,n\in\mathbb{Z}}$. For $k, n \in \mathbb{Z}$, let

$$E_{nk}(x,\xi) = e^{2\pi i n x}\, e^{-2\pi i k\xi}, \qquad \text{for } (x,\xi) \in \mathbb{R}^2.$$

The Zak transform has the property that

$$Z(M_n T_k g)(x,\xi) = E_{nk}(x,\xi)\, Zg(x,\xi) = e^{2\pi i n x}\, e^{-2\pi i k\xi}\, Zg(x,\xi).$$

Consequently, the image of the Gabor system at the critical density under Z is

$$Z\big(G(g,\mathbb{Z}^2)\big) = \{E_{nk}\, Zg\}_{(k,n)\in\mathbb{Z}^2}.$$

This is a two-dimensional version of the systems of weighted exponentials that we studied in Section 3. Thus we expect that similar results will hold, although there are some issues due to the higher-dimensional setting.

4.2 Applications

We will need the following lemma regarding the existence of certain functions in $L^2(Q)$. We will use the following notation for a "cone function" centered at (x_0, ξ_0):

$$\rho_{x_0,\xi_0}(x,\xi) = \sqrt{(x-x_0)^2 + (\xi-\xi_0)^2}.$$

Lemma 4.1 *Fix $(x_0,\xi_0) \in Q$ and $(a,b) \in \mathbb{Z}^2$. For $(n,k) \neq (a,b)$, let c_{nk} be the scalar of unit modulus such that $E_{nk}(x_0,\xi_0) + c_{nk}\, E_{ab}(x_0,\xi_0) = 0$. Then*

$$\frac{E_{nk} + c_{nk}\, E_{ab}}{\rho_{x_0,\xi_0}}$$

is bounded on $Q \setminus \{(x_0,\xi_0)\}$, and hence belongs to $L^2(Q)$.

Proof If $(x,\xi) \in Q$ and $(n,k) \neq (a,b)$, then a direct calculation shows that

$$|E_{nk}(x,\xi) + c_{nk}\, E_{ab}(x,\xi)| = |E_{n-a,k-b}(x-x_0, \xi-\xi_0) - 1|.$$

Therefore it suffices to show that if $(n, k) \neq (0, 0)$ then

$$\frac{E_{nk}(x - x_0, \xi - \xi_0) - 1}{\rho_{x_0,\xi_0}(x, \xi)} \quad \text{is bounded on } Q \setminus \{(x_0, \xi_0)\}. \tag{10}$$

First we note that

$$\begin{aligned} |E_{nk}(x, \xi) - 1| = |e^{2\pi i(nx - k\xi)} - 1| &\leq 2\pi\, |nx - k\xi| \\ &\leq 2\pi\, \big(|nx| + |k\xi|\big) \\ &\leq 2\pi\, \sqrt{k^2 + n^2}\, \sqrt{x^2 + \xi^2}. \end{aligned}$$

Therefore, for $(n, k) \neq (0, 0)$ we have that

$$|E_{nk}(x - x_0, \xi - \xi_0) - 1| \leq 2\pi\, \sqrt{k^2 + n^2}\; \rho_{x_0,\xi_0}(x, \xi),$$

and equation (10) follows from this. □

Now, we prove an analogue of Theorem 3.1 for Gabor systems at the critical density.

Theorem 4.2 *Let $g \in L^2(\mathbb{R})$ be such that $1/Zg \notin L^2(Q)$. Suppose there is some point $(x_0, \xi_0) \in Q$ such that $\rho_{x_0,\xi_0}/Zg \in L^2(Q)$. Then the following statements hold.*

(a) *The Gabor system $G(g, \mathbb{Z}^2 \setminus \{(a, b)\})$ is exact in $L^2(\mathbb{R})$ for every pair $(a, b) \in \mathbb{Z}^2$.*

(b) *For every pair $(a, b) \in \mathbb{Z}^2$, there is no ordering of $Z^2 \setminus \{(a, b)\}$ such that $G(g, \mathbb{Z}^2 \setminus \{(a, b)\})$ is a Schauder basis for $L^2(\mathbb{R})$.*

(c) *The system $G(g, \mathbb{Z}^2)$ does not have a reproducing partner.*

Proof The proof showing that $G(g, \mathbb{Z}^2 \setminus \{(a, b)\})$ is exact in $L^2(\mathbb{R})$ is similar to the proof of Theorem 3.1. Since $\langle f, M_n T_k g\rangle_{L^2(\mathbb{R})} = \langle Zf, E_{nk} Zg\rangle_{L^2(Q)}$, completeness follows immediately from the assumption that $1/Zg \notin L^2(Q)$.

Observe that

$$\frac{E_{nk} + c_{nk} E_{ab}}{\overline{Zg}} = \frac{E_{nk} + c_{nk} E_{ab}}{\rho_{x_0,\xi_0}} \cdot \frac{\rho_{x_0,\xi_0}}{\overline{Zg}}.$$

By Lemma 4.1, this is the product of a bounded function with a square-integrable function, and so it belongs to $L^2(Q)$. Therefore, since Z is unitary, there is a function $\widetilde{g}_{nk} \in L^2(\mathbb{R})$ such that

$$Z(\widetilde{g}_{nk}) = \frac{E_{nk} + c_{nk} E_{ab}}{\overline{Zg}}.$$

Since $\langle M_n T_k g, \widetilde{g}_{ml}\rangle_{L^2(\mathbb{R})} = \big\langle E_{nk} Zg,\ Z(\widetilde{g}_{ml})\big\rangle_{L^2(Q)}$, it follows that $\{\widetilde{g}_{ml}\}_{(m,l)\neq(a,b)}$ is biorthogonal to $G\big(g, \mathbb{Z}^2 \setminus \{(a, b)\}\big)$. Therefore that system is minimal.

By using the fact that Z is unitary, the proof that $G\big(g, \mathbb{Z}^2 \setminus \{(a, b)\}\big)$ is not a Schauder basis is very similar to the argument presented in the proof of Theorem 3.1.

Lastly, by Theorem 4.2, the Gabor system $G(g, \mathbb{Z}^2)$ is overcomplete by one element, and if we remove any one element then the resulting sequence is not a Schauder basis. Theorem 2.2 therefore implies that $G(g, \mathbb{Z}^2)$ does not have a reproducing partner. □

4.3 The Original Gabor System

We will show that the Gaussian Gabor system at the critical density does not possess a reproducing partner. We set $\varphi(t) = 2^{1/4} e^{-\pi t^2}$, and let $\Theta = Z\varphi$ be the Zak transform of the Gaussian function. Because φ is smooth and decays quickly, Θ is smooth on Q, and furthermore it has a single zero in Q, at the point $(1/2, 1/2)$. In fact, we have explicitly (compare [26]) that

$$\begin{aligned}\Theta(x,\xi) &= 2^{1/4} \sum_{k\in\mathbb{Z}} e^{-\pi(x-k)^2}\, e^{2\pi i k\xi} \\ &= -2^{1/4} i\, e^{-\pi(x-1/2)^2+\pi i(\xi-1/2)^2}\, \theta_1\big(\pi\big(\xi - \tfrac{1}{2} - i\big(x-\tfrac{1}{2}\big)\big),\, e^{-\pi}\big),\end{aligned}$$

where θ_1 is the first Jacobi theta function,

$$\theta_1(z,q) = -i \sum_{k\in\mathbb{Z}} (-1)^k\, q^{(k+1/2)^2}\, e^{(2k+1)iz},$$

see [41, Chap. 1]. We will implicitly assume henceforth that $q = e^{-\pi}$, and just write $\theta_1(z)$ instead of $\theta_1(z, q)$.

Corollary 4.3 *$1/\Theta \notin L^2(Q)$, but $\frac{\rho_{1/2,1/2}}{\Theta} \in L^2(Q)$. Consequently, $G(\varphi, \mathbb{Z}^2)$ does not possess a reproducing partner.*

Proof The Taylor series expansion of Θ about the point $(1/2, 1/2)$ is

$$\Theta(x,\xi) = -2^{1/4}\pi\, \theta_1'(0)\Big(\big(x-\tfrac{1}{2}\big) + i\big(\xi-\tfrac{1}{2}\big)\Big) + \mathcal{O}\Big(\big(x-\tfrac{1}{2}\big)^2 + \big(\xi-\tfrac{1}{2}\big)^2\Big), \tag{11}$$

where

$$\theta_1'(0) = \sum_{k\in\mathbb{Z}} (-1)^k\,(2k+1)\, e^{-\pi(k+1/2)^2} = 2\sum_{k=0}^{\infty} (-1)^k\,(2k+1)\, e^{-\pi(k+1/2)^2} \neq 0.$$

Therefore, there exist constants $C > 0$ and $0 < \delta < 1/2$ such that

$$|\Theta(x,\xi)| \geq C\sqrt{\big(x-\tfrac{1}{2}\big)^2 + \big(\xi-\tfrac{1}{2}\big)^2} = C\,\rho_{1/2,1/2}(x,\xi), \qquad \text{if } (x,\xi) \in B_\delta,$$

where B_δ is the open ball of radius δ centered at $(1/2, 1/2)$. Additionally, since Θ is continuous and its only zero in Q is at the point $(1/2, 1/2)$, there is some $c > 0$ such that

$$|\Theta(x,\xi)| \geq c, \qquad \text{if } (x,\xi) \in Q \setminus B_\delta.$$

Hence, we compute that

$$\begin{aligned}
&\iint_Q \frac{\rho_{1/2,1/2}(x,\xi)^2}{|\Theta(x,\xi)|^2}\,dx\,d\xi \\
&\quad = \iint_{Q\setminus B_\delta} \frac{\rho_{1/2,1/2}(x,\xi)^2}{|\Theta(x,\xi)|^2}\,dx\,d\xi + \iint_{B_\delta} \frac{\rho_{1/2,1/2}(x,\xi)^2}{|\Theta(x,\xi)|^2}\,dx\,d\xi \\
&\quad \leq \iint_{Q\setminus B_\delta} \frac{\rho_{1/2,1/2}(x,\xi)^2}{c^2}\,dx\,d\xi + \iint_{B_\delta} \frac{1}{C^2}\,dx\,d\xi \\
&\quad \leq \frac{1}{2c^2} + \frac{\pi\delta^2}{C^2} < \infty.
\end{aligned}$$

This shows that $\rho_{1/2,1/2}/\Theta$ is square-integrable on Q.

Moreover, by equation (11), there exist constants $D > 0$ and $0 < \eta < 1/2$ such that

$$|\Theta(x,\xi)| \leq D\sqrt{\left(x-\tfrac{1}{2}\right)^2 + \left(\xi-\tfrac{1}{2}\right)^2} = D\,\rho_{1/2,1/2}(x,\xi), \qquad \text{if } (x,\xi) \in B_\eta,$$

where B_η is the open ball of radius η centered at $(1/2, 1/2)$. Hence, we compute that

$$\begin{aligned}
\iint_Q \frac{1}{|\Theta(x,\xi)|^2}\,dx\,d\xi &\geq \frac{1}{D^2}\iint_{B_\eta} \frac{1}{\rho_{1/2,1/2}(x,\xi)^2}\,dx\,d\xi \\
&\geq \frac{1}{D^2}\int_0^{\frac{\pi}{2}}\int_0^{\eta} \frac{1}{r}\,dr\,d\theta = \infty.
\end{aligned}$$

Thus, $1/\Theta \notin L^2(Q)$. Theorem 4.2 therefore implies that $G(\varphi, \mathbb{Z}^2)$ is overcomplete by exactly one element, and if any one element is removed then the resulting system is not a Schauder basis for $L^2(\mathbb{R})$. Thus, it follows from Theorem 2.2 that $G(\varphi, \mathbb{Z}^2)$ does not possess a reproducing partner. □

5 Overcomplete by Finitely Many Elements

In this section we will generalize Theorem 2.2 to sequences that are overcomplete by $n > 1$ elements.

5.1 Lemmas

The following lemma will allow us to reduce to the case where the sets of overcomplete elements $\{\phi_0, \ldots, \phi_{n-1}\}$ and $\{\psi_0, \ldots, \psi_{n-1}\}$ are each linearly independent.

Lemma 5.1 *Let* $\{\phi_0, \ldots, \phi_{n-1}\}$ *and* $\{\psi_0, \ldots, \psi_{n-1}\}$ *be subset of* $\mathbb{H}$*. If either* $\{\phi_0, \ldots, \phi_{n-1}\}$ *or* $\{\psi_0, \ldots, \psi_{n-1}\}$ *is linearly dependent, then there exist elements* $\psi'_k, \phi'_k \in \mathbb{H}$ *such that*

$$\sum_{k=0}^{n-1} \langle f, \psi_k \rangle \langle \phi_k, g \rangle = \sum_{k=0}^{n-2} \langle f, \psi'_k \rangle \langle \phi'_k, g \rangle, \qquad \text{for all } f, g \in \mathbb{H}. \tag{12}$$

Further, we can choose these functions so that $\phi'_j \in \underset{k=0,\ldots,n-2}{\text{span}} \{\phi_k\}$ *and* $\psi'_j \in \underset{k=0,\ldots,n-2}{\text{span}} \{\psi_k\}$.

Proof Without loss of generality, assume that ψ is linearly dependent, so there exist coefficients c_j such that $\psi_{n-1} = \sum_{j=0}^{n-2} c_j \psi_j$. If we set $\psi'_k = \psi_k$ and $\phi'_k = \phi_k + \overline{c_k} \phi_{n-1}$ for $k = 0, \ldots, n-2$, then equation (12) holds. □

We also need a lemma that if $\{\phi_m\}_{m \in \mathcal{I}}$ is complete and $\{\psi_0, \ldots, \psi_{n-1}\}$ is linearly independent in $\mathbb{H}$, then we can create a particular sequence of vectors that is complete in $\mathbb{C}^n$.

Lemma 5.2 *Assume vectors* $\psi_0, \ldots, \psi_{n-1}$ *are linearly independent in* $\mathbb{H}$*, and* $\{\phi_m\}_{m \geq n}$ *is complete in* $\mathbb{H}$*. Set*

$$\mathbf{v}_m = \begin{bmatrix} \langle \psi_0, \phi_m \rangle \\ \vdots \\ \langle \psi_{n-1}, \phi_m \rangle \end{bmatrix}, \qquad \text{for } m \geq n.$$

Then $\{\mathbf{v}_m\}_{m \geq n}$ *is complete in* $\mathbb{C}^n$*, and hence spans* $\mathbb{C}^n$*.*

Proof Assume that $\mathbf{a} = (a_0, \ldots, a_{n-1}) \in \mathbb{C}^n$ is orthogonal to $\mathbf{v}_m$ for every $m \geq n$. Then for each $m \geq n$ we have that

$$\left\langle \sum_{j=0}^{n-1} \overline{a_j} \psi_j, \phi_m \right\rangle = \sum_{j=0}^{n-1} \overline{a_j} \langle \psi_j, \phi_m \rangle = \mathbf{v}_m \cdot \mathbf{a} = 0.$$

Since $\{\phi_m\}_{m \geq n}$ is complete, we therefore have $\sum_{j=0}^{n-1} \overline{a_j} \psi_j = 0$. But $\{\psi_0, \ldots, \psi_{n-1}\}$ is linearly independent, so $a_j = 0$ for $j = 0, \ldots, n-1$, and thus $\mathbf{a} = 0$. Since $\mathbb{C}^n$ is finite-dimensional, it follows that the finite linear span of $\{v_m\}_{m \geq n}$ is $\mathbb{C}^n$. □

5.2 Generalization of Theorem 2.2

Now we consider sequences that are overcomplete by $n > 1$ elements.

Theorem 5.3 *Assume that* $\Phi = \{\phi_k\}_{k \geq 0}$ *satisfies the following properties.*

(a) $\Phi' = \{\phi_k\}_{k \geq n}$ *is exact in* $\mathbb{H}$*.*

(b) *Φ has a reproducing partner $\Psi = \{\psi_k\}_{k\geq 0}$.*

Then Φ' is a Schauder basis for $\mathbb{H}$.

Proof By repeatedly applying Lemma 5.1 if necessary, we can assume that $\{\phi_0, \dots, \phi_{n-1}\}$ and $\{\psi_0, \dots, \psi_{n-1}\}$ are each linearly independent.

If $j \geq n$, then we have for all $f \in \mathbb{H}$ that

$$\begin{aligned}
\langle f, \widetilde{\phi}_j\rangle &= \sum_{k=n}^{\infty} \langle f, \psi_k\rangle \langle \phi_k, \widetilde{\phi}_j\rangle + \sum_{k=0}^{n-1} \langle f, \psi_k\rangle \langle \phi_k, \widetilde{\phi}_j\rangle && \text{(reproducing property)}\\
&= \langle f, \psi_j\rangle + \sum_{k=0}^{n-1} \langle f, \psi_k\rangle \langle \phi_k, \widetilde{\phi}_j\rangle && \text{(biorthogonality)}\\
&= \Big\langle f,\ \psi_j + \sum_{k=0}^{n-1} \langle \widetilde{\phi}_j, \phi_k\rangle\, \psi_k \Big\rangle.
\end{aligned}$$

Therefore

$$\widetilde{\phi}_j = \psi_j + \sum_{k=0}^{n-1} \langle \widetilde{\phi}_j, \phi_k\rangle\, \psi_k, \qquad \text{for every } j \geq n. \tag{13}$$

Hence, if $m \geq n$ and $g \in \mathbb{H}$, then

$$\begin{aligned}
\langle \phi_m, g\rangle &= \sum_{k=n}^{\infty} \langle \phi_m, \psi_k\rangle \langle \phi_k, g\rangle + \sum_{k=0}^{n-1} \langle \phi_m, \psi_k\rangle \langle \phi_k, g\rangle \quad \text{(reproducing property)}\\
&= \sum_{k=n}^{\infty} \Big\langle \phi_m,\ \widetilde{\phi}_k - \sum_{j=0}^{n-1} \langle \widetilde{\phi}_k, \phi_j\rangle\, \psi_j \Big\rangle \langle \phi_k, g\rangle\\
&\quad + \sum_{k=0}^{n-1} \langle \phi_m, \psi_k\rangle \langle \phi_k, g\rangle \quad \text{(by equation (13))}\\
&= \langle \phi_m, g\rangle - \sum_{k=n}^{\infty} \sum_{j=0}^{n-1} \langle \phi_m, \psi_j\rangle \langle \phi_j, \widetilde{\phi}_k\rangle \langle \phi_k, g\rangle + \sum_{k=0}^{n-1} \langle \phi_m, \psi_k\rangle \langle \phi_k, g\rangle.
\end{aligned} \tag{14}$$

Consequently, if we let $\mathbf{v}_m$ be as in Lemma 5.2 and let

$$\mathbf{u} = \begin{bmatrix} \langle \phi_0, g\rangle \\ \vdots \\ \langle \phi_{n-1}, g\rangle \end{bmatrix} \quad \text{and} \quad \mathbf{w}_k = \begin{bmatrix} \langle \phi_0, \widetilde{\phi}_k\rangle \langle \phi_k, g\rangle \\ \vdots \\ \langle \phi_{n-1}, \widetilde{\phi}_k\rangle \langle \phi_k, g\rangle \end{bmatrix},$$

then we have for all $m \geq n$ that

$$
\begin{aligned}
\mathbf{u} \cdot \mathbf{v}_m &= \sum_{k=0}^{n-1} \langle \phi_k, g \rangle \overline{\langle \psi_k, \phi_m \rangle} = \sum_{k=0}^{n-1} \langle \phi_m, \psi_k \rangle \langle \phi_k, g \rangle \\
&= \sum_{k=n}^{\infty} \sum_{j=0}^{n-1} \langle \phi_m, \psi_j \rangle \langle \phi_j, \widetilde{\phi}_k \rangle \langle \phi_k, g \rangle \qquad \text{(by equation (14))} \\
&= \lim_{N\to\infty} \sum_{k=n}^{N} (\mathbf{w}_k \cdot \mathbf{v}_m) = \lim_{N\to\infty} \left(\sum_{k=n}^{N} \mathbf{w}_k \right) \cdot \mathbf{v}_m
\end{aligned}
$$

But $\{\mathbf{v}_m\}_{m\geq n}$ spans $\mathbb{C}^n$ by Lemma 5.2, so this implies that $\sum_{k=n}^{N} \mathbf{w}_k$ converges weakly to $\mathbf{u}$ in $\mathbb{C}^n$ as $N \to \infty$. Since weak convergence implies strong convergence in finite-dimensional normed spaces, it follows that $\mathbf{u} = \sum_{k=n}^{\infty} \mathbf{w}_k$, with convergence in the norm of $\mathbb{C}^n$. Therefore

$$
\langle \phi_k, g \rangle = \sum_{j=n}^{\infty} \langle \phi_k, \widetilde{\phi}_j \rangle \langle \phi_j, g \rangle, \qquad \text{for } k = 0, \dots, n-1. \tag{15}
$$

Finally, in order to show that $\{\phi_k\}_{k\geq n}$ is a Schauder basis, fix any vectors f and g in $\mathbb{H}$. Then,

$$
\begin{aligned}
\langle f, g \rangle &= \sum_{k=0}^{n-1} \langle f, \psi_k \rangle \langle \phi_k, g \rangle + \sum_{k=n}^{\infty} \langle f, \psi_k \rangle \langle \phi_k, g \rangle && \text{(reproducing property)} \\
&= \sum_{k=0}^{n-1} \langle f, \psi_k \rangle \sum_{j=n}^{\infty} \langle \phi_k, \widetilde{\phi}_j \rangle \langle \phi_j, g \rangle + \sum_{k=n}^{\infty} \langle f, \psi_k \rangle \langle \phi_k, g \rangle && \text{(by equation (15))} \\
&= \sum_{j=n}^{\infty} \Big\langle f, \sum_{k=0}^{n-1} \langle \widetilde{\phi}_j, \phi_k \rangle \psi_k \Big\rangle \langle \phi_j, g \rangle + \sum_{k=n}^{\infty} \langle f, \psi_k \rangle \langle \phi_k, g \rangle \\
&= \sum_{j=n}^{\infty} \langle f, \widetilde{\phi}_j - \psi_j \rangle \langle \phi_j, g \rangle + \sum_{k=n}^{\infty} \langle f, \psi_k \rangle \langle \phi_k, g \rangle && \text{(by equation (13))} \\
&= \sum_{k=n}^{\infty} \langle f, \widetilde{\phi}_k \rangle \langle \phi_k, g \rangle.
\end{aligned}
$$

Lemma 2.1 therefore implies that Φ' is a Schauder basis for $\mathbb{H}$. □

5.3 Applications.

Heil and Yoon proved the following theorem in [21].

Theorem 5.4 *Let $N \in \mathbb{N}$ and $F \subseteq \mathbb{Z}$ be such that $|F| = N$. Then the system $E(t^N, \mathbb{Z} \setminus F)$ is exact in $L^2(\mathbb{T})$.*

From this we obtain the following corollary.

Corollary 5.5 *The system $E(t^N, \mathbb{Z})$ does not have a reproducing partner for any $N \in \mathbb{N}$.*

Proof By Theorem 5.4, we have that $E(t^N, \mathbb{Z})$ is overcomplete by N elements. Specifically, for any $F \subseteq \mathbb{Z}$ with $|F| = N$, the system $E(t^N, \mathbb{Z} \setminus F)$ is exact. Additionally, by Corollary 1 in [27], the exact system $E(t^N, \mathbb{Z} \setminus F)$ is not a Schauder basis for $L^2(\mathbb{T})$. Theorem 5.3 therefore implies that $E(t^N, \mathbb{Z})$ does not have a reproducing partner. □

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