DUALS OF FRAME SEQUENCES

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Abstract. Frames provide unconditional basis-like, but generally nonunique, representations of vectors in a Hilbert space $\mathcal{H}$. The redundancy of frame expansions allows the flexibility of choosing different dual sequences to employ in frame representations. In particular, oblique duals, Type I duals, and Type II duals have been introduced in the literature because of the special properties that they possess. This paper proves that all Type I and Type II duals are oblique duals, but not conversely, and characterizes the existence of oblique and Type II duals in terms of direct sum decompositions of $\mathcal{H}$, as well as characterizing when the Type I, Type II, and oblique duals will be unique. These results are also applied to the case of shift-generated sequences that are frames for shift-invariant subspaces of $L^2(\mathbb{R}^d)$.

1. Introduction and Main Results

Let $\mathcal{H}$ be a separable Hilbert space over the complex field. A countable (finite or countably infinite) sequence $X = \{x_j\}_{j\in\mathbb{J}}$ is a *frame* for $\mathcal{H}$ if $|||x||| = \bigl(\sum |\langle x, x_j\rangle|^2\bigr)^{1/2}$ is an equivalent norm for $\mathcal{H}$. Every frame provides basis-like but generally nonunique representations of the elements of $\mathcal{H}$. Specifically, $\mathcal{H}$ contains at least one sequence $Y = \{y_j\}_{j\in\mathbb{J}}$ that is dual to X in the sense that

$$x = \sum_{j\in\mathbb{J}} \langle x, y_j\rangle\, x_j, \qquad x \in \mathcal{H}. \tag{1.1}$$

Moreover, the y_j can be chosen so that this series converges unconditionally for all $x \in \mathcal{H}$. Similar remarks apply to *frame sequences*, which are frames for their closed spans

$$\mathcal{H}_X := \overline{\mathrm{span}}(X)$$

within $\mathcal{H}$. The redundancy and flexibility offered by frames has spurred their application in a variety of areas throughout mathematics and engineering, such as wireless communications [SH03], σ-δ quantization [BPY06], and image processing [CD05].

In this paper we study the various kinds of dual sequences that a frame sequence can possess. Christensen and Eldar [CE04] and Hemmat and Gabardo [HG07] each defined and studied several types of duals (see also [GH04]). We will characterize the exact relationships that hold among these duals, as well as obtaining other new results, including results on the uniqueness of dual sequences.

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To describe these results, we introduce some terminology and review some basic facts. For details and background on frames, we refer to [Chr03], [Grö01], or [HW89].

A *Bessel sequence* in $\mathcal{H}$ is a sequence $X = \{x_j\}_{j\in\mathbb{J}}$ such that the *analysis operator*

$$U_X(x) \;=\; \bigl\{\langle x, x_j\rangle\bigr\}_{j\in\mathbb{J}}, \qquad x \in X,$$

is a bounded mapping of $\mathcal{H}$ into $\ell^2(\mathbb{J})$. Equivalently, X is a Bessel sequence if the *synthesis* or *pre-frame operator* $T_X \colon \ell^2(\mathbb{J}) \to \mathcal{H}$ given by

$$T_X(c) \;=\; \sum_{j\in\mathbb{J}} c_j x_j, \qquad c \in \ell^2(\mathbb{J}),$$

is well-defined and bounded. In this case, $U_X = T_X^*$.

Every frame sequence X is a Bessel sequence, but not conversely. In fact, a Bessel sequence is a frame sequence if and only if $\mathrm{range}(T_X)$ is a closed subset of $\mathcal{H}$. In this case, we have

$$\mathcal{H}_X \;=\; \overline{\mathrm{span}}(X) \;=\; \mathrm{range}(T_X).$$

Each frame sequence has at least one *dual sequence*. The *frame operator* $S_X = T_X T_X^*$ is a positive operator, and is a bounded bijection of $\overline{\mathrm{span}}(X)$ onto itself. Letting $\dagger$ denotes the pseudoinverse of a bounded operator with closed range the sequence $\widetilde{X} = S_X^{\dagger}(X)$ is the *canonical* or *standard dual frame sequence* of X. Restricted to the domain $\mathcal{H}_X$, which S_X maps invertibly into itself, we have $S_X^{\dagger} = S_X^{-1}$. The canonical dual is itself a frame sequence, and $\mathcal{H}_{\widetilde{X}} = \overline{\mathrm{span}}(\widetilde{X}) = \overline{\mathrm{span}}(X) = \mathcal{H}_X$. Further, we have the reproducing properties

$$T_X T_{\widetilde{X}}^*|_{\mathcal{H}_X} \;=\; T_{\widetilde{X}} T_X^*|_{\mathcal{H}_X} \;=\; I|_{\mathcal{H}_X}, \tag{1.2}$$

where I is the identity operator on $\mathcal{H}$. Writing $\widetilde{X} = \{\widetilde{x}_j\}_{j\in\mathbb{J}}$, equation (1.2) is equivalent to

$$x \;=\; \sum_{j\in\mathbb{J}} \langle x, \widetilde{x}_j\rangle\, x_j \;=\; \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\, \widetilde{x}_j, \qquad x \in \mathcal{H}_X, \tag{1.3}$$

and furthermore these series converge unconditionally.

A frame X for which T_X is injective is called a *Riesz basis*, and likewise a frame sequence for which T_X is injective is called a *Riesz sequence*. Equivalently, a Riesz sequence is a frame sequence such that the series in equation (1.3) are the unique representations of $x \in \mathcal{H}_X$ in terms of X or $\widetilde{X}$. In this case $\widetilde{X}$ is the only dual sequence to X that is contained within $\mathcal{H}_X$, although there may be other dual sequences contained in $\mathcal{H}$.

Out of all the possible dual sequences, some have special properties. In particular, *Type I* and *Type II duals* were introduced by Hemmat and Gabardo [HG07], while Christensen and Eldar [CE04] introduced *oblique duals*. These are defined precisely as follows.

Definition 1.1. Let $X = \{x_j\}_{j\in\mathbb{J}}$ be a frame sequence and $Y = \{y_j\}_{j\in\mathbb{J}}$ a Bessel sequence in $\mathcal{H}$.

(a) Y is a *generalized* or *alternate dual* of X, or simply a *dual* for short, if

$$T_X T_Y^*|_{\mathcal{H}_X} \;=\; I|_{\mathcal{H}_X}. \tag{1.4}$$

(b) Y is a *Type I dual of* X if Y is a dual and $\mathrm{range}(T_Y) \subset \mathrm{range}(T_X)$.

(c) Y is a *Type II dual of* X if Y is a dual and $\mathrm{range}(T_Y^*) \subset \mathrm{range}(T_X^*)$.

(d) Y is an *oblique dual* of X if Y is a frame sequence that is dual to X, and also X is dual to Y.

Note that while X and Y play symmetric roles in the definition of oblique dual, the definition of Type I and Type II duals does not seem to be symmetric in X and Y. The next theorem demonstrates that the Type I and Type II definitions are indeed symmetric, and in fact are special cases of oblique duals. We write $\oplus$ denote an orthogonal direct sum of closed subspaces, and $\dot{+}$ a direct sum of closed subspaces whose intersection is zero. Part (a) of this theorem is a known fact, see [Chr03, Lem. 5.6.2]. Part (b) is new and will be proved in Section 3.

Theorem 1.2. Let X be a frame sequence and Y a Bessel sequence in $\mathcal{H}$.

(a) If Y is a Type I dual of X, then Y is a frame sequence and X is a Type I dual of Y. In this case, X and Y are oblique duals, $\mathcal{H}_Y = \mathrm{range}(T_Y) = \mathrm{range}(T_X) = \mathcal{H}_X$, and $\mathcal{H} = \mathrm{range}(T_X) \oplus \mathrm{range}(T_Y)^{\perp}$.

(b) If Y is a Type II dual of X, then Y is a frame sequence and X is a Type II dual of Y. In this case, X and Y are oblique duals, $\mathrm{range}(T_Y^*) = \mathrm{range}(T_X^*)$, and $\mathcal{H} = \mathrm{range}(T_X) \dot{+} \mathrm{range}(T_Y)^{\perp}$.

Theorem 1.2 clarifies the interest of these types of duals. An oblique dual Y is one where X and Y play completely complementary roles, with each being a frame sequence that is dual to the other. Type I and Type II duals are oblique duals with further restrictions. In particular, if it is important that X and Y have the same closed spans, then we should look for a Type I dual. On the other hand, the flexibility offered by frame expansions can often be better utilized by looking for a dual that is not within $\mathcal{H}_X$, e.g., see [LO05], [CK07]. A Type II dual will instead require that $\ker(T_X)$ and $\ker(T_Y)$ be equal, rather than the ranges of T_X and T_Y. The canonical dual is simultaneously a Type I and a Type II dual.

The relations that hold between the various types of duals are explained in our next result, which will be proved in Section 5.

Theorem 1.3. There are examples of a frame sequence X with dual Y such that:

(a) Y is not a frame sequence;

(b) Y is a frame sequence but not a Type I dual;

(c) Y is a frame sequence but not a Type II dual;

(d) Y is a Type I dual but not a Type II dual of X;

(e) Y is a Type II dual but not a Type I dual of X;

(f) Y is an oblique dual but not a Type I dual of X;

(g) Y is an oblique dual but not a Type II dual of X.

Next, we characterize the existence of oblique or Type II duals in terms of direct sum decompositions of $\mathcal{H}$. The proof of Theorem 1.4 will be given in Section 6.

Theorem 1.4. Let U, V be closed subspaces of $\mathcal{H}$, and let X be a frame for U. Then the following statements are equivalent.

(a) $\mathcal{H} = U \dotplus V^{\perp}$.

(b) There is a frame Y for V that is a Type II dual of X.

(c) There is a frame Y for V that is an oblique dual of X.

In case these hold, PX is a frame for V and we can take Y to be the canonical dual frame of PX in V, where $P = P_V|_U$.

We show in Section 7 that by using Theorem 1.4 we can recover [HG07, Prop. 3] on the uniqueness of Type I or Type II duals of a frame sequence.

For the case of shift-invariant spaces and shift-generated duals, we have the following further refinement of Theorem 1.4. The definitions and notation regarding shift-invariant spaces that are used here are explained in Section 4, and the proof is given in Section 6.

Theorem 1.5. Let U, V be shift-invariant subspaces of $L^2(\mathbb{R}^d)$, and let $E(\Phi)$ be a frame for U for some countable $\Phi \subset L^2(\mathbb{R}^d)$. Then the following statements are equivalent.

(a) $L^2(\mathbb{R}^d) = U \dotplus V^{\perp}$.

(b) There exists $\Psi \subset V$ with $|\Phi| = |\Psi|$ such that $V = S(\Psi)$ and $E(\Psi)$ is a Type II dual of $E(\Phi)$.

(c) There exists $\Psi \subset V$ with $|\Phi| = |\Psi|$ such that $V = S(\Psi)$ and $E(\Psi)$ is an oblique dual of $E(\Phi)$.

Our final main result deals with the uniqueness of shift-generated duals, and generalizes both [HG07, Thm. 6] and [CE04, Thm. 4.3].

Theorem 1.6. Let $\Phi \subset L^2(\mathbb{R}^d)$ be countable, and assume that $E(\Phi)$ is a frame for $U := S(\Phi)$. If V is a shift-invariant subspace of $L^2(\mathbb{R}^d)$ such that $L^2(\mathbb{R}^d) = U \dotplus V^{\perp}$, then the following statements are equivalent.

(a) There is a unique shift-generated dual of $E(\Psi)$ of $E(\Phi)$ such that $|\Psi| = |\Phi|$ and $E(\Psi)$ is a frame for V.

(b) $E(\Phi)$ is quasi-stable, i.e., $\widehat{\Phi}_{||\xi}$ is a Riesz basis of $\widehat{U}_{||\xi}$ for a.e. $\xi \in \sigma(U)$.

For the uniqueness of shift-generated duals of Type I or II, see Theorems 6 and 7 in [HG07].

The rest of this article is organized in the following manner. Section 2 contains some notation and basic results that will be needed throughout. The proof of Theorem 1.2 is given in Section 3, and Section 4 presents some additional background results that will be needed for the proofs of those results of ours that deal with shift-invariant spaces. The proof of Theorem 1.3 is then given in Section 5, and the proofs of Theorems 1.4 and 1.5 appear in Section 6. Finally, Section 7 contains results dealing with the uniqueness of oblique duals as well as the proof of Theorem 1.6.

2. Preliminaries

In this section we collect facts and background information that will be needed for the proofs of our main results.

2.1. **General Notation.** We use a Fourier transform normalized as $\widehat{f}(\xi) := \int f(t)\, e^{-2\pi i \xi \cdot t}$ for $f \in L^1(\mathbb{R}^d)$. This extends in the usual way to a unitary operator on $L^2(\mathbb{R}^d)$.

$\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the torus group, identified with $[0,1)^d$. Functions on $\mathbb{T}^d$ are identified with their 1-periodic extensions to $\mathbb{R}^d$.

The Fourier series of a sequence $a \in \ell^1(\mathbb{Z}^d)$ is $\widehat{a}(\xi) := \sum_{k \in \mathbb{Z}^d} a(k)\, e^{-2\pi i k \cdot \xi}$ for $\xi \in \mathbb{T}^d$.

The translation operator is denoted $T_a f(x) = f(x-a)$, where $a \in \mathbb{R}^d$.

2.2. **Oblique Projections.** A bounded linear operator $P \colon \mathcal{H} \to \mathcal{H}$ satisfying $P^2 = P$ is called an *oblique projection*, or simply a *projection* for short. In this case, we have $\mathcal{H} = \operatorname{range}(P) \dotplus \ker(P)$. Oblique projections are also known as *idempotent operators*.

Conversely, if $\mathcal{H} = U \dotplus V$ for some closed subspaces U, V, then there exists a unique projection $P_{U,V}$ such that $\operatorname{range}(P) = U$ and $\ker(P) = V$ [Con90, Chap. II]. We have that $P_{U,V}|_U = I|_U$, and the adjoint of $P_{U,V}$ is

$$P_{U,V}^{*} \;=\; P_{V^\perp, U^\perp}. \tag{2.1}$$

The orthogonal projection of $\mathcal{H}$ onto a closed subspace W is $P_W := P_{W,W^\perp}$.

2.3. **The Inf-Angle between Subspaces.** The *inf-angle* $R(U,V)$ between closed subspaces U, V of $\mathcal{H}$ with U nontrivial is

$$R(U,V) \;=\; \inf_{u \in U \setminus \{0\}} \frac{\|P_V u\|}{\|u\|},$$

see [UA94]. For $U = \{0\}$ we define $R(\{0\}, V) := 1$.

The following proposition relates the inf-angle to projections. The equivalence of parts (a), (e), and (f) is stated in [Tan00]. The equivalence of these parts with (b), (c), and (d) is proved in [BHKL08]. We write $U \cong V$ to mean that U is isomorphic to V, which is the case if and only if U and V have the same dimension. Note in the statement of this proposition that $P_U|_V$ denotes the restriction of the orthogonal projection P_U to V, and should not be confused with the oblique projection $P_{U,V}$.

Proposition 2.1. Let U, V be closed subspaces of $\mathcal{H}$ with at least one nontrivial. Then the following statements are equivalent.

(a) $0 < R(U,V)$ and $0 < R(V,U)$.
(b) $0 < R(U,V) = R(V,U)$.
(c) $P_V|_U : U \to V$ is invertible.
(d) $P_U|_V : V \to U$ is invertible.
(e) $\mathcal{H} = U \dotplus V^\perp$.
(f) $\mathcal{H} = V \dotplus U^\perp$.

Moreover, in case these hold, we have $U \cong V$, $V^\perp \cong U^\perp$, and

$$0 \;<\; R(U,V) \;=\; R(V,U) \;=\; \|(P_V|_U)^{-1}\|^{-1} \;=\; \|(P_U|_V)^{-1}\|^{-1}.$$

We will need the following lemma in Section 7.

Lemma 2.2. If U is a proper closed subspace of a separable Hilbert space $\mathcal{H}$, then there exists a closed subspace $V \neq U$ such that $R(U,V) = R(V,U) > 0$.

Proof. Let $\{u_n\}_{n\in\mathbb{J}}$ be an orthonormal basis for U, where either $\mathbb{J} = \{1,\dots,N\}$ or $\mathbb{J} = \mathbb{N}$. Let $f \in U^\perp$ be any unit vector. Define $v_1 := 2^{-1/2}\,(u_1 + f)$ and set $v_n = u_n$ for $n \geq 2$. Then $\{v_n\}_{n\in\mathbb{J}}$ is an orthonormal basis for $V := \overline{\text{span}}\{v_n\}$. If $u \in U$, then

$$P_V u \;=\; \sum_{n\in\mathbb{J}} \langle u, v_n\rangle\, v_n \;=\; 2^{-1/2}\,\langle u, u_1\rangle\, v_1 + \sum_{n>1} \langle u, v_n\rangle\, v_n,$$

so $\|P_V u\| \geq 2^{-1/2}\,\|u\|$. Similarly, if $v \in V$ then $\|P_U v\| \geq 2^{-1/2}\,\|v\|$. This shows that $R(U,V) > 0$ and $R(V,U) > 0$, and consequently $R(U,V) = R(V,U)$ by Proposition 2.1. ☐

2.4. **Pseudoinverses.** Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $A\colon \mathcal{H} \to \mathcal{K}$ a bounded linear operator with closed range, and let $V = \text{range}(A) = \ker(A^*)^\perp$. Then $U = \text{range}(A^*) = \ker(A)^\perp$ is closed in $\mathcal{H}$, and $A\colon U \to V$ is a bounded bijection. The *pseudoinverse* or *Moore–Penrose generalized inverse* of A is

$$A^\dagger \;=\; \big(A|_U\big)^{-1}\, P_V,$$

see [Gro77]. *Restricted to* V, we have that $A^\dagger\colon V \to U$ is the inverse of $A\colon U \to V$. Further,

$$A^\dagger A \;=\; P_U, \tag{2.2}$$

the orthogonal projection onto $U = \text{range}(A^*)$.

2.5. **The Canonical Dual.** As discussed in the Introduction, if X is a frame sequence in $\mathcal{H}$, then its canonical dual frame sequence is $\widetilde{X} = S_X^\dagger(X)$, where $S_X = T_X T_X^*$. Some facts about the canonical dual are given in the next lemma.

Lemma 2.3. Let X be a frame sequence with canonical dual frame sequence $\widetilde{X}$.

(a) $\mathcal{H}_{\widetilde{X}} = \overline{\text{span}}(\widetilde{X}) = \text{range}(T_X) = \text{range}(T_{\widetilde{X}}) = \overline{\text{span}}(X) = \mathcal{H}_X$.

(b) $T_{\widetilde{X}}^* = T_X^* S_X^\dagger = T_X^*(T_X T_X^*)^\dagger$.

(c) $\text{range}(T_X^*) = \text{range}(T_{\widetilde{X}}^*)$, and this is a closed subspace of $\mathcal{H}$.

3. Proof of Theorem 1.2

In this section we prove the new portion of Theorem 1.2.

Proof of Theorem 1.2(b). Suppose that X is a frame sequence in $\mathcal{H}$, and $Y \subset \mathcal{H}$ is a Bessel sequence that is a Type II dual of X. Let $\widetilde{X}$ be the canonical dual frame sequence of X. Then, *restricted to* $\mathcal{H}_X$, we have

$$\begin{aligned} T_Y^* \;&=\; P_{\text{range}(T_Y^*)} T_Y^* \\ &=\; P_{\text{range}(T_X^*)} T_Y^* && \text{since } Y \text{ is a Type II dual} \\ &=\; T_X^\dagger T_X T_Y^* && \text{by equation (2.2)} \\ &=\; T_X^\dagger && \text{since } Y \text{ is a dual of } X \end{aligned}$$

$$
\begin{aligned}
&= T_X^\dagger T_X T_{\widetilde{X}}^* && \text{since } \widetilde{X} \text{ is a dual of } X \\
&= P_{\mathrm{range}(T_X^*)} T_{\widetilde{X}}^* && \text{by equation (2.2)} \\
&= P_{\mathrm{range}(T_{\widetilde{X}}^*)} T_{\widetilde{X}}^* && \text{by Lemma 2.3} \\
&= T_{\widetilde{X}}^*.
\end{aligned}
$$

That is, $T_Y^*|_{\mathcal{H}_X} = T_{\widetilde{X}}^*|_{\mathcal{H}_X}$.

Now, since T_X^* has closed range, $T_X^* \colon \mathcal{H}_X = \mathrm{range}(T_X) \to \mathrm{range}(T_X^*)$ is a bijection, and likewise $T_{\widetilde{X}}^* \colon \mathcal{H}_X = \mathrm{range}(T_{\widetilde{X}}) \to \mathrm{range}(T_{\widetilde{X}}^*)$ is a bijection. Therefore,

$$
\begin{aligned}
\mathrm{range}(T_X^*) &= \mathrm{range}(T_{\widetilde{X}}^*) && \text{by Lemma 2.3} \\
&= \mathrm{range}(T_{\widetilde{X}}^*|_{\mathcal{H}_X}) \\
&= \mathrm{range}(T_Y^*|_{\mathcal{H}_X}) \\
&\subset \mathrm{range}(T_Y^*) \\
&\subset \mathrm{range}(T_X^*) && \text{since } Y \text{ is a Type II dual.}
\end{aligned}
$$

Therefore $\mathrm{range}(T_Y^*) = \mathrm{range}(T_X^*)$ is closed, and hence $\mathrm{range}(T_Y)$ is closed as well. Consequently Y is a frame sequence in $\mathcal{H}$, and we have

$$\mathcal{H}_Y := \overline{\mathrm{span}}(Y) = \mathrm{range}(T_Y).$$

Next, we claim that $T_X T_Y^* = P_{\mathcal{H}_X, \mathcal{H}_Y^\perp}$. Since $T_X T_Y^* = I|_{\mathcal{H}_X}$ and $\mathrm{range}(T_X) = \mathcal{H}_X$, we have both $(T_X T_Y^*)^2 = T_X T_Y^*$ and $\mathrm{range}(T_X T_Y^*) = \mathcal{H}_X$. If $f \in \ker(T_X T_Y^*)$, then $T_X T_Y^* f = 0$ and therefore

$$T_Y^* f \in \mathrm{range}(T_Y^*) \cap \ker(T_X) = \mathrm{range}(T_X^*) \cap \ker(T_X) = \{0\}.$$

Thus $\ker(T_X T_Y^*) \subset \ker(T_Y^*)$, and the opposite inclusion is trivial. Therefore

$$\ker(T_X T_Y^*) = \ker(T_Y^*) = \mathrm{range}(T_Y)^\perp = \mathcal{H}_Y^\perp.$$

Thus $T_X T_Y^* = P_{\mathcal{H}_X, \mathcal{H}_Y^\perp}$, and also the fact that this oblique projection exists implies that $\mathcal{H} = \mathcal{H}_X \dot{+} \mathcal{H}_Y^\perp$.

Finally, taking adjoints and using equation (2.1), it therefore follows that

$$T_Y T_X^* = (T_X T_Y^*)^* = P_{\mathcal{H}_X, \mathcal{H}_Y^\perp}^* = P_{\mathcal{H}_Y, \mathcal{H}_X^\perp}.$$

In particular, $T_Y T_X^*|_{\mathcal{H}_Y} = P_{\mathcal{H}_Y, \mathcal{H}_X^\perp}|_{\mathcal{H}_Y} = I|_{\mathcal{H}_Y}$, so X is a dual of Y. □

Remark 3.1. If in the statement of Theorem 1.2(b) we have that $\mathcal{H} = \mathrm{range}(T_X) \oplus \mathrm{range}(T_Y)^\perp$, then $\mathrm{range}(T_X) = \mathrm{range}(T_Y)$ and consequently Y is a Type I dual of X.

4. Background on Shift-Invariant Spaces

In this section we briefly review the theory of shift-invariant subspaces of $L^2(\mathbb{R}^d)$ [BDR94], [Bow00], [Hel64] and the fiberization technique developed by Ron and Shen [RS95].

4.1. **Shift-Invariant Spaces and Fibers.** A closed subspace S of $L^2(\mathbb{R}^d)$ is *shift-invariant* if $T_k f \in S$ for each $k \in \mathbb{Z}^d$ and $f \in S$, where T_k is the translation operator.

Given $\Phi \subset L^2(\mathbb{R}^d)$, we define

$$E(\Phi) \;:=\; \{T_k\varphi : k \in \mathbb{Z}^d,\, \varphi \in \Phi\} \qquad \text{and} \qquad S(\Phi) \;:=\; \overline{\text{span}}\big(E(\Phi)\big).$$

We call $S(\Phi)$ the shift-invariant space *generated by* Φ. If $\Phi = \{\varphi\}$ then we simply write $E(\varphi)$ and $S(\varphi)$.

Given $f \in L^2(\mathbb{R}^d)$, the *fiber of f at $\xi \in \mathbb{T}^d$* is the sequence

$$\widehat{f}_{||\xi} \;:=\; \big\{\widehat{f}(\xi + k)\big\}_{k\in\mathbb{Z}^d}.$$

This is well-defined for almost every $\xi \in \mathbb{T}^d$. If $\Phi \subset L^2(\mathbb{R}^d)$ is countable, then

$$\widehat{\Phi}_{||\xi} \;:=\; \big\{\widehat{\varphi}_{||\xi} : \varphi \in \Phi\big\}$$

exists for almost every $\xi \in \mathbb{T}^d$.

Given a shift-invariant subspace S of $L^2(\mathbb{R}^d)$, it is known that there exists a countable set that generates S [Hel64], [BDR94]. Consequently,

$$\widehat{S}_{||\xi} \;:=\; \big\{\widehat{f}_{||\xi} : f \in S\big\}$$

is well-defined for a.e. $\xi \in \mathbb{T}^d$. Moreover, if $S = S(\Phi)$, then

$$\widehat{S}_{||\xi} \;=\; \overline{\text{span}}\big(\widehat{\Phi}_{||\xi}\big) \;=\; \overline{\text{span}}\big\{\widehat{\varphi}_{||\xi} : \varphi \in \Phi\big\}, \qquad \text{a.e. } \xi \in \mathbb{T}^d.$$

We call $\widehat{S}_{||\xi}$ the *fiber space of S at ξ*. The *spectrum* $\sigma(S)$ of S is

$$\sigma(S) \;:=\; \{\xi \in \mathbb{T}^d : \widehat{S}_{||\xi} \neq \{0\}\}.$$

It is known that two shift-invariant spaces S_1, S_2 are equal if and only if $\widehat{S_1}_{||\xi} = \widehat{S_2}_{||\xi}$ for a.e. $\xi \in \mathbb{T}^d$ [BDR94], [Bow00].

The following proposition illustrates the importance of the fiberization technique. For proof, see [RS95], [Bow00].

Proposition 4.1. Let $S = S(\Phi)$, where Φ is a countable subset of $L^2(\mathbb{R}^d)$. Then the following statements hold.

(a) $E(\Phi)$ is a Bessel sequence in $L^2(\mathbb{R}^d)$ with a Bessel bound B if and only if $\widehat{\Phi}_{||\xi}$ is a Bessel sequence in $\ell^2(\mathbb{Z}^d)$ with a Bessel bound B for a.e. $\xi \in \mathbb{T}^d$.

(b) $E(\Phi)$ is a frame for S with frame bounds A and B if and only if $\widehat{\Phi}_{||\xi}$ is a frame for $\widehat{S}_{||\xi}$ with frame bounds A and B for a.e. $\xi \in \sigma(S)$.

(c) $E(\Phi)$ is a Riesz basis for S with Riesz bounds A and B if and only if $\widehat{\Phi}_{||\xi}$ is a Riesz basis for $\widehat{S}_{||\xi}$ with frame bounds A and B for a.e. $\xi \in \mathbb{T}^d$.

4.2. **Shift-Generated Duals.** Let Φ, $\Psi \subset L^2(\mathbb{R}^d)$ be countable and of the same cardinality. If $E(\Psi)$ is a dual of $E(\Phi)$, then we say that $E(\Psi)$ is a *shift-generated dual of* $E(\Phi)$. In particular, the canonical dual frame sequence of $E(\Phi)$ is always a shift-generated dual.

We will need the following fiber theorem for shift-generated duals. This result is [HG07, Thm. 5(a)], and there are similar results for Type I and Type II shift-generated duals, see [HG07, Thm. 5(b,c)].

Theorem 4.2. Suppose that $E(\Phi)$ is a frame sequence and $E(\Psi)$ is a Bessel sequence in $L^2(\mathbb{R}^d)$, where Φ, Ψ are countable subsets of $L^2(\mathbb{R}^d)$ with the same cardinality. Then $E(\Psi)$ is a dual of $E(\Phi)$ if and only if $\widehat{\Psi}_{||\xi}$ is a dual of $\widehat{\Phi}_{||\xi}$ for a.e. $\xi \in \mathbb{T}^d$.

4.3. **Gramians.** Suppose that $\Phi := \{\varphi_j\}_{j\in\mathbb{J}} \subset L^2(\mathbb{R}^d)$ is countable and $E(\Phi)$ is a Bessel sequence in $L^2(\mathbb{R}^d)$. Then the *Gramian of* Φ *at* $\xi \in \mathbb{T}^d$ is the $|\mathbb{J}| \times |\mathbb{J}|$ matrix $G_\Phi(\xi)$ whose i-j entry is

$$G_\Phi(\xi)_{i,j} \;:=\; \big\langle \widehat{\varphi_j}_{||\xi},\, \widehat{\varphi}_{i||\xi} \big\rangle_{\ell^2} \;=\; \sum_{k\in\mathbb{Z}^d} \widehat{\varphi_j}(\xi+k)\, \overline{\widehat{\varphi_i}(\xi+k)}.$$

The matrix $G_\Phi(\xi)$ is defined for almost every ξ.

If $\Psi := \{\psi_i\}_{i\in\mathbb{I}} \subset L^2(\mathbb{R}^d)$ is also countable and $E(\Psi)$ is a Bessel sequence, then the *mixed Gramian of* Φ *and* Ψ *at* $\xi \in \mathbb{T}^d$ is the $|\mathbb{I}| \times |\mathbb{J}|$ matrix whose i-j entry is

$$G_{\Phi,\Psi}(\xi)_{i,j} \;:=\; \big\langle \widehat{\varphi_j}_{||\xi},\, \widehat{\psi}_{i||\xi} \big\rangle_{\ell^2} \;=\; \sum_{k\in\mathbb{Z}^d} \widehat{\varphi_j}(\xi+k)\, \overline{\widehat{\psi}_i(\xi+k)}.$$

We have the following convenient characterization of finitely generated shift-invariant Bessel sequences and frame sequences in terms of Gramians. This proposition is proved in [RS95].

Proposition 4.3. Let Φ be a finite subset of $L^2(\mathbb{R}^d)$.

(a) $E(\Phi)$ is a Bessel sequence in $L^2(\mathbb{R})$ if and only if there exists $B > 0$ such that for a.e. $\xi \in \sigma(S(\Phi))$, each nonzero eigenvalue λ of $G_\Phi(\xi)$ satisfies $\lambda \le B$.

(b) $E(\Phi)$ is a frame sequence in $L^2(\mathbb{R})$ if and only if there exist A, $B > 0$ such that for a.e. $\xi \in \sigma(S(\Phi))$, each nonzero eigenvalue λ of $G_\Phi(x)$ satisfies $A \le \lambda \le B$.

In particular, we have the following for the case of singly-generated shift-invariant spaces.

Corollary 4.4. Let $\varphi \in L^2(\mathbb{R}^d)$ be given.

(a) $E(\varphi)$ is a Bessel sequence if and only if there exists $B > 0$ such that

$$\sum_{k\in\mathbb{Z}^d} |\widehat{\varphi}(\xi+k)|^2 \;\le\; B, \qquad \text{a.e. } \xi \in \sigma(S(\varphi)).$$

(b) $E(\varphi)$ is a frame sequence if and only if there exists A, $B > 0$ such that

$$A \;\le\; \sum_{k\in\mathbb{Z}^d} |\widehat{\varphi}(\xi+k)|^2 \;\le\; B, \qquad \text{a.e. } \xi \in \sigma(S(\varphi)).$$

We will need the following proposition on the Gramian and mixed Gramian matrices. Part (a) of Proposition 4.5 is [KKL05, Lem. 5.2(3)], and part (b) is [KL07, Lem. 3.6]. For shift-generated sequences $E(\Phi)$, it is convenient to view $T_{E(\Phi)}$ as a mapping from $(\ell^2(\mathbb{Z}^d))^\Phi$ into $L^2(\mathbb{R}^d)$.

Proposition 4.5. Suppose that $E(\Phi)$ and $E(\Psi)$ are Bessel sequences in $L^2(\mathbb{R}^d)$, where Φ and Ψ are finite subsets of $L^2(\mathbb{R}^d)$ such that $|\Phi| = |\Psi|$.

(a) If $E(\Phi)$ is a frame sequence, then $E(\Psi)$ is a dual of $E(\Phi)$ if and only if $G_\Phi(\xi)G_{\Phi,\Psi}(\xi) = G_\Phi(\xi)$ for a.e. $\xi \in \mathbb{T}^d$.

(b) $c \in (\ell^2(\mathbb{Z}^d))^\Phi$ belongs to $\ker(T_{E(\Phi)})$ if and only if $\widehat{c}(\xi) := (\widehat{c_\varphi}(\xi))_{\varphi\in\Phi} \in \ker(G_\Phi(\xi))$ for a.e. $\xi \in \mathbb{T}^d$.

The following corollary follows directly. Part (a) of Corollary 4.6 is [CE04, Thm. 4.1].

Corollary 4.6. Suppose that $E(\varphi)$ is a frame sequence and $E(\psi)$ is a Bessel sequence in $L^2(\mathbb{R})$ for some φ, $\psi \in L^2(\mathbb{R})$.

(a) $E(\psi)$ is a dual of $E(\varphi)$ if and only if

$$\sum_{k\in\mathbb{Z}} \widehat{\varphi}(\xi+k)\,\overline{\widehat{\psi}(\xi+k)} \;=\; 1, \qquad \text{a.e. } \xi \in \sigma(S(\varphi)).$$

(b) $c \in \ell^2(\mathbb{Z})$ belongs to $\ker(T_{E(\varphi)})$ if and only if $\operatorname{supp}(\widehat{c}) \subset \mathbb{T} \setminus \sigma(S(\varphi))$.

5. Proof of Theorem 1.3

Now we exhibit the counterexamples claimed in Theorem 1.3. All of these counterexamples will be constructed in singly or doubly generated shift-invariant subspaces of $L^2(\mathbb{R})$.

Proof of Theorem 1.3. Let $\{e_k\}_{k\in\mathbb{Z}}$ denote the standard orthonormal basis of $\ell^2(\mathbb{Z})$.

(a) Define φ and ψ by

$$\widehat{\varphi} := \chi_{[0,\frac12)} \qquad\text{and}\qquad \widehat{\psi}(\xi) := \chi_{[0,\frac12)}(\xi) + (2-2\xi)^{1/2}\,\chi_{[\frac12,1)}(\xi).$$

Then $\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi+k)|^2 = \chi_{[0,\frac12)}(\xi)$ and $\sigma(S(\varphi)) = [0,\frac12)$, so $E(\varphi)$ is a frame sequence by Corollary 4.4. Also, $\sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi+k)|^2 = \chi_{[0,\frac12)}(\xi) + (2-2\xi)\,\chi_{[\frac12,1)}(\xi)$, so $E(\psi)$ is a Bessel sequence but not a frame sequence. Finally, $\sum_{k\in\mathbb{Z}} \widehat{\varphi}(\xi+k)\,\overline{\widehat{\psi}(\xi+k)} = \chi_{[0,\frac12)}(\xi)$, so $E(\psi)$ is dual to $E(\varphi)$ by Corollary 4.6.

(b), (c) Define $\widehat{\varphi} := \chi_{[0,\frac12)}$ and $\widehat{\psi} := \chi_{[0,1)}$. Again applying Corollaries 4.4 and 4.6, we see that $E(\varphi)$ and $E(\psi)$ are both frame sequences, and $E(\psi)$ is dual to $E(\varphi)$.

We have $\sigma(S(\varphi)) = [0,\frac12)$ and $\sigma(S(\psi)) = [0,1)$. By Corollary 4.6, $\ker(T_{E(\varphi)}) \neq \ker(T_{E(\psi)})$, and therefore $\operatorname{range}(T^*_{E(\varphi)}) \neq \operatorname{range}(T^*_{E(\psi)})$. Hence $E(\psi)$ is not a Type II dual of $E(\varphi)$. On the other hand, we have $S(\varphi) \neq S(\psi)$ since $\sigma(S(\varphi)) \neq \sigma(S(\psi))$, so $E(\psi)$ is not a Type I dual of $E(\varphi)$ either.

(e), (f) Let $\widehat{\varphi} := \chi_{[0,\frac{1}{2})}$ and $\widehat{\psi} := \chi_{[0,\frac{1}{2})\cup[1,\frac{3}{2})}$. By Corollaries 4.4 and 4.6, we have that $E(\varphi)$ and $E(\psi)$ are frame sequences, each dual to the other, and hence are oblique duals. Further,

$$\widehat{\varphi}_{||\xi} = \begin{cases} e_0, & \xi \in [0,\frac{1}{2}), \\ 0, & \xi \in [\frac{1}{2},1), \end{cases} \qquad \widehat{\psi}_{||\xi} = \begin{cases} e_0 + e_1, & \xi \in [0,\frac{1}{2}), \\ 0, & \xi \in [\frac{1}{2},1). \end{cases}$$

so $\sigma(S(\varphi)) = [0,\frac{1}{2}) = \sigma(S(\psi))$. Corollary 4.6 therefore implies that $E(\psi)$ is a Type II dual of $E(\varphi)$. However, for $\xi \in [0,\frac{1}{2})$ we have that

$$\widehat{S(\varphi)}_{||\xi} = \operatorname{span}\{\widehat{\varphi}_{||\xi}\} \neq \operatorname{span}\{\widehat{\psi}_{||\xi}\} = \widehat{S(\psi)}_{||\xi},$$

so $S(\varphi) \neq S(\psi)$. Therefore $E(\psi)$ is not a Type I dual of $E(\varphi)$.

(d), (g) For this example we need two generators. Set

$$\begin{cases} \widehat{\varphi_1} := \chi_{[0,1)}, \\ \widehat{\varphi_2} := \frac{1}{2}\chi_{[0,1)}, \end{cases} \qquad \text{and} \qquad \begin{cases} \widehat{\psi_1} := \chi_{[0,1)}, \\ \widehat{\psi_2} := 0, \end{cases} .$$

Then we have for all $\xi \in \mathbb{T}$ that

$$\begin{cases} \widehat{\varphi_1}_{||\xi} = e_0, \\ \widehat{\varphi_2}_{||\xi} = \frac{1}{2}e_0, \end{cases} \qquad \text{and} \qquad \begin{cases} \widehat{\psi_1}_{||\xi} = e_0, \\ \widehat{\psi_2}_{||\xi} = 0. \end{cases} .$$

Direct calculations show that for all $\xi \in \mathbb{T}$,

$$G_\Phi(\xi) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}, \qquad G_{\Phi,\Psi}(\xi) = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}, \qquad G_\Psi(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The nonzero eigenvalue of $G_\Phi(\xi)$ is 5/4, while that of $G_\Psi(\xi)$ is 1. Proposition 4.3 therefore implies that $E(\Phi)$ and $E(\Psi)$ are frame sequences. Further, since $G_\Phi(\xi)G_{\Phi,\Psi}(\xi) = G_\Phi(\xi)$ for each $\xi \in \mathbb{T}$, we have by Proposition 4.5 that $E(\Psi)$ is a dual of $E(\Phi)$. Moreover, for $\xi \in [0,1)$,

$$\widehat{S(\Phi)}_{||\xi} = \operatorname{span}\{\widehat{\varphi_1}_{||\xi}, \widehat{\varphi_2}_{||\xi}\} = \operatorname{span}\{e_0\} = \operatorname{span}\{\widehat{\psi_1}_{||\xi}, \widehat{\psi_2}_{||\xi}\} = \widehat{S(\Psi)}_{||\xi}.$$

Therefore, $S(\Phi) = S(\Psi)$, and consequently $E(\Psi)$ is a Type I dual of $E(\Phi)$, and hence is also an oblique dual. However, $\ker(G_\Phi(\xi)) = \operatorname{span}\{(1,-2)^T\}$ while $\ker(G_\Psi(\xi)) = \operatorname{span}\{(0,1)^T\}$, so $\ker(T_{E(\Phi)}) \neq \ker(T_{E(\Psi)})$ by Proposition 4.5. Therefore $E(\Psi)$ is not a Type II dual of $E(\Phi)$. □

6. Proof of Theorems 1.4 and 1.5

Proof of Theorem 1.4. (a) ⇒ (b). Suppose that $\mathcal{H} = U \dotplus V^\perp$ and $X := \{x_j\}_{j\in\mathbb{J}}$ is a frame for U. We will construct a frame for V that is a Type II dual of X. Recall from Proposition 2.1 that $P := P_V|_U : U \to V$ is an isomorphism, so $P^{-1} : V \to U$ is also an isomorphism.

Therefore PX is a frame for V. Let $Y := \{y_j\}_{j\in\mathbb{J}}$ be the canonical dual frame of PX in V. Given any $u \in U$ we have $P_V u = Pu$. Therefore, since P_V is self-adjoint,

$$\begin{aligned}\sum_{j\in\mathbb{J}} \langle u, y_j\rangle\, x_j &= P^{-1}\biggl(\sum_{j\in\mathbb{J}} \langle u, P_V y_j\rangle\, Px_j\biggr)\\ &= P^{-1}\biggl(\sum_{j\in\mathbb{J}} \langle P_V u, y_j\rangle\, Px_j\biggr)\\ &= P^{-1}\biggl(\sum_{j\in\mathbb{J}} \langle Pu, y_j\rangle\, Px_j\biggr) \;=\; P^{-1}Pu \;=\; u.\end{aligned}$$

Hence Y is a dual of X, and furthermore Y is a frame for V.

It remains to show that $\mathrm{range}(T_Y^*) = \mathrm{range}(T_X^*)$. Since the range of the synthesis operator T_X is contained in U, if we consider T_X to be a mapping of $\ell^2(\mathbb{J})$ into U then the synthesis operator of PX is $T_{PX} = PT_X$. The corresponding frame operator $S_{PX}\colon V \to V$ for PX is $S_{PX} = T_{PX}T_{PX}^* = PT_XT_X^*P^*$, and this is an invertible mapping of V onto itself. The canonical dual frame Y of PX is therefore given by

$$Y \;=\; S_{PX}^{-1}PX \;=\; (P^*)^{-1}S_X^{-1}P^{-1}PX \;=\; (P^*)^{-1}S_X^{-1}X.$$

Further, by Lemma 2.3, the synthesis operator for Y is

$$T_Y \;=\; S_{PX}^{-1}T_{PX} \;=\; (P^*)^{-1}S_X^{-1}P^{-1}PT_X \;=\; (P^*)^{-1}S_X^{-1}T_X.$$

Since S_X and P are both invertible, we conclude that $\ker(T_Y) = \ker(T_X)$, and consequently $\mathrm{range}(T_Y^*) = \mathrm{range}(T_X^*)$.

(b) $\Rightarrow$(c). This follows immediately from Theorem 1.2.

(c) $\Rightarrow$ (a). Suppose that $Y \subset V$ is an oblique dual of X such that Y is also a frame for V. Set $T := T_XT_Y^*$. Then since Y is dual to X and since $U = \mathcal{H}_X = \mathrm{range}(T_X)$, we have $T|_U = I|_U$.

We will show that $T = P_{U,V^\perp}$. By definition of dual, we have $T^2 = T$ and $\mathrm{range}(T) = U$. Also, since Y is an oblique dual we have that X is dual to Y, and therefore $T^*|_V = T_YT_X^*|_V = I_V$. Hence $\mathrm{range}(T^*) = V$, so $\ker(T) = V^\perp$. This shows that $T = P_{U,V^\perp}$, and therefore $\mathcal{H} = U \dotplus V^\perp$. □

Proof of Theorem 1.5. Suppose that U and V are shift-invariant subspaces of $L^2(\mathbb{R}^d)$ and that $X := E(\Phi)$ is a frame for U, where Φ is a countable subset of $L^2(\mathbb{R}^d)$. In this case P_V commutes with integer translations, so $PX = P_VE(\Phi) = E(P_V\Phi) = E(P\Phi)$ is a shift-generated frame for V, where $P = P_V|_U$. The frame operator of a shift-generated frame also commutes with integer translations, so the canonical dual frame of PX in V is also shift-generated. Combining these facts with Theorem 1.4 yields the proof. □

7. Uniqueness of Oblique Duals and the Proof of Theorem 1.6

We will need two lemmas from [CE04] in order to prove Theorem 1.6. The following is Lemma 3.1 in [CE04].

Lemma 7.1. Suppose that $\mathcal{H} = U \dotplus V^{\perp}$, and that X, Y are Bessel sequences in $\mathcal{H}$ such that $\mathcal{H}_X = U$ and $\mathcal{H}_Y = V$. Then the following are equivalent:

(a) $T_X T_Y^*|_U = I|_U$,

(b) $T_Y T_X^*|_V = I|_V$,

(c) $P_{U,V^{\perp}} = T_X T_Y^*$,

(d) $P_{V,U^{\perp}} = T_Y T_X^*$.

Moreover, in this case, X is a frame for U, Y is a frame for V, and X and Y are oblique dual frames.

The following is Lemma B.2 in [CE04].

Lemma 7.2. Suppose that $\mathcal{H} = U \dotplus V^{\perp}$ and $X := \{x_j\}_{j\in\mathbb{J}}$ is a frame for U. Then the following statements are equivalent:

(a) $T : \ell^2(\mathbb{J}) \to \mathcal{H}$ is bounded, range$(T) = V$, and $TT_X^* = P_{V,U^{\perp}}$,

(b) there exists a bounded $\widetilde{T} : \ell^2(\mathbb{J}) \to V$ such that

$$T = P_{V,U^{\perp}} S_X^{\dagger} T_X + \widetilde{T}\big(I_{\ell^2(\mathbb{J})} - T_X^* S_X^{\dagger} T_X\big). \tag{7.1}$$

This gives us the following uniqueness result.

Proposition 7.3. Suppose that $\mathcal{H} = U \dotplus V^{\perp}$ and X is a frame for U. Then the following statements are equivalent:

(a) there exists a unique oblique dual frame Y for X such that $\mathcal{H}_Y = V$,

(b) X is a Riesz sequence.

Proof. Lemmas 7.1 and 7.2 imply that, for any bounded $\widetilde{T}\colon \ell^2(\mathbb{J}) \to V$, the sequence $Y = T(\ell^2(\mathbb{J}))$ is an oblique dual frame for X such that $\mathcal{H}_Y = V$, where T is as in (7.1). In particular, if we let $\widetilde{T} = 0$, then $P_{V,U^{\perp}} S_X^{\dagger} T_X(\ell^2(\mathbb{J}))$ is an oblique dual frame for X such that $\mathcal{H}_Y = V$. Hence the uniqueness of such an oblique dual frame for X is equivalent to the fact that the second term in the right-hand side of (7.1) is 0 for any $\widetilde{T}$. That is, uniqueness is equivalent to the condition that

$$\begin{aligned} I_{\ell^2(\mathbb{J})} &= T_X^* S_X^{\dagger} T_X \\ &= T_X^* (T_X T_X^*)^{\dagger} T_X \\ &= T_X^* (T_X^*)^{\dagger} T_X^{\dagger} T_X \quad \text{(by [Chr95, Cor. 2.3])} \\ &= T_X^* (T_X^{\dagger})^* T_X^{\dagger} T_X \quad \text{(by [Chr95, Lem. 2.4])} \end{aligned}$$

$$= (T_X^\dagger T_X)^* T_X^\dagger T_X$$
$$= P_{\mathrm{range}(T_X^*)} \qquad \text{(since } T_X^\dagger T_X = P_{\mathrm{range}(T_X^*)}\text{)},$$

and this holds if and only if $\mathrm{range}(T_X^*) = \ell^2(\mathbb{J})$, which is itself equivalent to the requirement that T_X be injective. Hence, there is a unique oblique dual frame for X in V if and only if X is a Riesz sequence. □

Next we recover the following result, which is [HG07, Prop. 3].

Proposition 7.4. Let X be a frame sequence in a Hilbert space $\mathcal{H}$.

(a) X has a unique dual if and only if X is a Riesz basis for $\mathcal{H}$.

(b) X has a unique Type I dual if and only if X is a Riesz sequence.

(c) X has a unique Type II dual if and only if X is complete, i.e, $\mathcal{H}_X = \overline{\mathrm{span}}(X) = \mathcal{H}$.

Proof. (b) By Theorem 1.2(a), a Bessel sequence Y is a Type I dual of X if and only if it is an oblique dual of X and $\mathcal{H}_Y = \mathcal{H}_X$. Since $\mathcal{H} = \mathcal{H}_X \oplus \mathcal{H}_X^\perp$, it follows from Proposition 7.3 that a Type I dual of X is unique if and only if X is a Riesz sequence.

(c) Suppose that X is not complete. Then $U = \mathcal{H}_X$ is a proper closed suspace of $\mathcal{H}$. Lemma 2.2 therefore implies that there exists a closed subspace $V \neq U$ such that $R(U, V) = R(V, U) > 0$. By Proposition 2.1, we therefore have $\mathcal{H} = U \dotplus V^\perp$. Theorem 1.4 then implies that there exists a frame Y for V that is a Type II dual of X. Since the canonical dual frame sequence $\widetilde{X}$ is a frame for U that is also a Type II dual of X, we conclude that X does not have a unique Type II dual.

Conversely, suppose that X is complete and that Y is a Type II dual of X. Then by Theorem 1.2 (b), we have $\mathcal{H} = \mathcal{H}_X \dotplus \mathcal{H}_Y^\perp$. Since $\mathcal{H}_X = \mathcal{H}$, we conclude that $\mathcal{H}_Y = \mathcal{H}$ as well. We will show that Y must be the canonical dual frame of X.

Suppose that $c \in \ell^2(\mathbb{J}) = \mathrm{range}(T_X^*) \oplus \ker(T_X)$, and write $c = T_X^* f + d$ where $f \in \mathcal{H}$ and $d \in \ker(T_X)$. Since Y is a Type II dual, we have $\mathrm{range}(T_X^*) = \mathrm{range}(T_Y^*)$, and therefore $\ker(T_X) = \ker(T_Y)$. Since $T_Y T_X^* = I$, we therefore have that

$$S_X T_Y c \;=\; T_X T_X^* T_Y (T_X^* f + d) \;=\; T_X T_X^* f \;=\; T_X (T_X^* f + d) \;=\; T_X c.$$

Letting $\{e_n\}_{n\in\mathbb{J}}$ denote the standard basis for $\ell^2(\mathbb{J})$, we have in particular that $S_X y_n = S_X T_Y e_n = T_X e_n = x_n$ for every $n \in \mathbb{J}$. Hence Y is the canonical dual frame of X.

(a) As pointed out in [HG07], this follows from statements (b) and (c). □

Now we can prove our final main result.

Proof of Theorem 1.6. Assuming the hypotheses of Theorem 1.6, it follows from [KKL05, Lem. 4.9] and its proof that $\ell^2(\mathbb{Z}^d) = \widehat{U}_{||\xi} \dotplus (\widehat{V}_{||\xi})^\perp$ for a.e. $\xi \in \mathbb{T}^d$, and also $\sigma(U) = \sigma(V)$. By Theorem 1.5, there exists a sequence $\Psi = \{\psi_j\}_{j\in\mathbb{J}}$ such that $E(\Psi)$ is an oblique dual of $E(\Phi)$ and $V = S(\Psi)$. By Proposition 4.1, for a.e. $\xi \in \sigma(U) = \sigma(V)$ we have that $\widehat{\Phi}_{||\xi}$ is a frame for $\widehat{U}_{||\xi}$ and $\widehat{\Psi}_{||\xi}$ is a frame for $\widehat{V}_{||\xi}$, and by Theorem 4.2 they are dual to each other. Therefore $\widehat{\Phi}_{||\xi}$ and $\widehat{\Psi}_{||\xi}$ are oblique dual frames for almost every $\xi \in \sigma(U)$.

(b) ⇒ (a). Suppose that $E(\Phi)$ is quasi-stable, i.e., $\widehat{\Phi}_{||\xi}$ is a Riesz basis for $\widehat{U}_{||x}$ for almost every $\xi \in \sigma(U)$. Let $E(\Psi)$ be an oblique dual of $E(\Phi)$ such that $S(\Psi) = V$. Since for a.e. $\xi \in \sigma(U)$ we have that $\widehat{\Phi}_{||\xi}$ is a Riesz basis for $\widehat{U}_{||\xi}$, it follows from Lemma 7.2 and the above discussion that for a.e. $\xi \in \sigma(U)$ there is a unique oblique dual $\widehat{\Psi}_{||x}$ of $\widehat{\Phi}_{||\xi}$ in $\widehat{V}_{||\xi}$. On the other hand, if $\xi \in \mathbb{T}^d \setminus \sigma(U)$, then $\widehat{U}_{||\xi} = \widehat{V}_{||\xi} = \{0\}$, and therefore $\widehat{\Phi}_{||\xi} = \widehat{\Psi}_{||\xi} = \{0\}$. It follows that $E(\Psi)$ is the unique shift-generated dual of $E(\Phi)$ satisfying $V = S(\Psi)$.

(a) ⇒ (b). Suppose that $E(\Phi)$ is not quasi-stable, and write $\Phi := \{\varphi_j\}_{j\in\mathbb{J}}$. By following the same method as in the proof of [HG07, Thm. 6], we can construct a sequence $\Delta := \{\delta_j\}_{j\in\mathbb{J}}$ in $L^2(\mathbb{R}^d)$ satisfying:

(i) $\Delta \neq \{0\}$,
(ii) $E(\Delta)$ is a Bessel sequence,
(iii) $S(\Delta) \subset S(\Psi)$, and
(iv) $\operatorname{range}(T^*_{E(\Phi)}) \perp \operatorname{range}(T^*_{E(\Delta)})$.

In particular, to obtain property (iii), we modify the proof of [HG07, Thm. 6] by using $P_{\widehat{V}_{||\xi}}$ instead of $P_{\widehat{U}_{||\xi}}$. We are able to do this because of the fact that $\sigma(S(\Phi)) = \sigma(S(\Psi))$.

Now, given $f,\, g \in L^2(\mathbb{R}^d)$, we have

$$T^*_{E(\Phi)} g \;=\; \big\{\langle g, T_k\varphi_j\rangle\big\}_{j\in\mathbb{J},\, k\in\mathbb{Z}^d} \qquad \text{and} \qquad T^*_{E(\Delta)} f \;=\; \big\{\langle f, T_k\delta_j\rangle\big\}_{j\in\mathbb{J},\, k\in\mathbb{Z}^d}.$$

Property (iv) therefore implies that

$$\sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{Z}} \langle f, T_k\delta_j\rangle\, \langle T_k\varphi_j, g\rangle \;=\; \big\langle T^*_{E(\Delta)} f,\, T^*_{E(\Phi)} g\big\rangle \;=\; 0.$$

Consequently, for all $f,\, g \in L^2(\mathbb{R}^d)$ we have

$$\sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{Z}} \langle f, T_k\delta_j\rangle\, T_k\varphi_j \;=\; 0 \;=\; \sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{Z}} \langle g, T_k\varphi_j\rangle\, T_k\delta_j. \tag{7.2}$$

Now let $\Xi := \{\psi_j + \delta_j\}_{j\in\mathbb{J}}$. Then Ξ is a Bessel sequence that is distinct from Ψ, and by combining (7.2) with the fact that $E(\Psi)$ is an oblique dual of $E(\Phi)$, we see that

$$T_{E(\Phi)} T^*_{E(\Xi)}|_U \;=\; I_U \qquad \text{and} \qquad T_{E(\Xi)} T^*_{E(\Phi)}|_V \;=\; I_V.$$

In particular, we have $S(\Psi) = V \subset \operatorname{range}(T_{E(\Xi)}) = S(\Xi)$. Also, by property (ii) we have $S(\Xi) \subset S(\Psi) = V$, so $S(\Xi) = V$. It therefore follows from Lemma 7.1 that $E(\Xi)$ is an oblique dual of $E(\Phi)$ with $V = S(\Xi)$. □

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