MODULATION SPACES AS SYMBOL CLASSES FOR PSEUDODIFFERENTIAL OPERATORS

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Abstract. We investigate the Weyl calculus of pseudodifferential operators with the methods of time-frequency analysis. As symbol classes we use the modulation spaces, which are the function spaces associated to the short-time Fourier transform and the Wigner distribution. We investigate the boundedness and Schatten-class properties of pseudodifferential operators, and furthermore we study their mapping properties between modulation spaces.

1. Introduction

In the language of physics, the Weyl transform is a method of quantization, but it also has a wide range of applications in mathematics and engineering. The goal is to associate to each function or distribution σ defined on the phase-space $\mathbb{R}^{2d}$ a corresponding operator L_σ, which is the so-called quantization. The mapping from the symbol σ to its quantization L_σ should satisfy the requirements of the laws of quantum mechanics. We will focus in this article on the quantization proposed by Weyl. However, quantizations cannot be unique, and other quantizations play important roles in other areas. For example, the Kohn–Nirenberg quantization is especially important in partial differential equations. One useful feature of the techniques that we will describe in this article is that most results can be easily transferred among different quantizations.

The Weyl transform is usually analyzed using methods from hard analysis, and there is a rich literature on this subject, which we will not attempt to survey. Since the main ingredients in the Weyl transform are also the fundamental objects in the area of mathematics known as time-frequency analysis, we have pursued a complimentary study of the Weyl transform using time-frequency techniques in [HRT97], [GH99], [Grö01], [Heil03]. This has led to new results, including some which improve classical results of Hörmander, Daubechies, or Calderón–Vaillancourt. In this article we will present some of these results, especially those related to boundedness of the Weyl transform, with some new developments from [GH02] indicated at the end of the article. An exposition focusing on some results related to spectral properties of the operator L_σ is available in [Heil03].

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2. Background and Discussion

The fundamental operations of time-frequency analysis are the *translation operators*

$$T_x f(t) = f(t-x), \qquad t, x \in \mathbb{R}^d,$$

and the *modulation operators*

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t), \qquad t, \omega \in \mathbb{R}^d.$$

The compositions $T_x M_\omega f(t) = e^{2\pi i \omega\cdot(t-x)} f(t-x)$ or $M_\omega T_x f(t) = e^{2\pi i\omega\cdot t} f(t-x)$ are called *time-frequency shifts.*

The Weyl transform of a symbol σ on $\mathbb{R}^{2d}$ is defined formally by

$$L_\sigma f = \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\xi, u)\, e^{\pi i \xi\cdot u}\, M_\xi T_{-u} f \,\mathrm{d}u\,\mathrm{d}\xi\,.$$

Thus, $L_\sigma f$ is obtained as a superposition of time-frequency shifts of f, with $\hat{\sigma}(\xi, u)$ controlling the weighting of the amount of each shift. It therefore seems natural to study this operator using the tools associated with the mathematics of time-frequency shifts (see [How80] for one of the first approaches in this direction).

The problems arising from the theory of partial differential equations suggest using the classical symbol classes defined by

$$S^N_{\delta,\rho} = \{\sigma \in C^\infty(\mathbb{R}^{2d}) : |D^\alpha_x D^\beta_\omega \sigma(x,\omega)| \le C_{\alpha\beta}(1+|\omega|)^{N+\delta|\alpha|-\rho|\beta|} \text{ for all multi-indices } \alpha, \beta \ge 0\}\,. \quad (1)$$

On the other hand, the definition of the Weyl transform as a superposition of time-frequency shifts suggests instead that we measure symbols in terms of their time-frequency concentrations, and investigate the question how a pseudodifferential operator affects the time-frequency concentration of a function. Technically, this is done by investigating the class of function spaces known as the *modulation spaces.* Modulation space norms are quantitative measures of the time-frequency concentration of a function or distribution, and have proven useful in the study of many aspects of time-frequency analysis. In this article we will study the mapping properties of the Weyl transform between modulation spaces, and in particular the boundedness and Schatten-class properties on $L^2(\mathbb{R}^d)$. This question is not entirely new, for instance, Tachizawa used the classical symbol classes and found conditions under which the corresponding pseudodifferential operators map modulation spaces into each other [Tac94]. We employ the modulation spaces in two contexts: first, the modulation spaces on $\mathbb{R}^{2d}$ serve as symbol classes to which the symbol σ belongs, and second, the modulation spaces on $\mathbb{R}^d$ are the spaces on which the quantization operator L_σ acts.

We will summarize in this article some results on time-frequency methods and pseudodifferential operators obtained by the authors in the past few years. Some new developments are indicated at the end. We will mostly focus on results due to the authors and do not attempt to provide an exhaustive survey of pseudodifferential operator theory. Recently we have become aware that some of the results we will present are also accessible by means of hard analysis. In particular, similar conclusions might be derived by using the work of Sjöstrand [Sjö94] and Toft [Toft02].

3. Time-Frequency Representations

We discuss a few standard time-frequency representations and and their main properties. For a detailed discussion of time-frequency analysis see [Fol89], [Grö01]. In our presentation we follow mostly [Grö01], and most statements without proof are taken from that reference.

The Fourier transform will be normalized as $\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t)\, e^{-2\pi i \omega\cdot t}\, \mathrm{d}t$ where $x \cdot t$ is the standard inner product on $\mathbb{R}^d$.

Definition 3.1. Fix a non-zero window $g \in L^2(\mathbb{R}^d)$. Then the *short-time Fourier transform* (STFT) of f with respect to g is defined to be

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\, \overline{g(t-x)}\, e^{-2\pi i t\cdot\omega}\, \mathrm{d}t\,, \qquad x,\omega \in \mathbb{R}^d\,,$$

and the *cross Wigner distribution* of f and $g \in L^2(\mathbb{R}^d)$ is

$$W(f,g)(x,\omega) = \int_{\mathbb{R}^d} f(x+\tfrac{t}{2})\, \overline{g(x-\tfrac{t}{2})}\, e^{-2\pi i \omega\cdot t}\, \mathrm{d}t\,, \qquad x,\omega \in \mathbb{R}^d\,. \tag{2}$$

The quadratic expression $Wf := W(f,f)$ is called the Wigner distribution of f.

We will employ STFTs both on $\mathbb{R}^d$ and on $\mathbb{R}^{2d}$, making the appropriate changes in the definition in the latter case.

The STFT can be written in a number of equivalent ways, for example:

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle = (f \cdot T_x \bar{g})\hat{\ }(\omega)\,.$$

Clearly, in this formulation, the STFT can be extended to many dual pairs. In particular, if $g \in \mathcal{S}(\mathbb{R}^d)$, then $V_g f$ is defined for any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$. In this way the STFT becomes an instrument to measure the time-frequency concentration of distributions.

We will need to use the following operators:

Partial Fourier transform: $\mathcal{F}_1 F(x,\omega) = \int_{\mathbb{R}^d} F(t,\omega)\, e^{-2\pi i x\cdot t}\, \mathrm{d}t\,,$

Partial Fourier transform: $\mathcal{F}_2 F(x,\omega) = \int_{\mathbb{R}^d} F(x,t)\, e^{-2\pi i \omega\cdot t}\, \mathrm{d}t\,,$

Coordinate change: $\mathcal{T}_a F(x,t) = F(t, t-x)\,,$

Coordinate change: $\mathcal{T}_s F(x,t) = F(x+\tfrac{t}{2}, x-\tfrac{t}{2})\,,$

Reflection: $\mathcal{I} g(x) = g(-x)\,.$

We will use the following notation for the tensor product of functions:

$$(f \otimes g)(x,y) = f(x)g(y).$$

The following lemma gives some relationships between and different representations of these time-frequency representations.

Lemma 3.2. *Let $f,\ g \in L^2(\mathbb{R}^d)$. Then the following statements hold.*

(a) Relationship between the STFT and the Wigner distribution:

$$W(f,g)(x,\omega) = 2^d\, e^{4\pi i x\cdot\omega}\, V_{\mathcal{I}g} f(2x, 2\omega)\,, \qquad (x,\omega) \in \mathbb{R}^{2d}\,.$$

(b) Factorization of the STFT:

$$V_g f = \mathcal{F}_2 \mathcal{T}_a (f \otimes \bar{g})\,.$$

(c) Factorization of the Wigner distribution:

$$W(f,g) = \mathcal{F}_2 \mathcal{T}_s (f \otimes \bar{g})\,.$$

Proof. Statements (b) and (c) follow immediately from the definitions. To prove part (a), we set $u = x + \frac{t}{2}$ in (2) and obtain thereby

$$\begin{aligned} W(f,g)(x,\omega) &= \int_{\mathbb{R}^d} f(x+\tfrac{t}{2})\, \overline{g(x-\tfrac{t}{2})}\, e^{-2\pi i \omega\cdot t}\, \mathrm{d}t \\ &= 2^d \int_{\mathbb{R}^d} f(u)\, \overline{g(-(u-2x))}\, e^{-4\pi i \omega\cdot(u-x)}\, du \\ &= 2^d e^{4\pi i x\cdot\omega}\, V_{\mathcal{I}g} f(2x, 2\omega)\,. \end{aligned}$$ □

These factorizations make the proof of the following fundamental properties of the STFT almost trivial.

Lemma 3.3 (Orthogonality Relations)**.**

(a) *If $f_1,\ f_2,\ g_1,\ g_2 \in L^2(\mathbb{R}^d)$, then $V_{g_j} f_j \in L^2(\mathbb{R}^{2d})$ for $j = 1,\ 2$, and we have*

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)}\, \overline{\langle g_1, g_2 \rangle}_{L^2(\mathbb{R}^d)}\,.$$

(b) *In particular, if $\|g\|_2 = 1$, then V_g is an isometry from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$, and we have*

$$\langle f, h \rangle_{L^2(\mathbb{R}^d)} = \langle V_g f, V_g h \rangle_{L^2(\mathbb{R}^{2d})}\,.$$

Proof. Statement (b) is an immediate consequence of (a). To prove (a), we use the fact that the operators $\mathcal{F}_2$ and $\mathcal{T}_a$ are unitary on $L^2(\mathbb{R}^{2d})$ to calculate that

$$\begin{aligned} \langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} &= \big\langle \mathcal{F}_2 \mathcal{T}_a (f_1 \otimes \overline{g_1}),\ \mathcal{F}_2 \mathcal{T}_a (f_2 \otimes \overline{g_2}) \big\rangle_{L^2(\mathbb{R}^{2d})} \\ &= \big\langle f_1 \otimes \overline{g_1},\ f_2 \otimes \overline{g_2} \big\rangle_{L^2(\mathbb{R}^{2d})} \\ &= \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)}\, \overline{\langle g_1, g_2 \rangle}_{L^2(\mathbb{R}^d)}\,. \end{aligned}$$ □

In several places in this article we will prove a statement only for the case of test functions in the Schwartz class $\mathcal{S}$, and then appeal to routine density techniques to extend the argument to the entire space. In this context the following (well-known) observation is important, cf. [Fol89, Prop. 1.42].

Lemma 3.4.

(a) *If* $f,\, g \in \mathcal{S}(\mathbb{R}^d)$*, then* $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$ *and* $W(f,g) \in \mathcal{S}(\mathbb{R}^{2d})$*.*

(b) *If* $f,\, g \in \mathcal{S}'(\mathbb{R}^d)$*, then* $V_g f \in \mathcal{S}'(\mathbb{R}^{2d})$ *and* $W(f,g) \in \mathcal{S}'(\mathbb{R}^{2d})$*.*

Proof. Note that each of the operators $\mathcal{F}_2$, $\mathcal{T}_a$, and $\mathcal{T}_s$ are isomorphisms of $\mathcal{S}(\mathbb{R}^{2d})$ onto itself and of $\mathcal{S}'(\mathbb{R}^{2d})$ onto itself. Thus, if f and $g \in \mathcal{S}(\mathbb{R}^{2d})$, then $f \otimes \bar{g} \in \mathcal{S}(\mathbb{R}^{2d})$ and so $V_g f = \mathcal{F}_2 \mathcal{T}_a (f \otimes \bar{g}) \in \mathcal{S}(\mathbb{R}^{2d})$. The other statements follow similarly. □

It follows that the orthogonality relations extend from $L^2 \times L^2$ to other pairs of dual spaces. In particular, we have the following statement, where we use the convention that $\langle \cdot, \cdot \rangle$ always extends the inner product on L^2 and thus is conjugate linear in the second coordinate.

Lemma 3.5. *Assume that* $g \in \mathcal{S}(\mathbb{R}^d)$ *with* $\|g\|_2 = 1$*. Then we have for all* $f \in \mathcal{S}'(\mathbb{R}^d)$ *and* $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *that*

$$\langle f, \varphi \rangle = \langle V_g f, V_g \varphi \rangle \,. \tag{3}$$

Lemma 3.5 extends the isometry property of the STFT from $L^2 \times L^2$ to $\mathcal{S}' \times \mathcal{S}$. Note that we have $V_g \varphi \in \mathcal{S}(\mathbb{R}^{2d})$ and $V_g f \in \mathcal{S}'(\mathbb{R}^{2d})$ by Lemma 3.4, so the second duality in (3) is well-defined.

In many applications it is important to calculate the STFT of a Wigner distribution or of a STFT. For appropriately chosen windows this calculation leads to the following "magic formula" that is immensely useful. We refer to [Grö01], [Jam98], [Jan98], [CG02] for applications.

Lemma 3.6. *Let* $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *and set* $\Phi = W\varphi = W(\varphi,\varphi) \in \mathcal{S}(\mathbb{R}^{2d})$*. Writing* $z = (z_1, z_2)$*,* $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$*, we have*

$$V_\Phi\big(W(g,f)\big)(z,\zeta) = e^{-2\pi i z_2 \cdot \zeta_2}\, \overline{V_\varphi f(z_1 + \tfrac{\zeta_2}{2}, z_2 - \tfrac{\zeta_1}{2})}\, V_\varphi g(z_1 - \tfrac{\zeta_2}{2}, z_2 + \tfrac{\zeta_1}{2})\,. \tag{4}$$

Proof. In the calculation below we distinguish between V_Φ, which is the short-time Fourier transform of functions on $\mathbb{R}^{2d}$, and V_φ, which is the short-time Fourier transform of functions on $\mathbb{R}^d$. To prove (4), we will use the covariance properties of the STFT and the Wigner distribution, namely

$$W(T_{z_1} M_{z_2} \varphi, T_{z_1} M_{z_2} \varphi)(x,\omega) = W(\varphi,\varphi)(x - z_1, \omega - z_2)$$

and

$$V_{T_{\zeta_1} M_{\zeta_2} h}(T_{\zeta_1} M_{\zeta_2} g)(x,\omega) = e^{2\pi i (x\cdot\zeta_2 - \omega\cdot\zeta_1)} V_h g(x,\omega)\,.$$

The first step is to write $V_\Phi(W(g,f))$ as an inner product of two STFTs on $\mathbb{R}^{2d}$ as follows, by using the definition of the STFT on $\mathbb{R}^{2d}$, the covariance properties given above, and Lemma 3.2(a):

$$\begin{aligned} &V_\Phi\big(W(g,f)\big)(z,\zeta) \\ &= \iint_{\mathbb{R}^{2d}} W(g,f)(x,\omega)\, \overline{W(\varphi,\varphi)(x - z_1, \omega - z_2)}\, e^{-2\pi i (x\cdot\zeta_1 + \omega\cdot\zeta_2)}\, \mathrm{d}x\, \mathrm{d}\omega \end{aligned}$$

$$= 2^{2d} \iint_{\mathbb{R}^{2d}} V_{\mathcal{I}f} g(2x, 2\omega) \, \overline{V_{\mathcal{I}T_{z_1}M_{z_2}\varphi}(T_{z_1}M_{z_2}\varphi)(2x, 2\omega)} \, e^{-2\pi i (x\cdot\zeta_1 + \omega\cdot\zeta_2)} \, dx \, d\omega$$

$$= \iint_{\mathbb{R}^{2d}} V_{\mathcal{I}f} g(x, \omega) \, e^{2\pi i (-x\cdot\frac{\zeta_1}{2} - \omega\cdot\frac{\zeta_2}{2})} \cdot \overline{V_{\mathcal{I}T_{z_1}M_{z_2}\varphi}(T_{z_1}M_{z_2}\varphi)(x, \omega)} \, dx \, d\omega$$

$$= \iint_{\mathbb{R}^{2d}} V_{T_{\frac{\zeta_2}{2}} M_{-\frac{\zeta_1}{2}} \mathcal{I}f}(T_{\frac{\zeta_2}{2}} M_{-\frac{\zeta_1}{2}} g)(x, \omega) \cdot \overline{V_{\mathcal{I}T_{z_1}M_{z_2}\varphi}(T_{z_1}M_{z_2}\varphi)(x, \omega)} \, dx \, d\omega$$

$$= \Big\langle V_{T_{\frac{\zeta_2}{2}} M_{-\frac{\zeta_1}{2}} \mathcal{I}f}(T_{\frac{\zeta_2}{2}} M_{-\frac{\zeta_1}{2}} g), \, V_{\mathcal{I}T_{z_1}M_{z_2}\varphi}(T_{z_1}M_{z_2}\varphi) \Big\rangle .$$

After these easy (but ugly) reformulations, we can apply the orthogonality relations (Lemma 3.3) to obtain

$$V_\Phi(W(g,f))(z,\zeta) = \Big\langle T_{\frac{\zeta_2}{2}} M_{-\frac{\zeta_1}{2}} g, \, T_{z_1}M_{z_2}\varphi \Big\rangle \cdot \overline{\Big\langle T_{\frac{\zeta_2}{2}} M_{-\frac{\zeta_1}{2}} \mathcal{I}f, \, \mathcal{I}T_{z_1}M_{z_2}\varphi \Big\rangle} . \tag{5}$$

After commuting translations and modulations to the correct positions, the first factor on the right side of (5) can be rewritten as

$$\Big\langle g, \, M_{\frac{\zeta_1}{2}} T_{z_1 - \frac{\zeta_2}{2}} M_{z_2}\varphi \Big\rangle = e^{2\pi i z_2\cdot(z_1 - \frac{\zeta_2}{2})} \, V_\varphi g\big(z_1 - \tfrac{\zeta_2}{2}, z_2 + \tfrac{\zeta_1}{2}\big) .$$

Since $\mathcal{I}(T_u M_\eta f) = T_{-u} M_{-\eta} \mathcal{I} f$ and since $\mathcal{I}$ is unitary, the second factor on the right side of (5) can be rewritten as

$$\overline{\Big\langle \mathcal{I}T_{-\frac{\zeta_2}{2}} M_{\frac{\zeta_1}{2}} f, \, \mathcal{I}T_{z_1}M_{z_2}\varphi \Big\rangle} = \overline{\Big\langle f, \, M_{-\frac{\zeta_1}{2}} T_{z_1 + \frac{\zeta_2}{2}} M_{z_2}\varphi \Big\rangle}$$
$$= e^{-2\pi i z_2\cdot(z_1 + \frac{\zeta_2}{2})} \, \overline{V_\varphi f\big(z_1 + \tfrac{\zeta_2}{2}, \, z_2 - \tfrac{\zeta_1}{2}\big)} .$$

The formula for $V_\Phi(W(g,f))$ then follows. □

4. Modulation Spaces

The modulation space norms provide a quantitative measure of time-frequency concentration. We use the STFT to define them, but we could just as well use the Wigner distribution or some other time-frequency representation. For simplicity, in this article we will only consider weights with polynomial growth, i.e., we assume that m is a non-negative function satisfying the condition

$$m(z_1 + z_2) \le C(1 + |z_1|^2)^{s/2} \, m(z_2) , \qquad z_1, z_2 \in \mathbb{R}^{2d} ,$$

for some constant $C > 0$.

Definition 4.1. Fix a nonzero *window function* $g \in \mathcal{S}(\mathbb{R}^d)$, and let $1 \le p, q \le \infty$. Then the modulation space $M^{p,q}_m(\mathbb{R}^d)$ is the subspace of the tempered distributions consisting of all $f \in \mathcal{S}'(\mathbb{R}^d)$ for which

$$\|f\|_{M^{p,q}_m} = \|V_g f\|_{L^{p,q}_m}$$
$$= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p \, m(x,\omega)^p \, dx \right)^{q/p} d\omega \right)^{1/q} < \infty , \tag{6}$$

with the usual adjustments if p or q is ∞. If $m = 1$ then we omit writing it.

Modulation spaces can also be defined for weights that grow faster than polynomially; for the necessary modifications see [Grö01, Ch. 11.4].

The modulation spaces were invented and extensively investigated by Hans Feichtinger over the period 1980–1995, with some of the main references being [Fei80], [Fei81], [FG89a], [FG89b], [FG97]. However, the first official publication is due to Triebel [Tri83]. While the original motivation was to investigate smoothness spaces in analogy to the Besov spaces, the main purpose of modulation spaces is their applications to time-frequency analysis. The modulation spaces are the appropriate function spaces for time-frequency analysis, and they occur in any mathematical analysis of problems where the time-frequency shifts $M_\omega T_x$ occur.

For a demonstration that the modulation spaces are well-defined see, for example, [Grö01, Prop. 11.3.2]. In fact, if m is a weight function with polynomial growth, then the definition of $M^{p,q}_m$ is independent of the choice of the window $g \in \mathcal{S}(\mathbb{R}^d)$, and different windows g yield equivalent norms on $M^{p,q}_m$.

For a detailed development of the theory of modulation spaces we refer to the original literature mentioned above and to [Grö01, Ch. 11–13]. The modulation spaces possess an elegant structure theory and possess atomic decompositions similar to the Besov spaces.

Among the classical function spaces the following spaces can be identified as modulation spaces.

Proposition 4.2 (Identification of classical spaces as modulation spaces)**.** *We have the following for $s \in \mathbb{R}$.*

(a) $M^{2,2} = L^2$.

(b) *If* $m(x,\omega) = (1+|x|)^s$, *then* $L^2_s := \{f \in \mathcal{S}'(\mathbb{R}^d) : fm \in L^2(\mathbb{R}^d)\} = M^{2,2}_m$.

(c) *If* $m(x,\omega) = (1+|\omega|^2)^{s/2}$, *then* $H^s := \{f \in \mathcal{S}'(\mathbb{R}^d) : \hat{f}m \in L^2(\mathbb{R}^d)\} = M^{2,2}_m$.

(d) *If* $v_s(x,\omega) = (1+|x|^2+|\omega|^2)^{s/2}$, *then* $M^{2,2}_{v_s} = L^2_s \cap H^s$. *Furthermore,* $M^{2,2}_{v_s}$ *coincides with the Shubin class* Q^s [Shu01].

(e) *The space* $M^{1,1} = \{f \in L^2(\mathbb{R}^d) : V_g f \in L^1(\mathbb{R}^{2d})\}$ *coincides with Feichtinger's algebra* S_0.

(f) *The Schwartz class of test functions can be identified as* $\mathcal{S} = \bigcap_{s\geq 0} M^{p,p}_{v_s}$.

(g) *The space of tempered distribution is* $\mathcal{S}' = \bigcup_{s\leq 0} M^{\infty,\infty}_{v_s}$.

Among the many properties of the modulation spaces, we note the following; see also Section 6.2 below for inclusion and interpolation properties.

Theorem 4.3 (Duality)**.** *If* $1 \leq p, q < \infty$, *then the dual space of* $M^{p,q}_m$ *is isometrically isomorphic to* $M^{p',q'}_{1/m}$, *where* $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$. *In short,* $(M^{p,q}_m)^* = M^{p',q'}_{1/m}$.

5. Representations of Pseudodifferential Operators

5.1. **The Weyl calculus.** The Weyl transform associates an operator to each function or distribution on the time-frequency plane. The Weyl calculus is a particular form for representing general pseudodifferential operators.

Definition 5.1. Given a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, the pseudodifferential operator L_σ is defined to be

$$L_\sigma f = \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\xi, u)\, e^{-\pi i \xi \cdot u}\, T_{-u} M_\xi f \, \mathrm{d}u \, \mathrm{d}\xi \,. \tag{7}$$

The operator L_σ is called the *Weyl transform* of the (Weyl) *symbol* σ, and $\hat{\sigma}$ is called the *spreading function*.

We first explain how the integral (7) should be understood. If $\hat{\sigma} \in L^1(\mathbb{R}^{2d})$, then L_σ is of course well-defined as an absolutely convergent integral in operator norm, and we have

$$\|L_\sigma\|_{\mathrm{op}} \le \|\hat{\sigma}\|_1 \,,$$

where the operator norm is measured on $L^2(\mathbb{R}^d)$. However, L_σ is well-defined even for arbitrary symbols in $\mathcal{S}'(\mathbb{R}^{2d})$ if we interpret the integral in (7) in a weak sense. That is, for $f,\, g \in \mathcal{S}(\mathbb{R}^d)$, we interpret (7) as meaning that

$$\langle L_\sigma f, g\rangle = \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\xi, u)\, e^{-\pi i \xi \cdot u}\, \langle T_{-u} M_\xi f, g\rangle \, \mathrm{d}u \, \mathrm{d}\xi \,, \qquad f, g \in \mathcal{S}(\mathbb{R}^d) \,. \tag{8}$$

Hence with $f \in \mathcal{S}(\mathbb{R}^d)$ we have $L_\sigma f$ defined as a functional on $\mathcal{S}(\mathbb{R}^d)$. In fact, this functional is continuous on $\mathcal{S}(\mathbb{R}^d)$, and so $L_\sigma f$ is a tempered distribution. The following lemma makes this statement precise.

Lemma 5.2. *Let $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$. If $f \in \mathcal{S}(\mathbb{R}^d)$, then* (8) *defines a tempered distribution $L_\sigma f \in \mathcal{S}'(\mathbb{R}^d)$. Further, L_σ is a continuous mapping of $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$.*

Proof. Since σ is a tempered distribution on $\mathbb{R}^{2d}$, its Fourier transform $\hat{\sigma}$ is also a tempered distribution on $\mathbb{R}^{2d}$. If we define $A_{f,g}(\xi, u) = e^{-\pi i \xi \cdot u}\, \langle T_{-u} M_\xi f, g\rangle = e^{-\pi i \xi \cdot u}\, V_g f(u, -\xi)$ (this is the *cross ambiguity function* of f and g), then the weak interpretation of the integral in (8) is simply the statement that

$$\langle L_\sigma f, g\rangle = \left\langle \hat{\sigma},\, \overline{A_{f,g}} \right\rangle \,. \tag{9}$$

Now, since $f,\, g \in \mathcal{S}(\mathbb{R}^d)$, we have by Lemma 3.4 that $A_{f,g} \in \mathcal{S}(\mathbb{R}^{2d})$, so (9) is well-defined and in fact defines $L_\sigma f$ as a tempered distribution on $\mathbb{R}^d$. Thus we have that $f \mapsto L_\sigma$ maps $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$, and it can be shown that this mapping is continuous. □

Our next goal is to express L_σ as an integral operator with a distributional kernel, and then to derive a formula for L_σ directly in terms of the symbol σ rather than in terms of $\hat{\sigma}$.

To derive these formulas, assume first that all integrals in the following calculation converge absolutely. For example, this is the case when $\hat{\sigma} \in L^1(\mathbb{R}^{2d})$ and $f \in \mathcal{S}(\mathbb{R}^d)$. We compute that

$$\begin{aligned}
L_\sigma f(x) &= \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\xi, u)\, e^{-\pi i u\cdot\xi}\, T_{-u} M_\xi f(x)\, \mathrm{d}u\, \mathrm{d}\xi \\
&= \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\xi, u)\, e^{2\pi i \xi\cdot(x+\frac{u}{2})}\, f(u+x)\, \mathrm{d}u\, \mathrm{d}\xi \\
&\underset{y=u+x}{=} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \hat{\sigma}(\xi, y-x)\, e^{2\pi i \xi\cdot\frac{x+y}{2}}\, \mathrm{d}\xi \right) f(y)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^{2d}} k(x,y)\, f(y)\, \mathrm{d}y\,.
\end{aligned}$$

This expresses L_σ as an integral operator. The integral kernel

$$k(x,y) = \int_{\mathbb{R}^d} \hat{\sigma}(\xi, y-x)\, e^{2\pi i \xi\cdot\frac{x+y}{2}}\, \mathrm{d}\xi$$

is an inverse partial Fourier transform in the first variable of $\hat{\sigma}$, followed by a coordinate transformation. Let us rewrite this in order to obtain a relationship directly between k and σ. First, note that

$$\mathcal{F}_1^{-1}\hat{\sigma} = \mathcal{F}_1^{-1}\mathcal{F}_1\mathcal{F}_2\sigma = \mathcal{F}_2\sigma.$$

Second, observe that the inverse of the symmetric coordinate transformation $\mathcal{T}_s = F(x+\frac{t}{2}, x-\frac{t}{2})$ is $\mathcal{T}_s^{-1}F(x,y) = F(\frac{x+y}{2}, x-y)$. Hence,

$$\begin{aligned}
k(x,y) &= \mathcal{F}_1^{-1}\hat{\sigma}\big(\tfrac{x+y}{2}, y-x\big) \\
&= \mathcal{F}_2\sigma\big(\tfrac{x+y}{2}, y-x\big) \\
&= \mathcal{F}_2^{-1}\sigma\big(\tfrac{x+y}{2}, x-y\big) \\
&= \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\sigma(x,y)\,. \qquad (10)
\end{aligned}$$

Since $\mathcal{T}_s$ and $\mathcal{F}_2$ are isomorphisms on $\mathcal{S}'(\mathbb{R}^{2d})$, we can use equation (10) as a definition that allows us to interpret arbitrary pseudodifferential operators L_σ with distributional symbols $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ as integral operators with distributional kernels. The following proposition makes this statement precise, cf. [Fol89].

Proposition 5.3. *Let $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$. Then the Weyl transform can be written as an integral operator with the distributional kernel*

$$k = \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\sigma \in \mathcal{S}'(\mathbb{R}^{2d})\,, \qquad (11)$$

in the sense that

$$\langle L_\sigma f, g\rangle = \big\langle k,\, g\otimes \bar{f}\big\rangle \qquad \text{for } f, g \in \mathcal{S}(\mathbb{R}^d)\,.$$

Furthermore,

$$\langle L_\sigma f, g\rangle = \langle \sigma,\, W(g,f)\rangle \qquad \text{for } f, g \in \mathcal{S}(\mathbb{R}^d)\,. \qquad (12)$$

Proof. Formula (11) is the distributional extension of (10) from the case $\hat{\sigma} \in L^1(\mathbb{R}^{2d})$ to the case $\hat{\sigma} \in \mathcal{S}'(\mathbb{R}^{2d})$. Equation (12) follows from the calculation

$$\langle L_\sigma f, g\rangle = \langle k, g \otimes \bar{f}\rangle = \langle \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\sigma, g \otimes \bar{f}\rangle = \langle \sigma, \mathcal{F}_2\mathcal{T}_s(g \otimes \bar{f})\rangle = \langle \sigma, W(g,f)\rangle \,,$$

the last step following from Lemma 3.2. □

Proposition 5.3 underlines the importance of the Wigner distribution in the analysis of pseudodifferential operators. In particular, formula (12) can be used to study how the time-frequency concentration of a function is transformed by a pseudodifferential operator.

5.2. **The Kohn–Nirenberg correspondence.** The Kohn–Nirenberg correspondence assigns to a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ the operator K_σ defined by

$$K_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\, \hat{f}(\omega)\, e^{2\pi i x\cdot\omega}\, \mathrm{d}\omega \,.$$

This is the classical version of pseudodifferential operators that is used in the investigation of partial differential operators, cf. [Hör90]. As in the case of the Weyl calculus, it can be shown that K_σ is a continuous mapping from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. Furthermore, K_σ can be represented as a superposition of time-frequency shifts as follows:

$$\begin{aligned} K_\sigma f(x) &= \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\eta, y-x)\, e^{2\pi i \eta\cdot x}\, f(y)\, \mathrm{d}\eta\, \mathrm{d}y \\ &= \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\eta, u)\, e^{2\pi i \eta\cdot x}\, f(x+u)\, \mathrm{d}u\, \mathrm{d}\eta \\ &= \iint_{\mathbb{R}^{2d}} \hat{\sigma}(\eta, u)\, \big(M_\eta T_{-u} f\big)(x)\, \mathrm{d}u\, \mathrm{d}\eta \,. \end{aligned} \tag{13}$$

Comparing (13) with (7) reveals that K_σ and L_σ differ in their definitions only by the "chirp" $e^{\pi i u\cdot\xi}$. More precisely, define the operator $\mathcal{U}$ acting on symbols by

$$(\mathcal{U}\sigma)\hat{}\,(\xi,u) = e^{\pi i u\cdot\xi}\, \hat{\sigma}(\xi,u)\,.$$

Then $\mathcal{U}$ is a unitary operator on $L^2(\mathbb{R}^{2d})$ and is an isomorphism on $\mathcal{S}(\mathbb{R}^{2d})$ and on $\mathcal{S}'(\mathbb{R}^{2d})$. The Kohn–Nirenberg and the Weyl correspondences are then related by the formula

$$K_{\mathcal{U}\sigma} = L_\sigma \,. \tag{14}$$

This connection makes it easy to switch from the Kohn–Nirenberg calculus to the Weyl calculus and vice versa. We will only treat the Weyl transform. Using (14), it is not hard to show that the results of this paper also hold for the Kohn–Nirenberg calculus (and in fact for a family of other pseudodifferential operators). See [GH99], [Grö01], [Sjö94], [Toft01], [Toft02] for details.

6. Pseudodifferential Operators on $L^2(\mathbb{R}^d)$

For pseudodifferential operators with respect to the traditional symbol classes, the mapping properties between modulation spaces were studied by Tachizawa and Rochberg [Tac94], [RT97] (see also [CR02]). Modulation spaces figure implicitly in the work of Heil, Ramanathan, and Topiwala [HRT97], and implicitly in Sjöstrand [Sjö94], Boulkhemair [Bou99], and Toft [Toft01]. Modulation spaces enter explicitly in [GH99], where $M^{\infty,1}$ in particular is used as a symbol class to establish boundedness results of pseudodifferential operators on $M^{p,p}$. Further developments in this spirit were obtained by Labate [Lab01a], [Lab01b], and recently the boundedness of bilinear pseudodifferential operators on the modulation spaces has been investigated by Bényi and Okoudjou [BO02].

To deal with boundedness of pseudodifferential operators, we will use the techniques developed in [Grö01, Ch. 14.5].

6.1. **Endpoint results.**

We first establish some natural endpoint results.

In the following, $\mathcal{I}_p$ will denote for $1 \le p < \infty$ the p-Schatten class, which is the Banach space of all compact operators on $L^2(\mathbb{R}^d)$ whose singular values lie in ℓ^p [BS87], [DS88], [Sim79]. The singular values of a compact operator $A\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ can be defined by spectral theory as the square-roots of the eigenvalues of the positive, compact, self-adjoint operator A^*A, i.e.,

$$s_j(A) = \sqrt{\lambda_j(A^*A)}, \qquad j = 1, 2, 3, \dots .$$

Alternatively, since $L^2(\mathbb{R}^d)$ is a Hilbert space, the singular values of A coincide with its approximation numbers, which are defined in terms of best approximations of A by finite-rank operators. Specifically,

$$s_j(A) = \inf\bigl\{\|A - T\|_{\mathrm{op}} : \mathrm{rank}(T) < j\bigr\}, \qquad j = 1, 2, 3, \dots .$$

For a compact, self-adjoint operator on $L^2(\mathbb{R}^d)$, the singular values are simply the absolute values of the eigenvalues. The Schatten class $\mathcal{I}_p$ is the Banach space of all compact operators A for which the norm

$$\|A\|_{\mathcal{I}_p} = \|\{s_j(A)\}\|_{\ell^p} = \left(\sum_{j=1}^{\infty} s_j(A)^p\right)^{1/p}$$

is finite. The space $\mathcal{I}_2$ is called the space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^d)$, and $\mathcal{I}_1$ is the space of trace-class operators. Although not a standard notation, for convenience we will denote the Banach space of all bounded operators on $L^2(\mathbb{R}^d)$ by $\mathcal{I}_\infty$, with norm $\|A\|_{\mathcal{I}_\infty} = \|A\|_{\mathrm{op}}$.

We now establish several useful results for the cases $p = 2$, 1, and ∞, respectively.

Hilbert–Schmidt operators. The Hilbert–Schmidt operators can be exactly characterized in terms of the modulation space $M^{2,2} = L^2$.

Theorem 6.1 (Pool)**.** *The Weyl transform $\sigma \longmapsto L_\sigma$ is a unitary map $L^2(\mathbb{R}^{2d}) \to \mathcal{I}_2$.*

Proof. Recall that an operator A is a Hilbert–Schmidt operator on $L^2(\mathbb{R}^d)$ if and only if A is an integral operator of the form $Af(x) = \int_{\mathbb{R}^d} k(x,y)\, f(y)\, \mathrm{d}y$ with a kernel $k \in L^2(\mathbb{R}^{2d})$. In this case, $\|A\|_{\mathcal{I}_2} = \|k\|_{L^2(\mathbb{R}^{2d})}$. By Theorem 5.3, A can be written as a pseudodifferential operator L_σ with Weyl symbol $\sigma = \mathcal{F}_2\mathcal{T}_s k$. Since both $\mathcal{T}_s$ and $\mathcal{F}_2$ are unitary operators on $L^2(\mathbb{R}^d)$, we deduce that $\|A\|_{\mathcal{I}_2} = \|\sigma\|_{L^2(\mathbb{R}^{2d})}$. □

Trace Class Operators. The simplest trace class result using modulation spaces is due to Feichtinger and Sjöstrand, with the first proof published in [Grö96]. This result gives a sufficient but not necessary condition on the symbol σ which ensures that the corresponding operator L_σ is trace-class. For simple proofs, see [GH99, Prop. 4.1] or [Heil03, Prop. 6.1].

Theorem 6.2. *If $\sigma \in M^{1,1}$, then $L_\sigma \in \mathcal{I}_1$.*

Theorem 6.2 should be compared to the trace-class results that were obtained independently by Daubechies [Dau80] and Hörmander [Hör79]. Using the identification of Lemma 4.2(d), their results can be formulated as follows: *If $\sigma \in M^{2,2}_{v_s}$ for $s > 2d$, then $L_\sigma \in \mathcal{I}_1$.* That Theorem 6.2 is slightly sharper can be seen from the elementary embedding

$$M^{2,2}_{v_s} \hookrightarrow M^{1,1} \quad \Longleftrightarrow \quad s > 2d.$$

So far the best result along these lines seems to be due to Heil, Ramanathan, and Topiwala [HRT97]: *If $\sigma \in M^{2,2}_{v_s}$ for $s > d$, then L_σ, $K_\sigma \in \mathcal{I}_1$.* See [Heil03] for an exposition of the proof and some extensions, and [PS75] for a related result.

Boundedness. In order to analyze the boundedness of the pseudodifferential operator L_σ, we fix any window $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\|\varphi\|_2 = 1$, and define a window on $\mathbb{R}^{2d}$ by $\Phi = W(\varphi,\varphi)$. Then we use Proposition 5.3 and Lemma 3.5 to write

$$\langle L_\sigma f, g\rangle = \langle \sigma,\, W(g,f)\rangle = \langle V_\Phi \sigma,\, V_\Phi W(g,f)\rangle\,. \tag{15}$$

Note that for f, $g \in \mathcal{S}(\mathbb{R}^d)$ and $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, the duality in (15) is well-defined, since by Lemma 3.4 we have that $W(g,f) \in \mathcal{S}(\mathbb{R}^{2d})$, and therefore $V_\Phi\sigma \in \mathcal{S}'(\mathbb{R}^{4d})$ and $V_\Phi W(g,f) \in \mathcal{S}(\mathbb{R}^{4d})$.

The formulas above literally demand the use of modulation space norms. Since we have already calculated the STFT of a Wigner distribution in Lemma 3.6, the following boundedness result is almost immediate. This result generalizes and simplifies [GH99, Thm 1.1].

Theorem 6.3. *If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$, then L_σ is a bounded operator from $M^{p,q}(\mathbb{R}^d)$ to $M^{p,q}(\mathbb{R}^d)$ for each $1 \le p,q \le \infty$, with a uniform estimate*

$$\|L_\sigma\|_{\mathrm{op}} \le \|\sigma\|_{M^{\infty,1}}\,.$$

In particular, L_σ is bounded on $L^2(\mathbb{R}^d)$.

Proof. If f, $g \in \mathcal{S}(\mathbb{R}^d)$, then we have $W(g,f) \in \mathcal{S}(\mathbb{R}^{2d})$ by Lemma 3.4, and hence another application of Lemma 3.4 yields that $V_\Phi(W(g,f)) \in \mathcal{S}(\mathbb{R}^{4d})$. On the other hand, since $\sigma \in M^{\infty,1}(\mathbb{R}^{2d}) \subset M^{\infty,\infty}(\mathbb{R}^{2d})$, we have by definition of $M^{\infty,\infty}$ that the short-time Fourier transform $V_\Phi\sigma$ is a bounded function. Hence the duality in (15) is given by the integral

$$\langle L_\sigma f, g\rangle = \langle V_\Phi\sigma,\, V_\Phi W(g,f)\rangle = \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} V_\Phi\sigma(z,\zeta)\,\overline{V_\Phi(W(g,f))(z,\zeta)}\,\mathrm{d}z\,\mathrm{d}\zeta\,. \quad (16)$$

The integral converges absolutely, and in particular we can interchange the order of integration as we like. We apply a special case of Hölder's inequality for mixed norm spaces, namely,

$$|\langle F, G\rangle| \le \|F\|_{L^{\infty,1}}\,\|G\|_{L^{1,\infty}}\,,$$

to (16), to obtain that

$$\begin{aligned}|\langle L_\sigma f, g\rangle| &\le \|V_\Phi\sigma\|_{L^{\infty,1}(\mathbb{R}^{4d})}\ \ \|V_\Phi(W(g,f))\|_{L^{1,\infty}(\mathbb{R}^{4d})}\\ &= \|\sigma\|_{M^{\infty,1}(\mathbb{R}^{2d})}\cdot \sup_{\zeta\in\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}\big|V_\Phi\big(W(g,f)\big)(z,\zeta)\big|\,\mathrm{d}z\,. \quad (17)\end{aligned}$$

To estimate the integral over $z=(z_1,z_2)$ appearing in (17), we use Lemma 3.6. Write $\tilde\zeta = (-\frac{\zeta_2}{2},\frac{\zeta_1}{2})\in\mathbb{R}^{2d}$, so that $V_\varphi f(z_1+\frac{\zeta_2}{2}, z_2-\frac{\zeta_1}{2}) = T_{\tilde\zeta}V_\varphi f(z)$. Then, using Lemma 3.6 and Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^{2d}}\big|V_\Phi\big(W(g,f)\big)(z,\zeta)\big|\,\mathrm{d}z &= \int_{\mathbb{R}^{2d}}|T_{\tilde\zeta}V_\varphi f(z)|\,|T_{-\tilde\zeta}V_\varphi g(z)|\,\mathrm{d}z\\ &\le \|T_{\tilde\zeta}V_\varphi f\|_{L^{p,q}(\mathbb{R}^{2d})}\,\|T_{-\tilde\zeta}V_\varphi g\|_{L^{p',q'}(\mathbb{R}^{2d})}\\ &= \|V_\varphi f\|_{L^{p,q}(\mathbb{R}^{2d})}\,\|V_\varphi g\|_{L^{p',q'}(\mathbb{R}^{2d})}\\ &= \|f\|_{M^{p,q}(\mathbb{R}^{d})}\,\|g\|_{M^{p',q'}(\mathbb{R}^{d})}\,, \quad (18)\end{aligned}$$

independently of $\zeta\in\mathbb{R}^{2d}$. Combining (17) and (18), we have proved that

$$|\langle L_\sigma f, g\rangle| \le \|\sigma\|_{M^{\infty,1}(\mathbb{R}^{2d})}\,\|f\|_{M^{p,q}(\mathbb{R}^{d})}\,\|g\|_{M^{p',q'}(\mathbb{R}^{d})} \quad (19)$$

for $f,\ g\in\mathcal{S}(\mathbb{R}^d)$. This extends by a standard density argument to all $f\in M^{p,q}$ and $g\in M^{p',q'} = (M^{p,q})^*$. Consequently, we have $L_\sigma f\in M^{p,q}$, and taking the supremum in (19) over all g with $\|g\|_{M^{p',q'}}=1$ yields

$$\|L_\sigma f\|_{M^{p,q}(\mathbb{R}^d)} \le \|\sigma\|_{M^{\infty,1}(\mathbb{R}^{2d})}\,\|f\|_{M^{p,q}(\mathbb{R}^d)}\,.$$

Hence L_σ is bounded on $M^{p,q}$, and its operator norm is at most $\|\sigma\|_{M^{\infty,1}}$. □

Thus, the modulation space $M^{\infty,1}(\mathbb{R}^{2d})$ arises naturally as a symbol class for the study of pseudodifferential operators, and moreover is particularly well-suited for the purposes of time-frequency analysis. To compare Theorem 6.3 with analogous results using standard function spaces, we quote the following embedding theorem. Here $C_b^{d+1}(\mathbb{R}^d)$ denotes the space of all functions which are differentiable up to order $d+1$ with bounded derivatives.

Theorem 6.4. *$C_b^{d+1}(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d)$ and there exists $C > 0$ such that*

$$\|f\|_{M^{\infty,1}} \le C\,\|f\|_{C^{d+1}} = C \sum_{|\alpha|\le d+1} \|D^\alpha f\|_\infty \,.$$

Remarks 6.5. a. In particular, it follows from this embedding that *if $\sigma \in C_b^{2d+1}(\mathbb{R}^{2d})$, then L_σ is bounded on $M^{p,q}$ for every $1 \le p, q \le \infty$.* The statement that L_σ is bounded on $L^2 = M^{2,2}$ when $\sigma \in C_b^{2d+1}(\mathbb{R}^{2d})$ is in the spirit of the celebrated *Calderón–Vaillancourt Theorem* [CV72], cf. [Fol89, Thm. 2.73].

b. A more elaborate argument of Heil, Ramanathan, and Topiwala shows the stronger embedding result $C^s(\mathbb{R}^{2d}) \subset M^{\infty,1}(\mathbb{R}^{2d})$ for $s > 2d$, where C^s is the Hölder–Lipschitz space of order s [HRT97, Prop. 6.3]. A more direct proof of this embedding has recently been found by Okoudjou, as a special case of a class of embeddings of Besov and Triebel–Lizorkin spaces into the modulation spaces [Oko02]. A different set of embeddings of Besov spaces into modulation spaces has also been found by Toft [Toft02] and by Gröbner in his earlier, unpublished thesis [Gröb91].

c. In contrast to the standard theory of pseudodifferential operators, which works with C^∞-symbols, the hypotheses of Theorem 6.3 do not require smoothness of σ in order to obtain the boundedness of L_σ on L^2 or on $M^{p,q}$. See [GH99] for examples.

d. A special case of Theorem 6.3 was proved by Sjöstrand [Sjö94], who was apparently unaware of the extended theory of modulation spaces that was available. Among hard analysts, the space $M^{\infty,1}$ is sometimes known as Sjöstrand's class.

Using relation (14) between the Kohn–Nirenberg correspondence and the Weyl calculus, it is easy to transfer Theorem 6.3 to the Kohn–Nirenberg symbol, giving the following result [Grö01, Cor. 14.5.5].

Corollary 6.6. *If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$, then K_σ is a bounded operator on $M^{p,q}$ for each $1 \le p, q \le \infty$.*

A boundedness result similar to Theorem 6.3 holds for weighted modulation spaces as well. The proof is a modification of the proof of Theorem 6.3 and will be omitted, see [Grö01, Ch. 14.5].

Theorem 6.7. *Let v be a submultiplicative weight on $\mathbb{R}^{2d}$ and set $w(z,\zeta) = v(\tilde{\zeta})^2 = v(-\frac{\zeta_2}{2}, \frac{\zeta_1}{2})^2$ on $\mathbb{R}^{4d}$. If $\sigma \in M_w^{\infty,1}(\mathbb{R}^{2d})$, then L_σ is bounded on $M_m^{p,q}(\mathbb{R}^{2d})$ for all $1 \le p, q \le \infty$ and all v-moderate weights m.*

6.2. **Refinements.** To refine the endpoint results of the previous section, we use further properties of the modulation spaces, namely their basic inclusion and interpolation properties.

The following inclusion result is trivial using the atomic decompositions associated to the modulation spaces (Gabor frames or Wilson bases), see [Grö01, Thm. 12.2.2].

Theorem 6.8 (Inclusion Theorem)**.** *$M^{p_1,q_1} \subset M^{p_2,q_2} \iff p_1 \le p_2,\ q_1 \le q_2$.*

In addition, the modulation spaces interpolate similarly to the mixed-norm $L^{p,q}$-spaces. For the case of complex interpolation, we have the following result [Fei81].

Theorem 6.9 (Complex Interpolation)**.**

(a) $[M^{1,1}, M^{2,2}]_\theta = M^{p,p}$ *for* $1 \le p \le 2$.

(b) $[M^{2,2}, M^{\infty,1}]_\theta = M^{p,p'}$ *for* $2 \le p \le \infty$.

Since the p-Schatten classes also interpolate like L^p-spaces, namely, $[\mathcal{I}_1, \mathcal{I}_\infty]_\theta = \mathcal{I}_p$ for $1 \le p \le \infty$, we can now prove the following statement, see also [GH99, Cor. 3.5].

Theorem 6.10.

(a) *If* $1 \le p, q \le 2$ *and* $\sigma \in M^{p,q}$, *then* $L_\sigma \in \mathcal{I}_{\max\{p,q\}}$.

(b) *If* $p \ge 2$ *and* $q \le p'$, *and if* $\sigma \in M^{p,q}$, *then* $L_\sigma \in \mathcal{I}_p$.

(c) *In particular, if* $q \le 2$ *and either* $1 \le p \le 2$ *or* $p \le q'$, *then* L_σ *is a bounded operator on* $L^2(\mathbb{R}^d)$.

Proof. (a) We first use interpolation. Fix $1 \le p \le 2$. Then by combining Theorem 6.9 with the interpolation of the Schatten classes, we obtain that if $\sigma \in [M^{1,1}, M^{2,2}]_\theta = M^{p,p}$, then $L_\sigma \in [\mathcal{I}_1, \mathcal{I}_2]_\theta = \mathcal{I}_p$.

Next, we use the inclusion relations. Fix $1 \le p, q \le 2$, and set $\mu = \max\{p, q\}$. Then by Theorem 6.8, we have $M^{p,q} \subset M^{\mu,\mu}$ and therefore $L_\sigma \in \mathcal{I}_\mu$, so part (a) is proved.

(b) Again we use interpolation and then inclusions. If $2 \le p \le \infty$ and $\sigma \in [M^{2,2}, M^{\infty,1}]_\theta = M^{p,p'}$, then $L_\sigma \in [\mathcal{I}_2, \mathcal{I}_\infty]_\theta = \mathcal{I}_p$. When $p \ge 2$ and $q \le p'$, we have $M^{p,q} \subset M^{p,p'}$, and consequently $L_\sigma \in \mathcal{I}_p$.

(c) The hypotheses on p, q here are just a reformulation of the conditions in parts (a) and (b), so this part is immediate. □

It is shown in [GH02] that Theorem 6.10 is sharp in the following sense.

Theorem 6.11.

(a) *If* $q > 2$, *then there exist* $\sigma \in M^{p,q}$ *such that* L_σ *is unbounded on* $L^2(\mathbb{R}^d)$.

(b) *If* $p \ge 2$ *and* $p > q'$, *then there exist* $\sigma \in M^{p,q}$ *such that* L_σ *is unbounded on* $L^2(\mathbb{R}^d)$.

For statement (a) it is possible to find simple counter-examples in the form of "rank one" operators. However, statement (b) is more subtle and requires the full machinery of the modulation spaces, see [GH02].

7. Mapping Properties between Modulation Spaces

Using the techniques of Theorem 6.3, it is easy to investigate the mapping properties of the Weyl transform between modulation spaces. The following result is obtained by technical modifications to the proof of Theorem 6.3.

Theorem 7.1. *Assume that $1 \le q \le p \le \infty$ and that $\sigma \in M^{p,q}(\mathbb{R}^{2d})$. If $\mu_j \ge p'$, $\nu_j \le p$, and*

$$\frac{1}{\mu_j} - \frac{1}{\nu_j} = \frac{1}{q'} - \frac{1}{p}, \qquad j = 1,2, \tag{20}$$

then L_σ is a bounded operator from M^{μ_1,μ_2} to M^{ν_1,ν_2}.

Proof. As in (15), we represent L_σ as

$$\langle L_\sigma f, g\rangle = \langle \sigma, W(g,f)\rangle = \langle V_\Phi \sigma, V_\Phi W(g,f)\rangle,$$

where as usual we fix $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\|\varphi\|_2 = 1$ and set $\Phi = W(\varphi,\varphi)$. Now we apply Hölder's inequality for the mixed-norm spaces $L^{p,q}$ to obtain that

$$|\langle L_\sigma f, g\rangle| \le \|V_\Phi \sigma\|_{L^{p,q}(\mathbb{R}^{4d})} \; \|V_\Phi(W(g,f))\|_{L^{p',q'}(\mathbb{R}^{4d})}. \tag{21}$$

Note that $\|V_\Phi\sigma\|_{L^{p,q}(\mathbb{R}^{4d})} = \|\sigma\|_{M^{p,q}}$ is the exactly the appropriate modulation-space norm of σ. To estimate the norm of the STFT of the Wigner distribution appearing in (21) we use Lemma 3.6. Set $\tilde{\zeta} = (-\zeta_2, \zeta_1) \in \mathbb{R}^{2d}$, and define $F = |V_\varphi f|$ and $G = |V_\varphi g|$. Then by Lemma 3.6,

$$|V_\Phi(W(g,f))(z,\zeta)| = |V_\varphi f(z - \tfrac{\tilde{\zeta}}{2})|\,|V_\varphi g(z + \tfrac{\tilde{\zeta}}{2})| = F(z - \tfrac{\tilde{\zeta}}{2})\,G(z + \tfrac{\tilde{\zeta}}{2}).$$

Hence,

$$\begin{aligned}
\|V_\Phi(W(g,f)\|_{L^{p',q'}} &= \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} F(z - \tilde{\zeta}/2)^{p'}\, G(z + \tilde{\zeta}/2)^{p'}\, dz\right)^{q'/p'} d\zeta\right)^{1/q'} \\
&= \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} F(z)^{p'}\, \mathcal{I}G(-z - \tilde{\zeta})^{p'}\, dz\right)^{q'/p'} d\zeta\right)^{1/q'} \\
&= \left(\int_{\mathbb{R}^{2d}} (F^{p'} * \mathcal{I}G^{p'})(-\tilde{\zeta})^{q'/p'}\, d\zeta\right)^{1/q'} \\
&= \|F^{p'} * \mathcal{I}G^{p'}\|_{q'/p'}^{1/p'}.
\end{aligned}$$

Set $t = q'/p'$, and note that $t \ge 1$ since $q \le p$. Let r_j, s_j be numbers, defined precisely in a moment, which satisfy

$$r_j, s_j \ge 1 \qquad \text{and} \qquad \frac{1}{r_j} + \frac{1}{s_j} = 1 + \frac{1}{t}, \qquad \text{for } j = 1,2. \tag{22}$$

Then Young's convolution theorem in its version for mixed-norm spaces [BP61, Thm. 10.1] yields

$$\|F^{p'} * \mathcal{I}G^{p'}\|_t^{1/p'} \le \|F^{p'}\|_{L^{r_1,r_2}}^{1/p'}\, \|G^{p'}\|_{L^{s_1,s_2}}^{1/p'}.$$

Now,

$$\|F^{p'}\|_{L^{r_1,r_2}}^{1/p'} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_\varphi f(x,\omega)|^{p'r_1}\,\mathrm{d}x\right)^{p'r_2/(p'r_1)}\mathrm{d}\omega\right)^{1/(p'r_2)} = \|f\|_{M^{p'r_1,p'r_2}}\,,$$

and likewise,

$$\|G^{p'}\|_{L^{s_1,s_2}}^{1/p'} = \|g\|_{M^{p's_1,p's_2}}\,,$$

so by combining all these estimates we conclude that for all $f \in M^{p'r_1,p'r_2}$ and $g \in M^{p's_1,p's_2}$, we have

$$|\langle L_\sigma f, g\rangle| \leq \|\sigma\|_{M^{p,q}}\,\|f\|_{M^{p'r_1,p'r_2}}\|g\|_{M^{p's_1,p's_2}}\,. \tag{23}$$

Taking the supremum over g with unit norm leads to the conclusion that L_σ maps $M^{p'r_1,p'r_2}$ into $M^{(p's_1)',(p's_2)'}$.

The result now follows by choosing the proper values for r_j, s_j and attending to some bookkeeping. Define $r_j = \mu_j/p'$ and $s_j = \nu_j'/p'$. By hypothesis, $\mu_j \geq p'$ and $\nu_j \leq p$, so we have r_j, $s_j \geq 1$, and it follows from (20) and a little calculation that (22) holds, and hence the result follows. $\square$

Remarks 7.2. a. Theorem 7.1 was announced by the authors at several conferences, including FSDONA (Teistungen, July 2001) and the International Conference on Wavelets and their Applications (Chennai, January 2002). Recently, it was proved independently by J. Toft [Toft02].

b. Analogous results can be proved for symbols in the spaces $\mathcal{F}M^{p,q}$. The proofs are similar, the only modification required is a change in the order of integration.

c. By combining the proof of Theorem 6.7 with the proof of Theorem 7.1, one can derive mapping properties of pseudodifferential operators between weighted modulation spaces. These theorems quantify precisely how the time-frequency distribution of a function changes under the action of a pseudodifferential operator with a symbol in $M^{p,q}_m$. These results will be elaborated upon in later work.

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