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Necessary conditions for the existence of multivariate multiscaling functions

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ABSTRACT

In this paper we outline the main ideas behind the recent proof of the authors that if a multivariate, multi-function refinement equation with an arbitrary dilation matrix has a continuous, compactly supported solution which has independent lattice translates, then the joint spectral radius of certain matrices restricted to an appropriate subspace is strictly less than one.

Keywords: Joint spectral radius, multiresolution analysis, multiwavelets, refinement equations, tiles, wavelets

1. INTRODUCTION

In this paper we will sketch the essential ideas behind recent results on necessary conditions for the existence of continuous multivariate multiscaling functions with arbitrary dilation matrices.

The result we describe is somewhat surprising when compared to most other well-known necessary conditions for one-dimensional or single-function refinement equations. In particular, for the classical one-dimensional, single-function refinement equation $f(x) = \sum_{k=0}^{N} c_k\, f(2x-k)$, it is known that a continuous, compactly supported solution exists if and only if all infinite products of two appropriate matrices T_0, T_1 restricted to a certain invariant subspace $\mathcal{W}$ converge.[1–3] The entries of the matrices T_0, T_1 are the coefficients c_k in a certain explicit order, but the subspace $\mathcal{W}$ is determined only implicitly by the c_k. With the additional hypothesis that integer translates of f are independent, this subspace $\mathcal{W}$ can be shown to coincide with the subspace $\mathcal{V} = (1, \dots, 1)^{\perp}$,[4,5] which is independent of the c_k. However, even in the one-dimensional case, as soon as *multi*scaling functions are considered it is easy to see that the appropriate necessary and sufficient space $\mathcal{W}$ can be considerably smaller than the analogue of the co-dimension 1 space $\mathcal{V}$. Additionally, $\mathcal{W}$ can be very difficult to determine explicitly in the multi-function setting, even with the assumption of linear independence, while $\mathcal{V}$ is given explicitly. However, we will show in this paper that, even in the general multivariate, multi-function case with arbitrary dilation matrix, the infinite matrix products of the analogues of T_0, T_1 must actually converge on the bigger space $\mathcal{V}$, and not merely on $\mathcal{W}$. The proof in complete generality can be found in Cabrelli et. al[6] along with sufficient conditions and other results. Here we will try to elucidate the main ideas of the proof of the necessary conditions without the obscuring technical details.

There are three key ingredients to the proof:

- First, the proof of the necessary conditions by Wang[3] for the single-function, univariate case.
- Second, the theory of tilings of $\mathbb{R}^n$ by arbitrary tiles.
- Third, the application of self-similarity to this setting.

2. NOTATION, ATTRACTORS, AND TILINGS

We will assume throughout this paper that A is a fixed dilation matrix and that $\mathbb{Z}^n$ is its associated full-rank invariant lattice. That is, $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$ and every eigenvalue λ of A satisfies $|\lambda| > 1$. For clarity, we will only consider here the lattice $\mathbb{Z}^n$, but the results all carry over without loss of generality to any full-rank lattice Γ.[6]

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2.1. The Refinement Equation and the Refinement Operator

We will consider refinement equations of multiplicity r of the form

$$f(x) \;=\; \sum_{k\in\Lambda} c_k\, f(Ax-k), \qquad x\in\mathbb{R}^n, \tag{1}$$

where Λ is a fixed finite subset of $\mathbb{Z}^n$ and the c_k are fixed $r\times r$ matrices. We let $c_k=0$ for $k\notin\Lambda$. A solution of the refinement equation is a vector-valued function $f=(f_1,\dots,f_r)^{\mathrm{t}}\colon\mathbb{R}^n\to\mathbb{C}^r$ and is called a *vector scaling function* or a *refinable vector function*.

The *refinement operator* associated with this refinement equation is the mapping S acting on vector functions $g\colon\mathbb{R}^n\to\mathbb{C}^r$ defined by

$$Sg(x) \;=\; \sum_{k\in\Lambda} c_k\, g(Ax-k), \qquad x\in\mathbb{R}^n.$$

A vector scaling function is thus a fixed point of S. The *cascade algorithm* is the iteration

$$f^{(i+1)} \;=\; Sf^{(i)}.$$

Our main interest in this paper is in compactly supported solutions of the refinement equation, especially ones which satisfy the "minimal accuracy" condition defined in Section 2.8. We therefore note the following mild normalization condition on the coefficients c_k which ensures that a compactly supported solution to the refinement equation will exist, at least in the sense of distributions.[7]

Proposition 2.1. *If the $r\times r$ matrix $\Delta=\frac{1}{m}\sum_{k\in\Lambda}c_k$ has eigenvalues $\lambda_1=\cdots=\lambda_s=1$ and $|\lambda_{s+1}|,\dots,|\lambda_r|<1$ with the eigenvalue 1 nondegenerate, then there exist compactly supported distributions $f_1,\dots,f_r$ such that $f=(f_1,\dots,f_r)^{t}$ satisfies the refinement equation (1) in the sense of distributions. Furthermore, $\hat f(\omega)$ is a continuous vector function, and $\hat f(0)\neq 0$.*

2.2. Attractors

Since we require that $A(\mathbb{Z}^n)\subset\mathbb{Z}^n$, the dilation matrix A necessarily has integer determinant. We define

$$m \;=\; |\det(A)|$$

and let

$$D \;=\; \{d_1,\dots,d_m\}$$

be a *full set of digits* with respect to A and $\mathbb{Z}^n$, i.e., a complete set of representatives of the order-m group $\mathbb{Z}^n/A(\mathbb{Z}^n)$. Without loss of generality, we impose the condition that $0\in D$, i.e., the zero vector is one of the digits. Because D is a full set of digits, the lattice $\mathbb{Z}^n$ is partitioned into the disjoint cosets

$$\mathbb{Z}^n_d \;=\; A(\mathbb{Z}^n)-d \;=\; \{Ak-d : k\in\mathbb{Z}^n\}, \qquad d\in D. \tag{2}$$

The space $\mathcal{H}(\mathbb{R}^n)$ consisting of all nonempty, compact subsets of $\mathbb{R}^n$ is a complete metric space under the Hausdorff metric $h(\cdot,\cdot)$ defined by

$$h(B,C) \;=\; \inf\{\varepsilon>0 : B\subset C_\varepsilon \text{ and } C\subset B_\varepsilon\},$$

where

$$B_\varepsilon \;=\; \{x\in\mathbb{R}^n : \operatorname{dist}(x,B)<\varepsilon\}. \tag{3}$$

That is,

$$h(B,C)<\varepsilon \quad\Longleftrightarrow\quad B\subset C_\varepsilon \text{ and } C\subset B_\varepsilon.$$

Since all norms on $\mathbb{R}^n$ are equivalent, the definition of the Hausdorff metric is independent of the choice of norm on $\mathbb{R}^n$.

For each $k \in \mathbb{Z}^n$, let $w_k : \mathbb{R}^n \to \mathbb{R}^n$ be the affine map

$$w_k(x) \;=\; A^{-1}(x+k).$$

Since A^{-1} is contractive, each w_k is a contractive mapping on $\mathbb{R}^n$. For each finite subset $H \subset \mathbb{Z}^n$, define $w_H : \mathcal{H}(\mathbb{R}^n) \to \mathcal{H}(\mathbb{R}^n)$ by

$$w_H(B) \;=\; \bigcup_{k\in H} w_k(B) \;=\; A^{-1}(B+H). \tag{4}$$

Using the fact that each w_k is contractive on $\mathbb{R}^n$ under the Euclidean norm, it can be shown that w_H is contractive on $\mathcal{H}(\mathbb{R}^n)$ under the Hausdorff metric. The Contraction Mapping Theorem therefore implies that there exists a unique nonempty compact set $K_H \subset \mathbb{R}^n$ such that

$$w_H(K_H) \;=\; K_H.$$

That is, K_H is defined by the property

$$K_H \;=\; A^{-1}(K_H+H). \tag{5}$$

The set K_H is called the *attractor* of the iterated function system (IFS) generated by $\{w_k\}_{k\in H}$.[8] We can use equation (5) to obtain another expression for K_H. Iterating equation (5) k times, we see that $K_H = \sum_{j=1}^{k} A^{-j}(H) + A^{-k}(K_H)$. Then, using the fact that A^{-1} is a contraction, it can be shown that

$$K_H \;=\; \sum_{j=1}^{\infty} A^{-j}(H) \;=\; \Bigl\{\sum_{j=1}^{\infty} A^{-j}h_j : h_j \in H\Bigr\}. \tag{6}$$

The attractors K_Λ and $Q = K_D$ of the IFS's generated by $\{w_k\}_{k\in\Lambda}$ and $\{w_k\}_{k\in D}$, respectively, will play particularly important roles in this paper.

Recall that Λ is the subset of $\mathbb{Z}^n$ for which the coefficients c_k of the refinement equation (1) are nonzero. If we define the support of a vector-valued function $g = (g_1,\dots,g_r)^{\mathrm{t}} : \mathbb{R}^n \to \mathbb{C}^r$ to be the closure of $\{x \in \mathbb{R}^n : g(x) \neq 0\}$, then it follows from basic properties of attractors that a vector scaling function must be supported within K_Λ. Therefore, the support of the scaling function does not depend on the values of the coefficients c_k but rather on their placements.

2.3. Tiles

With $D = \{d_1,\dots,d_m\}$ a full set of digits with respect to A and $\mathbb{Z}^n$, the attractor $Q = K_D$ satisfies the following important properties.[9,10] Here Q° denotes the interior of Q, and $|Q|$ is the Lebesgue measure of Q.

Lemma 2.1. *The following statements hold.*

(a) $Q + \mathbb{Z}^n = \mathbb{R}^n$.

(b) *Q has nonempty interior, Q is the closure of Q°, and* $|\partial Q| = 0$.

(c) $|Q \cap (Q+k)| = 0$ *for all* $k \in \mathbb{Z}^n \setminus \{0\}$ *if and only if* $|Q| = 1$. *In this case,* $Q \cap (Q+k) \subset \partial Q$ *for each* $k \in \mathbb{Z}^n \setminus \{0\}$.

In other words, part (c) above says that if $|Q| = 1$, then Q is a *tile* in the sense that the $\mathbb{Z}^n$-translates $\{Q+k\}_{k\in\mathbb{Z}^n}$ cover $\mathbb{R}^n$ with overlaps of measure zero. It is not always true that for each dilation matrix A there exists a full set of digits D such that the corresponding attractor Q is a tile in this sense.[11,12] However, Lagarias and Wang have shown that this is the case if $n = 1, 2, 3$ or if $m = |\det(A)| > n$.[13–15] For $n = 4$ a counterexample exists.[11]

We will assume throughout this paper that whenever a dilation matrix A and choice of digits D are given, the corresponding attractor $Q = K_D$ is in fact a tile. That is, we always implicitly assume that the $\mathbb{Z}^n$-translates of Q cover $\mathbb{R}^n$ with overlaps of measure zero.

Example. A tile Q may have a fractal boundary. For example, for the dilation matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and digit set $D = \{(0,0),\,(1,0)\}$, the tile Q is the celebrated "twin dragon" fractal shown in Figure 1.

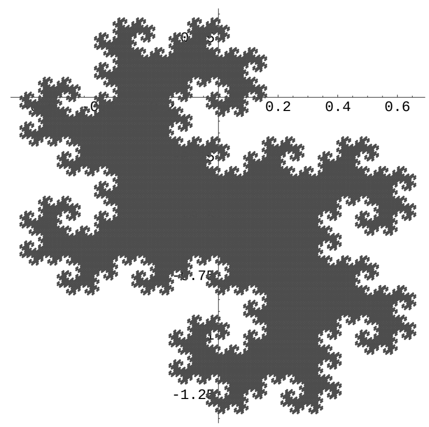

Figure 1. Twin Dragon attractor

For the attractor Q, by equation (6) we have

$$Q \;=\; K_D \;=\; \sum_{j=1}^{\infty} A^{-j}(D) \;=\; \Bigl\{ \sum_{j=1}^{\infty} A^{-j}\xi_j : \xi_j \in D \Bigr\}.$$

Thus, each point $x \in Q$ can be written $x = \sum_{j=1}^{\infty} A^{-j}\xi_j$ for some $\xi_j \in D$. We write $x = .\xi_1\xi_2\cdots$ in this case, and refer to this representation of x as an *A-nary expansion* of x. Note that A-nary expansions need not be unique.

We will define a function $\tau\colon Q \to Q$ that is analogous to the univariate $2x$ mod 1 map. Recall that, by definition, $Q = \bigcup_{i=1}^{m} w_{d_i}(Q)$. If $x \in Q$ is such that $x \in w_{d_i}(Q)$ for a *unique* digit d_i, then we set

$$\tau x \;=\; w_{d_i}^{-1}(x) \;=\; Ax - d_i. \tag{7}$$

Thus, if $x = .\xi_1\xi_2\cdots$ is an A-nary expansion of such an x, then $\xi_1 = d_i$ and $\tau x = .\xi_2\xi_3\cdots$. For other x, the meaning of equation (7) is ambiguous. We eliminate this ambiguity by "disjointizing" the sets $w_{d_i}(Q)$. Specifically, we define

$$Q_1 \;=\; w_{d_1}(Q) \qquad \text{and} \qquad Q_i \;=\; w_{d_i}(Q) \setminus \Bigl(\bigcup_{j<i} Q_j \Bigr) \qquad \text{for } i = 2, \dots, m. \tag{8}$$

Then $Q_i \subset w_{d_i}(Q)$, and Q is the union of the disjoint sets $Q_1, \dots, Q_m$. Hence each $x \in Q$ lies in a unique Q_i, and we define τx by equation (7) using that unique value of i.

The tile Q has the property that it covers $\mathbb{R}^n$ by lattice translates with overlaps of measure zero. However, for some proofs we need a subset $\tilde{Q}$ of Q which covers $\mathbb{R}^n$ by lattice translates *without* overlaps. For example, if $n = 1$ and the tile Q was the interval $[0, 1]$, then we could simply remove one endpoint to obtain a new set $\tilde{Q} = [0, 1)$ which tiles $\mathbb{R}$ without overlaps. In general, however, the attractor Q may have a fractal boundary, and it is not obvious that an analogous "peeling off" process is always possible. In particular, in the higher-dimensional case the Hausdorff dimension of the boundary is often strictly larger than $n - 1$!

To overcome this problem, we divide the lattice $\mathbb{Z}^n$ into three disjoint subsets $\mathbb{Z}^n_+$, $\mathbb{Z}^n_-$, and $\{0\}$ in such a way that $\mathbb{Z}^n_- = -\mathbb{Z}^n_+$ and both $\mathbb{Z}^n_+$ and $\mathbb{Z}^n_-$ are closed under vector addition. Specifically, we set

$$\mathbb{Z}^n_+ \;=\; \bigcup_{i=1}^{n} \{k \in \mathbb{Z}^n : k = (k_1, \dots, k_i, 0, \dots, 0),\ k_i > 0\}$$

and then define $\mathbb{Z}^n_- = -\mathbb{Z}^n_+$. Then we have the following result.[6]

Proposition 2.2. *Assume that Q is a tile, and define*

$$\tilde{Q} \;=\; Q \setminus \bigcup_{k \in \mathbb{Z}^n_+} (Q + k).$$

Then the $\mathbb{Z}^n$-translates of $\tilde{Q}$ cover $\mathbb{R}^n$ without overlaps, i.e., $\tilde{Q} + \mathbb{Z}^n = \mathbb{R}^n$ and $\tilde{Q} \cap (\tilde{Q} + k) = \emptyset$ for $k \in \mathbb{Z}^n \setminus \{0\}$. Further, $\tilde{Q} \cap \mathbb{Z}^n$ contains a single element.

2.4. Generalized Matrix Notation

It will be convenient to use a generalized matrix notation which allows matrices or vectors to be indexed by arbitrary countable sets. If desired, such generalized matrices can always be realized as ordinary matrices by choosing a specific ordering for the index set. The actual ordering used is not important, as long as the same ordering is used consistently. To be precise, let J and K be finite or countable index sets. Let $m_{j,k}$ be $r \times s$ matrices for $j \in J$ and $k \in K$. Then we say that $M = [m_{j,k}]_{j\in J,k\in K} \in (\mathbb{C}^{r\times s})^{J\times K}$ is a $J \times K$ matrix (with $r \times s$ block entries). If $N = [n_{k,\ell}]_{k\in K,\ell\in L} \in (\mathbb{C}^{s\times t})^{K\times L}$, then the product of the $J \times K$ matrix M with the $K \times L$ matrix N is the $J \times L$ matrix formally defined by

$$MN \;=\; \Bigl[\sum_{k\in K} m_{j,k}\, n_{k,\ell}\Bigr]_{j\in J,\ell\in L}.$$

Most summations encountered in this paper will contain only finitely many nonzero terms. A "column vector" is a $J \times 1$ matrix, which we will denote by $v = [v_j]_{j\in J}$. The entries v_j may be scalars or $r \times s$ blocks. In particular, $\mathbb{C}^r$ is the space of column vectors of length r. Analogously, a "row vector" is a $1 \times J$ matrix, which we will denote by $u = (u_j)_{j\in J}$. In particular, $\mathbb{C}^{1\times r}$ is the space of all row vectors of length r.

2.5. Admissibility

Recall that any compactly supported solution f to the refinement equation must be supported within the attractor K_Λ, which is a compact set in $\mathbb{R}^n$. Since Q is a tile, a finite number of lattice translates of Q will cover K_Λ. Let $\Omega \subset \mathbb{Z}^n$ denote any fixed set such that

$$K_\Lambda \;\subset\; Q + \Omega.$$

If the boundary of the tile Q is fractal-like, it may be difficult to construct such a set Ω. Algorithms for constructing such an Ω exist, although they do not necessarily produce the smallest possible such Ω.[6] For some results, no assumptions on Ω other than $K_\Lambda \subset Q + \Omega$ are required. However, the proof of the necessary conditions presented in this paper will require the following additional "admissibility" requirement on Ω.

Definition. Let H be a finite subset of $\mathbb{Z}^n$. Then we say that a nonempty, finite set $\Omega \subset \mathbb{Z}^n$ is *H-admissible* if

$$w_H(\Omega) \;\cap\; \mathbb{Z}^n \;\subset\; \Omega,$$

where $w_H(\Omega) = A^{-1}(\Omega + H)$ is as defined in equation (4).

The notion of admissibility arises naturally in the study of refinement equations.[16] For example, if Ω is Λ-admissible then the space

$$\ell(\Omega) \;=\; \{a = [a_k]_{k\in\mathbb{Z}^n} \in (\mathbb{C}^{r\times 1})^{\mathbb{Z}^n\times 1} : \operatorname{supp}(a) \subset \Omega\}$$

is right-invariant under the infinite matrix $L = [c_{Ai-j}]_{i,j\in\mathbb{Z}^n}$, and the right-eigenvectors of L corresponding to nonzero eigenvalues are necessarily elements of $\ell(\Omega)$. The eigenvalues and eigenvectors of L are intimately tied to the accuracy of the vector scaling function, a topic which is explored in more detail in Cabrelli et. al.[7]

We will need to consider sets $\Omega \subset \mathbb{Z}^n$ which are admissible with respect to the set

$$\Lambda - D \;=\; \{k - d : k \in \Lambda,\, d \in D\}.$$

Since we have assumed that $0 \in D$, it follows that $\Lambda \subset \Lambda - D$. For example, the set $\Omega_{\Lambda-D} = K_{\Lambda-D} \cap \mathbb{Z}^n$ is both Λ-admissible and $(\Lambda - D)$-admissible. By Lemma 4.8 in Cabrelli et al.,[7] every finite subset of $\mathbb{Z}^n$ is contained in a $(\Lambda - D)$-admissible set.

2.6. Matrix Form of the Refinement Operator

We will obtain a matrix form of the refinement operator in this section.

Consider any function $g\colon \mathbb{R}^n \to \mathbb{C}^r$ such that $\operatorname{supp}(g) \subset K_\Lambda$. Define the *folding* of g to be the function $\Phi g\colon Q \to (\mathbb{C}^{r\times 1})^{\Omega\times 1}$ given by

$$\Phi g(x) \;=\; \bigl[g(x+k)\bigr]_{k\in\Omega}, \qquad x \in Q.$$

If we write $(\Phi g)_k(x) = g(x+k)$ for the kth component of $\Phi g(x)$, then this folding has the property that $(\Phi g)_{k_1}(x_1) = (\Phi g)_{k_2}(x_2)$ whenever $x_1, x_2 \in Q$ and $k_1, k_2 \in \Omega$ are such that $x_1 + k_1 = x_2 + k_2$ (because Q is a tile, any such points x_1, x_2 would have to lie on ∂Q).

For each $d \in D$, define an $\Omega \times \Omega$ matrix T_d by

$$T_d \;=\; \big[c_{Aj-k+d}\big]_{j,k\in\Omega}.$$

Let $Q_1, \dots, Q_m$ be defined as in equation (8). Define an operator T acting on vector functions

$$u(x) = [u_k(x)]_{k\in\Omega} \colon Q \to (\mathbb{C}^{r\times 1})^{\Omega\times 1}$$

by

$$Tu(x) \;=\; \sum_{i=1}^{m} \chi_{Q_i}(x) \cdot T_{d_i} u(Ax - d_i).$$

Or, equivalently, T can be defined by

$$Tu(x) \;=\; T_{d_i} u(\tau x) \quad \text{if } x \in Q_i.$$

This operator T is related to the refinement operator S as follows.[6]

Proposition 2.3. *Let $\Omega \subset \mathbb{Z}^n$ be such that $K_\Lambda \subset Q + \Omega$. Assume that $g \colon \mathbb{R}^n \to \mathbb{C}^r$ satisfies*

$$\operatorname{supp}(g) \subset K_\Lambda \qquad \textit{and} \qquad g(x) = 0 \textit{ for } x \in \partial K_\Lambda.$$

Let $x \in Q$. Then for each $d \in D$ such that $Ax - d \in Q$, we have

$$\Phi S g(x) \;=\; T_d \Phi g(Ax - d).$$

Consequently,

$$\Phi S g \;=\; T \Phi g. \tag{9}$$

One point to be made about the proof of Proposition 2.3 is the following. Since $K_\Lambda \subset Q + \Omega$, if $y \in K_\Lambda$ then $y = x + k$ for some $x \in Q$ and $k \in \Omega$, but in general this x and k need not be unique. However, if y lies in the interior K_Λ° of K_Λ, then it can be shown that x and k are unique.

The equality in equation (9) is a pointwise everywhere equality. This is critical for later application of this result to the existence of continuous solutions of the refinement equation. If instead we we interested in, say, existence of L^p solutions then we would require only a conclusion of equality almost everywhere. In this case, the hypothesis in Proposition 2.3 that $g(x)$ vanish on the boundary of K_Λ would not be needed, because that boundary must have measure zero.

2.7. The Joint Spectral Radius

The spectral radius of a square matrix M is

$$\rho(M) \;=\; \lim_{\ell\to\infty} \|M^\ell\|^{1/\ell} \;=\; \max\{|\lambda| : \lambda \text{ is an eigenvalue of } M\}.$$

This value is independent of the choice of norm $\|\cdot\|$. For each $1 \le p \le \infty$, the *p-joint spectral radius* (p-JSR) of a finite collection of $s \times s$ matrices $\mathcal{M} = \{M_1, \dots, M_m\}$ is

$$\hat\rho_p(\mathcal{M}) \;=\; \begin{cases} \displaystyle\lim_{\ell\to\infty} \Big(\sum_{\Pi\in\mathcal{P}_\ell} \|\Pi\|^p\Big)^{1/p\ell}, & 1 \le p < \infty, \\ \displaystyle\lim_{\ell\to\infty} \max_{\Pi\in\mathcal{P}_\ell} \|\Pi\|^{1/\ell}, & p = \infty, \end{cases} \tag{10}$$

where

$$\mathcal{P}_0 \;=\; \{I\} \qquad \text{and} \qquad \mathcal{P}_\ell \;=\; \{M_{j_1}\cdots M_{j_\ell} : 1 \le j_i \le m\}.$$

It is easy to see that the limit in equation (10) exists and is independent of the choice of norm $\|\cdot\|$ on $\mathbb{C}^{s\times s}$. Note that if $p \ge q$, then $\hat{\rho}_p(\mathcal{M}) \le \hat{\rho}_q(\mathcal{M})$.

The ∞-JSR was introduced by Rota and Strang[17] and independently rediscovered by Daubechies and Lagarias,[18–20] who also were the first to apply the JSR to the study of refinement equations. The 1-JSR was introduced by Wang[21] and the p-JSR was independently introduced by Jia.[22]

We will be concerned here only with the ∞-JSR, which we will refer to as the *uniform joint spectral radius* or simply as the *joint spectral radius*. Berger and Wang[23] proved that $\hat{\rho}_\infty(\mathcal{M}) < 1$ if and only if every product $M_{j_1}\cdots M_{j_\ell}$ converges to the zero matrix as $\ell \to \infty$, and that

$$\hat{\rho}_\infty(\mathcal{M}) \;=\; \lim_{\ell\to\infty} \max_{\Pi\in\mathcal{P}_\ell} \rho(\Pi)^{1/\ell}.$$

While this equality is trivial for the case that $\mathcal{M}$ contains a single matrix, it is not at all trivial when $m \ge 2$.

2.8. Minimal Accuracy Conditions

To complete the setup for our main result, we need to recall what it means for a scaling function to satisfy the *minimal accuracy conditions*. The *accuracy* of a refinable vector function f is the largest integer $\kappa > 0$ such that every multivariate polynomial $q(x) = q(x_1,\dots,x_n)$ with $\deg(q) < \kappa$ can be written

$$q(x) \;=\; \sum_{k\in\mathbb{Z}^n} a_k f(x+k) \;=\; \sum_{k\in\mathbb{Z}^n}\sum_{i=1}^{r} a_{k,i} f_i(x+k) \text{ a.e.}, \qquad x \in \mathbb{R}^n,$$

for some row vectors $a_k = (a_{k,1},\dots,a_{k,r}) \in \mathbb{C}^{1\times r}$. If no polynomials are reproducible from translates of f then we set $\kappa = 0$. We say that translates of f along $\mathbb{Z}^n$ are *linearly independent* if $\sum_{k\in\mathbb{Z}^n} a_k f(x+k) = 0$ implies $a_k = 0$ for each k.

Conditions for accuracy are easy to formulate in the one-dimensional, single function case ($r = 1$, $n = 1$). As long as we only consider the conditions that imply reproduction of the constant polynomial alone, it is not too difficult to generalize these conditions to the case of higher dimensions or multiple functions. These can be summarized as follows.[24] The conditions for higher-order accuracy are more involved.[24]

Lemma 2.2. *Let f be a compactly supported distributional solution of the refinement equation. Let $\mathbb{Z}^n_d = A(\mathbb{Z}^n) - d$ denote the cosets defined in equation* (2).

(a) *If there exists a row vector $u_0 \in C^{1\times r}$ such that $u_0\hat{f}(0) \ne 0$ and*

$$u_0 \;=\; \sum_{k\in\mathbb{Z}^n_d} u_0 c_k \quad \textit{for each } d \in D, \tag{11}$$

then f has accuracy $\kappa \ge 1$, and

$$\sum_{k\in\mathbb{Z}^n} u_0 f(x+k) \;=\; 1 \;\textit{a.e.}$$

(b) *If f has accuracy $\kappa \ge 1$ and if f has independent translates, then there exists a row vector $u_0 \in C^{1\times r}$ such that $u_0\hat{f}(0) \ne 0$ and equation* (11) *holds.*

Suppose now that f is a compactly supported vector scaling function with accuracy $\kappa \ge 1$, and let u_0 be the row vector such that $\sum_{k\in\mathbb{Z}^n} u_0 f(x+k) = 1$ a.e. If $x \in Q$, then Lemma 4.11 in Cabrelli et al.[6] implies that the only nonzero terms in this series occur when $k \in \Omega$. Hence, if we set $e_0 = (u_0)_{k\in\Omega}$, i.e., e_0 is the row vector in $(\mathbb{C}^{1\times r})^{1\times\Omega}$ obtained by repeating the block u_0 once for each $k \in \Omega$, then

$$e_0\Phi f(x) \;=\; \sum_{k\in\Omega} u_0 f(x+k) \;=\; \sum_{k\in\mathbb{Z}^n} u_0 f(x+k) \;=\; 1 \text{ a.e.}, \qquad x \in Q.$$

Thus the values of $\Phi f(x)$ are constrained to lie in a particular hyperplane E_0 in $(\mathbb{C}^{r\times 1})^{\Omega\times 1}$, namely, the collection of vectors $v = [v_k]_{k\in\Omega}$ such that $e_0 v = \sum_{k\in\Omega} u_0 v_k = 1$. Further, the set of differences $V_0 = E_0 - E_0$ is the subspace consisting of vectors $v = [v_k]_{k\in\Omega}$ such that $e_0 v = \sum_{k\in\Omega} u_0 v_k = 0$. If we define the dot product of two column vectors $u = [u_k]_{k\in\Omega}$ and $v = [v_k]_{k\in\Omega} \in (\mathbb{C}^{r\times 1})^{\Omega\times 1}$ by

$$u \cdot v \;=\; u^* v \;=\; \sum_{k\in\Omega} u_k^* v_k \;=\; \sum_{k\in\Omega} \sum_{i=1}^{r} \bar{u}_{k,i} v_{k,i},$$

where u^* is the Hermitian, or conjugate transpose, of u, then $e_0 v = e_0^* \cdot v$, so V_0 is simply the orthogonal complement of the single column vector e_0^*. This space V_0 can be shown to be right-invariant under each matrix T_d for $d \in D$.

3. MAIN RESULT

It has been shown[6] that if the coefficients c_k of the refinement equation satisfy the conditions for minimal accuracy, then a sufficient condition for the existence of a continuous vector scaling function is that $\hat{\rho}(\{T_d|_{V_0}\}_{d\in D}) < 1$. Here, the matrices $T_d = [c_{Ai-j+d}]_{i,j\in\Omega}$ and the subspace $V_0 = (e_0^*)^{\perp}$ depend implicitly on the choice of $\Omega \subset \mathbb{Z}^n$, but there are no restrictions on Ω except that it be a finite subset of $\mathbb{Z}^n$ such that $K_\Lambda \subset Q + \Omega$.

In this section we will show that if the minimal accuracy conditions are satisfied and if in addition the lattice translates of f are "stable" and the set Ω is "admissible," then the condition $\hat{\rho}(\{T_d|_{V_0}\}_{d\in D}) < 1$ is also necessary for the existence of a continuous vector scaling function. Note that V_0 is a very "large" space; specifically, it has co-dimension 1. Moreover, it is easily determined: the row vector e_0 is obtained simply by repeating the row vector u_0 once for each $k \in \Omega$, and u_0 itself is by equation (11) simply the common left 1-eigenvector of the matrices $\sum_{k\in\mathbb{Z}^n_d} c_k$ for $d \in D$.

The definition of "stable translates" that we shall use is as follows.

Definition. A bounded vector function $g\colon \mathbb{R}^n \to \mathbb{C}^r$ is said to have L^∞-*stable translates* if there exist constants C_1, $C_2 > 0$ such that

$$C_1 \sup_{k\in\mathbb{Z}^n} \max_{1\le i\le r} |a_{k,i}| \;\le\; \Big\| \sum_{k\in\mathbb{Z}^n} a_k\, g(x+k) \Big\|_{L^\infty} \;\le\; C_2 \sup_{k\in\mathbb{Z}^n} \max_{1\le i\le r} |a_{k,i}|$$

for all finitely supported sequences of row vectors $\{a_k\}_{k\in\mathbb{Z}^n}$, where $a_k = (a_{k,1}, \dots, a_{k,r}) \in C^{1\times r}$ for $k \in \mathbb{Z}^n$. Equivalently, using the fact that all norms on a finite-dimensional vector space are equivalent, if $\|\cdot\|$ is any norm on $\mathbb{C}^{r\times r}$, then a vector function $g \in L^\infty(\mathbb{R}^n, \mathbb{C}^r)$ has L^∞-stable translates if and only if there exist constants C_1, $C_2 > 0$ such that

$$C_1 \sup_{k\in\mathbb{Z}^n} \|B_k\| \;\le\; \Big\| \sum_{k\in\mathbb{Z}^n} B_k\, g(x+k) \Big\|_{L^\infty} \;\le\; C_2 \sup_{k\in\mathbb{Z}^n} \|B_k\|$$

for all finitely supported sequences of matrices $\{B_k\}_{k\in\mathbb{Z}^n}$, where $B_k \in C^{r\times r}$ for $k \in \mathbb{Z}^n$.

Using the above notation, we can now formulate our major result as follows.[6]

Theorem 3.1. *Let f be a continuous, compactly supported solution to the refinement equation* (1) *such that f has L^∞-stable translates. Assume that there exists a row vector $u_0 \in C^{1\times r}$ such that*

$$u_0 \hat{f}(0) \neq 0 \qquad \textit{and} \qquad u_0 \;=\; \sum_{k\in\mathbb{Z}^n_d} u_0 c_k \;\textit{for}\; d \in D.$$

If $\Omega \subset \mathbb{Z}^n$ is any $(\Lambda - D)$-admissible set such that $K_\Lambda \subset Q + \Omega$, then $\hat{\rho}(\{T_d|_{V_0}\}_{d\in D}) < 1$.

The proof of Theorem 3.1 consists of two separate steps. First, one shows that the existence of a continuous solution to the refinement equation with stable translates implies that a matrix-valued version of the cascade algorithm converges pointwise everywhere when a specific starting function is used. Second, one shows that the convergence of this version of the cascade algorithm necessarily implies that the JSR in question is less than 1. Each of these stages is of interest in itself. Moreover, the first stage requires the assumption of stable translates but does not require any admissibility assumptions on the set Ω, while the second stage requires that the set Ω be $(\Lambda - D)$-admissible but does not require that f have stable translates. In particular, using the second stage of the proof one can conclude that whenever the cascade algorithm converges pointwise uniformly for a specific function, then the JSR of the matrices T_d restricted to V_0 is strictly less than 1.

The matrix version of the cascade algorithm referred to above is defined as follows. Let $\tilde{Q}$ be the subset of Q constructed in Proposition 2.2. This set $\tilde{Q}$ has the property that the $\mathbb{Z}^n$-translates of $\tilde{Q}$ cover $\mathbb{R}^n$ *without overlaps*. Further, $\tilde{Q}$ contains a unique element γ_0 of $\mathbb{Z}^n$, i.e.,

$$\tilde{Q} \cap \mathbb{Z}^n \;=\; \{\gamma_0\}.$$

Let $\varphi^{(0)}$ be the characteristic function of the unique translate of $\tilde{Q}$ that contains the origin times the $r \times r$ identity matrix I_r, i.e.,

$$\varphi^{(0)}(x) \;=\; \chi_{\tilde{Q}-\gamma_0}(x) \cdot I_r, \tag{12}$$

and let $\varphi^{(i)} \colon \mathbb{R}^n \to \mathbb{C}^{r \times r}$ be obtained by iterating the refinement operator S on $\varphi^{(0)}$, i.e.,

$$\varphi^{(i+1)}(x) \;=\; S\varphi^{(i)}(x) \;=\; \sum_{k \in \Lambda} c_k\, \varphi^{(i)}(Ax - k). \tag{13}$$

Note that we have abused notation in equation (13) since the refinement operator S is formally defined to act on vector-valued functions, while we are here applying it to matrix-valued functions. However, the abuse is slight and the intended meaning is clear. We will perform similar abuses throughout this section without further comment.

Suppose now that the minimal accuracy conditions are satisfied, i.e., there exists a row vector $u_0 \in C^{1\times r}$ such that $\sum_{k\in\mathbb{Z}^n} u_0 f(x+k) = 1$. The first stage of the proof of Theorem 3.1 is to show that if the translates of f are L^∞-stable, then the functions $\varphi^{(i)}$ obtained via the matrix cascade algorithm converge everywhere to the matrix-valued function $f(x)u_0$ and further that the convergence is uniform (i.e., in L^∞-norm). It will be important for the second stage of the proof of Theorem 3.1 that this convergence is *pointwise everywhere*, and not merely almost everywhere. In this first stage of the proof of Theorem 5.20 we do not require any admissibility assumptions on the set Ω. Note that the matrix-valued function $f(x)u_0$ has rank one for each x.

The first stage of the proof is summarized in the following result.[6]

Theorem 3.2. *Let f be a continuous, compactly supported solution to the refinement equation* (1) *such that f has L^∞-stable translates. Assume that there exists $u_0 \in C^{1\times r}$ such that $u_0\hat{f}(0) \neq 0$ and $u_0 = \sum_{k\in\mathbb{Z}^n_d} u_0 c_k$ for $d \in D$. Let $\varphi^{(0)}$ and $\varphi^{(i)}$ be defined by equations* (12) *and* (13)*. Then $\varphi^{(i)}$ converges everywhere to $f(x)u_0$ as $i \to \infty$ and this convergence is uniform.*

To obtain the pointwise everywhere convergence of the cascade algorithm, the proof of the above result uses in a crucial way the fact that $\tilde{Q}$ covers $\mathbb{R}^n$ by lattice translates *without* overlaps.[6]

For the second stage of the proof of Theorem 3.1, we require some auxiliary notation and results. We shall in the remainder of this section often encounter nested subsets of $\mathbb{Z}^n$ of the form

$$\Omega \;\subset\; \tilde{\Omega} \;\subset\; \mathbb{Z}^n.$$

We will use a tilde to denote the analogues for $\tilde{\Omega}$ of objects implicitly associated with Ω. For example, since $T_d = [c_{Aj-k+d}]_{j,k\in\Omega}$, we define $\tilde{T}_d = [c_{Aj-k+d}]_{j,k\in\tilde{\Omega}}$.

The need for these larger sets $\tilde{\Omega}$ arises because we will be applying the cascade algorithms to functions that are compactly supported but which need not be supported within the attractor K_Λ. However, as stated in the following lemma, it is possible to control the supports of the iterates of the cascade algorithm by observing that these supports must converge in the Hausdorff metric to K_Λ. For this purpose, recall the notation introduced in association with the Hausdorff metric, namely, $B_\varepsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x, B) < \varepsilon\}$.

Lemma 3.3. *Let $\Omega \subset \tilde{\Omega} \subset \Gamma$ be such that $K_\Lambda \subset Q + \Omega$ and $\tilde{\Omega}$ is $(\Lambda - D)$-admissible. If g is any function such that $\operatorname{supp}(g) \subset Q + \tilde{\Omega}$, then $\operatorname{supp}(Sg) \subset Q + \tilde{\Omega}$ as well. Further, given $\varepsilon > 0$ there exists $i_0 > 0$ such that $\operatorname{supp}(S^i g) \subset (Q+\Omega)_\varepsilon$ for all $i \geq i_0$.*

Now we can complete the second stage of the proof of Theorem 3.1. Specifically, we show next that the pointwise convergence of the matrix cascade algorithm implies a restriction on the uniform JSR. This result requires an admissibility assumption on Ω, but does not require that f have L^∞-stable translates.[6]

THEOREM 3.4. *Let f be a continuous, compactly supported solution to the refinement equation* (1). *Assume that there exists $u_0 \in C^{1\times r}$ such that $u_0\hat{f}(0) \neq 0$ and $u_0 = \sum_{k\in\mathbb{Z}^n_d} u_0 c_k$ for $d \in D$. Let Ω be any $(\Lambda - D)$-admissible subset of $\mathbb{Z}^n$ such that $K_\Lambda \subset Q + \Omega$. If the functions $\varphi^{(i)}$ defined by equations* (12) *and* (13) *converge pointwise everywhere to $f(x)u_0$, then $\hat{\rho}(\{T_d|_{V_0}\}_{d\in D}) < 1$.*

Omitting details, we will sketch the proof of this theorem. We will first prove that if $\{\xi_i\}_{i=1}^\infty$ is any sequence of digits $\xi_i \in D$, then the matrix product $T_{\xi_1}\cdots T_{\xi_i}$ converges as $i \to \infty$ to the rank-one matrix each of whose columns is $\Phi(f(x)u_0)$, where x is the point

$$x \;=\; .\xi_1\xi_2\cdots \;=\; \sum_{j=1}^{\infty} A^{-j}\xi_j \;\in\; Q.$$

From this fact we will then deduce that $\hat{\rho}(\{T_d|_{V_0}\}_{d\in D}) < 1$.

To begin, let a sequence of digits $\{\xi_i\}_{i=1}^\infty$ be fixed, and set $x = .\xi_1\xi_2\cdots \in Q$. By hypothesis, $\varphi^{(i)}(x) \to f(x)u_0$ when $\varphi^{(0)}(x) = \chi_{\tilde{Q}-\gamma_0}(x)\cdot I_r$. Given $\varepsilon > 0$ we fix a "large enough" $(\Lambda - D)$-admissible set $\tilde{\Omega}$ containing Ω; large enough means that $(Q+\Omega)_\varepsilon \subset (Q+\tilde{\Omega})^\circ$.

Let σ_h denote the translation operator, i.e., $\sigma_h g(x) = g(x-h)$. For each $h \in \tilde{\Omega}$, set

$$\varphi_h^{(0)}(x) \;=\; \big(\sigma_{h+\gamma_0}\varphi^{(0)}\big)(x) \;=\; \chi_{\tilde{Q}+h}(x)\cdot I_r,$$

and define

$$\varphi_h^{(i)}(x) \;=\; S^i\varphi_h^{(0)}(x) \;=\; S^i\big(\sigma_{h+\gamma_0}\varphi^{(0)}\big)(x) \;=\; \sigma_{A^{-i}(h+\gamma_0)}(S^i\varphi^{(0)})(x). \tag{14}$$

Note that $\mathrm{supp}(\varphi_h^{(0)}) \subset Q + h \subset Q + \tilde{\Omega}$ for each $h \in \tilde{\Omega}$.

Now fix any particular $h \in \Omega$ and consider the points

$$y_i \;=\; .\xi_{i+1}\xi_{i+2}\cdots \;\in\; Q.$$

Recall that $\tilde{Q}$ was defined to have the property that the $\mathbb{Z}^n$-translates of $\tilde{Q}$ cover $\mathbb{R}^n$ without overlaps. Therefore, the point $y_i + h$ must lie in some unique translate of $\tilde{Q}$. Hence, there exist unique points $q_i \in \tilde{Q}$ and $k_i \in \mathbb{Z}^n$ such that $y_i + h = q_i + k_i$, and by Lemma 4.11 in Cabrelli et al.,[6] we have $k_i \in \tilde{\Omega}$. Hence, if we let $\delta_{h,j}$ denote the Kronecker delta, then then folding of $\varphi_{k_i}^{(0)}$ satisfies

$$\tilde{\Phi}\varphi_{k_i}^{(0)}(y_i) \;=\; \big[\;\delta_{h,j}\cdot I_r\;\big]_{j\in\tilde{\Omega}}.$$

Fix any ordering on $\tilde{\Omega}$ such that the elements of Ω come first. Then since $\Omega \subset \tilde{\Omega}$ and since $T_d = [c_{Aj-k+d}]_{j,k\in\Omega}$, the matrix $\tilde{T}_d = [c_{Aj-k+d}]_{j,k\in\tilde{\Omega}}$ has the block form

$$\tilde{T}_d \;=\; \begin{bmatrix} T_d & B_d \\ 0 & C_d \end{bmatrix} \tag{15}$$

for some matrices B_d, C_d. The fact that the lower-left block of $\tilde{T}_d$ is zero is a consequence of the fact that $c_k = 0$ when $k \notin \Lambda$. Define

$$\Delta_h \;=\; \big[\;\delta_{h,j}\cdot I_r\;\big]_{j\in\Omega} \qquad\text{and}\qquad \tilde{\Delta}_h \;=\; \big[\;\delta_{h,j}\cdot I_r\;\big]_{j\in\tilde{\Omega}} \;=\; \begin{bmatrix} \Delta_h \\ 0 \end{bmatrix}.$$

Δ_h and $\tilde{\Delta}_h$ are generalized column vectors with the identity block I_r appearing in "row block h" and zeros elsewhere. Multiplication of a matrix by Δ_h or $\tilde{\Delta}_h$ on the right therefore selects out "column block h" from that matrix. Thus, by Proposition 2.3 we have for $i \geq i_0$ that

$$\tilde{\Phi}\varphi_{k_i}^{(i)}(x) \;=\; \tilde{\Phi}S^i\varphi_{k_i}^{(0)}(x) \;=\; \tilde{T}_{\xi_1}\cdots\tilde{T}_{\xi_i}\tilde{\Phi}\varphi_{k_i}^{(0)}(y_i) \;=\; \tilde{T}_{\xi_1}\cdots\tilde{T}_{\xi_i}\tilde{\Delta}_h, \tag{16}$$

which is "column block h" of $\tilde{T}_{\xi_1} \cdots \tilde{T}_{\xi_i}$. On the other hand, we have by hypothesis that

$$S^i \varphi^{(0)}(x) \;=\; \varphi^{(i)}(x) \;\to\; f(x)u_0. \tag{17}$$

Therefore,

$$\begin{aligned} \tilde{T}_{\xi_1} \cdots \tilde{T}_{\xi_i} \tilde{\Delta}_h \;&=\; \tilde{\Phi}\varphi_{k_i}^{(i)}(x) && \text{by (16)} \\ &=\; \tilde{\Phi}(\sigma_{A^{-i}(k_i+\gamma_0)} S^i \varphi^{(0)})(x) && \text{by (14)} \\ &\to\; \tilde{\Phi}(f(x)u_0), \end{aligned} \tag{18}$$

the conclusion on the preceding line following from equation (17), the contractivity of A^{-1}, and the fact that each k_i lies in the finite set $\tilde{\Omega}$. Thus, "column block h" of $\tilde{T}_{\xi_1} \cdots \tilde{T}_{\xi_i}$ converges to $\tilde{\Phi}(f(x)u_0)$. This is true for each $h \in \Omega$, whereas the column blocks of $\tilde{T}_{\xi_1} \cdots \tilde{T}_{\xi_i}$ are indexed by the larger set $\tilde{\Omega}$. Therefore let us examine the column blocks corresponding to indices $h \in \Omega$ in more detail. Since $\tilde{T}_d$ has the block form given by equation (15), we have

$$\tilde{T}_{\xi_1} \cdots \tilde{T}_{\xi_i} \tilde{\Delta}_h \;=\; \begin{bmatrix} T_{\xi_1} \cdots T_{\xi_i} & * \\ 0 & * \end{bmatrix} \begin{bmatrix} \Delta_h \\ 0 \end{bmatrix} \;=\; \begin{bmatrix} T_{\xi_1} \cdots T_{\xi_i} \Delta_h \\ 0 \end{bmatrix}. \tag{19}$$

Further,

$$\tilde{\Phi}(f(x)u_0) \;=\; \begin{bmatrix} \Phi(f(x)u_0) \\ * \end{bmatrix}, \tag{20}$$

so we conclude by combining equations (18)–(20) that

$$T_{\xi_1} \cdots T_{\xi_i} \Delta_h \;\to\; \Phi(f(x)u_0). \tag{21}$$

Since the columns blocks of $T_{\xi_1} \cdots T_{\xi_i}$ are indexed by Ω, equation (21) implies that each column block of $T_{\xi_1} \cdots T_{\xi_i}$ converges to $\Phi(f(x)u_0)$. Therefore, the product $T_{\xi_1} \cdots T_{\xi_i}$ converges to to the matrix $B(x)$ consisting of Ω column blocks each equal to $\Phi(f(x)u_0)$. That is,

$$T_{\xi_1} \cdots T_{\xi_i} \;\to\; B(x) \;=\; \big(\Phi(f(x)u_0)\big)_{k \in \Omega}.$$

This matrix $B(x)$ is rank-one because each column block $\Phi(f(x)u_0)$ consists of rows that are multiples of the $1 \times r$ row vector u_0.

Thus, we have demonstrated that $T_{\xi_1} \cdots T_{\xi_i}$ converges to a rank-one matrix for each sequence of digits $\{\xi_i\}_{i=1}^{\infty}$. Why is this enough to prove Theorem 3.4? The answer is that the coefficients c_k satisfy the conditions for minimal accuracy. Because of this, it can be seen that there exists an orthonormal basis $\mathcal{B}$ for $(\mathbb{C}^{r\times 1})^{\Omega \times 1}$ such that each matrix T_d has in this basis the block form

$$[T_d]_{\mathcal{B}} \;=\; \begin{bmatrix} 1 & 0 \\ * & C_d \end{bmatrix},$$

where 1 is the *scalar* 1, and $C_d = [T_d|_{V_0}]_{\mathcal{B}_0}$ is the matrix for T_d restricted to V_0 with respect to an orthonormal basis $\mathcal{B}_0$ for V_0. Consequently, working in this basis, we have for each i that

$$[T_{\xi_1} \cdots T_{\xi_i}]_{\mathcal{B}} \;=\; \begin{bmatrix} 1 & 0 \\ * & C_{\xi_1} \cdots C_{\xi_i} \end{bmatrix}.$$

Since $T_{\xi_1} \cdots T_{\xi_i}$ converges to a rank-one matrix, the product $C_{\xi_1} \cdots C_{\xi_i}$ must therefore converge to the zero matrix. This implies by Theorem I of Berger and Wang[23] that $\hat{\rho}(\{C_d\}_{d\in D}) < 1$, and completes the proof.

Finally, the proof of Theorem 3.1 follows by combining Theorems 3.2 and 3.4.

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