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Matrix Refinement Equations: Existence and Uniqueness

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ABSTRACT. Matrix refinement equations are functional equations of the form $f(x) = \sum_{k=0}^{N} c_k f(2x-k)$, where the coefficients c_k are matrices and f is a vector-valued function. Refinement equations play key roles in wavelet theory and approximation theory. Existence and uniqueness properties of scalar refinement equations (where the coefficients c_k are scalars) are known. This paper considers analogous questions for matrix refinement equations. Conditions for existence and uniqueness of compactly supported distributional solutions are given in terms of the convergence properties of an infinite product of the matrix $\Delta = \frac{1}{2}\sum c_k$ with itself. Fundamental differences between solutions of matrix equations and scalar refinement equations are examined. In particular, it is shown that "constrained" solutions of the matrix refinement equation can exist even when the infinite product diverges. The existence of constrained solutions is related to the eigenvalue structure of Δ; solutions are obtained from the convergence of Δ on its 1-eigenspace.

1. Introduction

Functional equations of the form

$$f(x) \;=\; \sum_{k=0}^{N} c_k\, f(2x-k), \tag{1}$$

where the coefficients $c_0, \dots, c_N$ are real or complex numbers, play a key role in wavelet theory and in subdivision schemes in approximation theory. Such equations are known variously as *refinement equations*, *dilation equations*, or *two-scale difference equations*. A solution f is a *scaling function* or a *refinable function*. Two excellent references are Daubechies [D1] for the connection to wavelets, and Cavaretta, Dahmen, and Micchelli [CDM] for the connection to subdivision.

A now-classic application of refinement equations is the construction of compactly supported orthonormal wavelet bases. Suppose f is an integrable scaling function that has the additional property that its integer translates $\{f(x-k)\}_{k\in\mathbf{Z}}$ form an orthonormal system in $L^2(\mathbf{R})$. Then f gives rise to a wavelet g via the recipe

$$g(x) \;=\; \sum_{k=0}^{N} (-1)^k\, c_{N-k}\, f(2x-k), \tag{2}$$

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and this wavelet has the property that the collection $\{2^{n/2}g(2^nx-k)\}_{n,k\in\mathbf{Z}}$ forms an orthonormal basis for $L^2(\mathbf{R})$. The first orthonormal wavelet basis was constructed by Haar [Ha] using different methods. The Haar wavelet is $g = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, corresponding in the refinement equation setting to $N = 1$, $c_0 = c_1 = 1$, and $f = \chi_{[0,1)}$. Daubechies constructed the first continuous, compactly supported orthonormal wavelets [D2]. For each odd N, she found $c_0, \dots, c_N$ so that the corresponding scaling function $f = D_N$ is orthogonal to its integer translates, and, hence, determines an orthonormal wavelet $g = W_N$. These scaling functions D_N increase linearly in smoothness with N (measured by the number of continuous derivatives and Hölder exponent of continuity of the last derivative). In addition, the *accuracy*, or *order of approximation*, increases with N; each of the polynomials $1, x, \dots, x^{(N-1)/2}$ can be written exactly as an infinite linear combination of the integer translates of D_N.

There are now many choices of wavelet bases available, each possessing various combinations of desirable properties such as orthogonality, compact support, smoothness, symmetry, or high accuracy. However, some of these properties are mutually exclusive. For example, no compactly supported orthogonal wavelet, other than the Haar wavelet, can be symmetric or antisymmetric [D1]. In addition, the accuracy and smoothness of the scaling function is tied to the number of coefficients in the refinement equation [DL]. Thus high accuracy or smoothness implies a large support for the scaling function, for if c_0, $c_N \neq 0$ then $[0, N]$ is the smallest connected interval containing $\mathrm{supp}(f)$. This reduces the locality of the wavelet representation, increases the computational complexity of the wavelet transform, and increases the difficulty in transforming the wavelet basis from a basis on the line to a basis for an interval.

Recently, attention has focused on "multiwavelet" bases. One advantage of these bases is that they do allow the simultaneous inclusion of desirable properties. The constructions of Goodman and Lee [GL] and Donovan, Geronimo, Hardin, Kessler, and Massopust [HKM, GHM, DGHM] were among the first in this direction. The simple idea is to allow multiple wavelets $g_1, \dots, g_r$, i.e., to construct a basis of the form $\{2^{n/2}g_i(2^nx-k)\}_{n,k\in\mathbf{Z},\, i=1,\dots,r}$. The price is additional theoretical and computational complexity. The advantage is the simultaneous inclusion of desirable properties.

Multiwavelets can be formulated in terms of a matrix version of the refinement equation. The scalar coefficients $c_0, \dots, c_N$ are replaced by $r \times r$ matrices. The scaling function f becomes a vector-valued function, i.e., $f = (f_1, \dots, f_r)^{\mathrm{t}}$. We refer to (1) with matrix c_k as a *matrix refinement equation* (MRE). It is shown in [DGHM] and [SS1] that if $\{f_i(x-k)\}_{k\in\mathbf{Z},\, i=1,\dots,r}$ is an orthonormal system, then a wavelet $g = (g_1, \dots, g_r)^{\mathrm{t}}$ can be obtained via an analogue of (2); i.e., there exist matrices d_k so that the collection $\{2^{n/2}g_i(2^nx-k)\}_{n,k\in\mathbf{Z},\, i=1,\dots,r}$ is an orthonormal basis for $L^2(\mathbf{R})$ if $g(x) = \sum_{k=0}^{N} d_k\, f(2x-k)$. However, the noncommutativity of the matrices c_k means that $d_k = (-1)^k\, c_{N-k}$ is no longer correct. The theory of paraunitary matrices reveals how to choose the d_k [SS1].

Example 1.1. This MRE is due to Geronimo, Hardin, and Massopust [GHM]. Let c_0, c_1, c_2, c_3 be the four 2×2 matrices such that

$$\frac{1}{2}\sum_{k=0}^{3} c_k\, z^k \;=\; \frac{1}{20}\begin{pmatrix} 6+6z & 8\sqrt{2} \\ (-1+9z+9z^2-z^3)/\sqrt{2} & -3+10z-3z^2 \end{pmatrix}.$$

The corresponding MRE is solved by $f = (f_1, f_2)^{\mathrm{t}}$, with f_1 and f_2 the symmetric, Lipschitz functions shown in Figure 1.1. The collection $\{f_i(x-k)\}_{k\in\mathbf{Z},\, i=1,2}$ is orthonormal, so there is a corresponding orthonormal wavelet $g = (g_1, g_2)^{\mathrm{t}}$. Noteworthy properties of these functions are symmetry, short support, accuracy $p = 1$ (both constant and linear functions can be obtained exactly as linear combinations of integer translates of f_1 and f_2), and easy conversion from a wavelet basis for $L^2(\mathbf{R})$ to a wavelet basis for $L^2[0, 1]$. $\square$

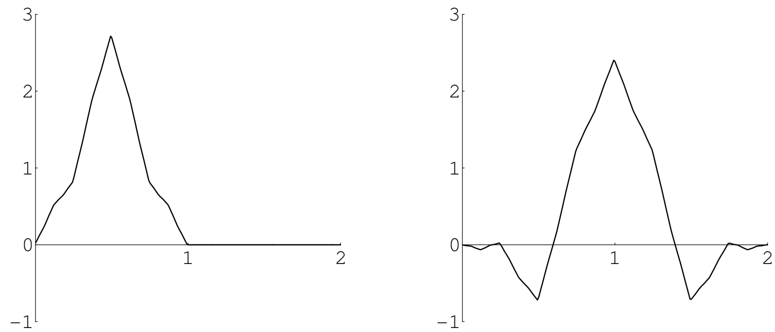

FIGURE 1.1. Scaling functions f_1 and f_2 from Example 1.1.

The theory of scalar refinement equations is considerably developed. Much work, often concentrating on continuous scaling functions with independent translates, is available in the approximation theory literature [CDM]. Many basic existence and uniqueness results were laid out by Daubechies and Lagarias [DL], along with major results on accuracy and local and global smoothness. The corresponding theory of MREs is still being developed. Some aspects of the scalar theory carry over easily. Others do not. For example, while determining the accuracy of a scalar refinement equation is relatively simple, the corresponding calculation for a MRE is surprisingly intricate yet still constructive and computable [HSS, P]. Smoothness and accuracy are distinct properties, although there are ties between them. Global smoothness for solutions of matrix refinement equations is explored in [CHM].

Fundamental existence and uniqueness theorems for integrable solutions of scalar refinement equations were proved by Daubechies and Lagarias [DL]. As remarked there, the results extend to compactly supported distributional solutions. We find this latter setting of *scaling distributions* a convenient framework in which to work. The key fact is that the Fourier transform converts the refinement equation (1) to the equivalent form

$$\hat{f}(2\gamma) \;=\; M(\gamma)\,\hat{f}(\gamma), \tag{3}$$

where M is the trigonometric polynomial

$$M(\gamma) \;=\; \frac{1}{2}\sum_{k=0}^{N} c_k\, e^{-2\pi i k\gamma}. \tag{4}$$

Iterating (3), we obtain

$$\hat{f}(\gamma) \;=\; \left(\prod_{j=1}^{n} M(2^{-j}\gamma)\right)\hat{f}(2^{-n}\gamma) \;=\; P_n(\gamma)\,\hat{f}(2^{-n}\gamma) \qquad \text{for all } n \in \mathbf{Z}. \tag{5}$$

An infinite product is implicated, and the value of

$$\Delta \;=\; M(0) \;=\; \frac{1}{2}\sum_{k=0}^{N} c_k \tag{6}$$

is critical. The fundamental results for scalar refinement equations can be outlined as follows [DL].

a. There exists a compactly supported scaling distribution if and only if $\Delta = 2^n$ for some integer $n \geq 0$. In this case, f is supported in the interval $[0, N]$. If f is an integrable function, it can have at most $N - 2$ continuous derivatives.

b. If $\Delta = 1$, then f is specified by $\hat{f}(\gamma) = \big(\prod_{j=1}^{\infty} M(2^{-j}\gamma)\big)\hat{f}(0)$, the infinite product converging uniformly on compact sets. In particular, $\hat{f}(0) \neq 0$ and f is determined uniquely up to scale.

c. If $\Delta = 2^n$ with $n > 0$, then $\hat{f}(0) = 0$ and f is the nth distributional derivative of a scaling distribution F for the refinement equation determined by the coefficients $2^{-n}c_0, \dots, 2^{-n}c_N$.

The contribution of this paper is to develop the existence and uniqueness theory for matrix refinement equations, focusing on compactly supported distributional solutions. Our notation and some basic observations are laid out in §2. Precise statements of our results follow in §3. Proofs of these results occupy §§4–7, and §8 contains some final observations on noncompactly supported solutions to MREs. The remainder of this section outlines the ideas and motivations behind our results.

We begin by adapting the facts in (3)–(6) to the matrix setting. If we define the Fourier transform componentwise, let M be the matrix-valued polynomial defined by (4), and let Δ be the matrix defined by (6); then we retain (3) as the Fourier version of the MRE. As each component of M is a bounded, infinitely differentiable function, (3) is fully equivalent to (1) in the sense of distributions. Unlike the scalar case, order in (5) is now important (products must be expanded on the right). To a certain extent the analysis for MREs now parallels that of the scalar case, but with important differences, which we now list.

The scalar condition $\Delta = 2^n$ for existence of solutions is replaced by the condition that 2^n be an eigenvalue for Δ. Solutions for $n > 0$ satisfy $\hat{f}(0) = 0$ and are distributional derivatives of solutions to the MRE determined by $2^{-n}c_0, \dots, 2^{-n}c_N$. (These results on derivatives are proved in §4.) Thus, the *fundamental* solutions occur when $n = 0$, i.e., when 1 is an eigenvalue of Δ. Fundamental solutions satisfy $\hat{f}(0) \neq 0$, and this vector $\hat{f}(0)$ is a 1-eigenvector for Δ. However, it is the remaining eigenvalues of Δ that determine the "stability" of this fundamental solution f. In the scalar case there are no other eigenvalues that can affect the solution—all solutions are "unconstrained."

This need not be the case for MREs. We will say that "unconstrained" solutions occur when the infinite matrix product

$$P(\gamma) \;=\; \prod_{j=1}^{\infty} M(2^{-j}\gamma) \;=\; \lim_{n\to\infty} P_n(\gamma) \tag{7}$$

converges for every γ. Considering (5), it is clear that in this case any distributional solution whose Fourier transform is a continuous function must have the form

$$\hat{f}(\gamma) \;=\; P(\gamma)\, v \tag{8}$$

for some vector v. We discuss the details of unconstrained convergence in §5. We show that unconstrained solutions occur exactly when 1 is a nondegenerate eigenvalue for Δ and all other eigenvalues are less than 1 in absolute value, i.e., $\Delta^\infty = \lim_{n\to\infty} \Delta^n$ exists and is nontrivial. In this case, the infinite product in (7) converges uniformly on compact sets, and (8) defines a compactly supported scaling distribution for each choice of v. All distributional solutions whose Fourier transforms are continuous functions are produced in this way, and the number of independent solutions is exactly the multiplicity of the eigenvalue 1 for Δ. Unlike scalar refinement equations, there can be several independent compactly supported solutions to the MRE.

We enter the "constrained," or "unstable," arena when 1 is not the largest eigenvalue of Δ. There is no analogue for scalar refinement equations. Now Δ^∞ does not exist, and the infinite product in (7) does not converge. We show in §6 that if the spectral radius $\rho(\Delta)$ is less than 2 with a single nondegenerate largest eigenvalue, and if 1 is an eigenvalue of Δ, then "constrained" compactly supported fundamental solutions exist. They are defined via a "constrained" convergence of the infinite product as

$$\hat{f}(\gamma) \;=\; \lim_{n\to\infty}\left(\left(\prod_{j=1}^{n} M(2^{-j}\gamma)\right)v\right) \;=\; \lim_{n\to\infty} P_n(\gamma)v, \tag{9}$$

the limit converging uniformly on compact sets when v is a 1-eigenvector of Δ. This algorithm is unstable; that is, the infinite matrix product alone does not converge, and the choice of v is limited.

In §7, we present numerical evidence that if $\rho(\Delta)$ is large enough, then even the unstable algorithm given by (9) may not converge. This evidence is supported in principle by consideration of the multiplicative ergodic theorem of Oseledec [ER]. We therefore conjecture that some MREs have "superconstrained" fundamental solutions that are not obtainable via the infinite product approach. That is, although there exists a solution f and we know that $P_n(\gamma)\hat{f}(2^{-n}\gamma)$ converges to $\hat{f}(\gamma)$ and that $\hat{f}(2^{-n}\gamma)$ converges to $\hat{f}(0)$, we cannot obtain $\hat{f}$ by replacing the unknown and varying $\hat{f}(2^{-n}\gamma)$ by the known and fixed $v = \hat{f}(0)$. Thus the infinite product approach does not provide a constructive means of obtaining $\hat{f}$ in this case.

For scalar refinement equations, there are no constrained or superconstrained solutions, for if $\Delta \neq 1$, then any solution must satisfy $\hat{f}(0) = 0$ and must be the nth derivative of a fundamental solution for a modified refinement equation. Moreover, because compactly supported solutions to scalar refinement equations are unique up to scale, distributional differentiation of scalar scaling functions must coincide with pointwise almost-everywhere differentiation when the latter exists and is nontrivial. These facts need not hold for MREs. The distributional derivative Df and pointwise derivative f' of a fundamental solution are both solutions of the MRE determined by the matrices $2c_0, \dots, 2c_N$, but in the matrix case they can be distinct (Example 3.6). Distributional derivatives cannot be fundamental; they must satisfy $(Df)^\wedge(0) = 0$. The pointwise almost-everywhere derivative can be fundamental; it may happen that $(f')^\wedge(0) \neq 0$. Constrained or superconstrained fundamental solutions may or may not be pointwise derivatives of other solutions.

We note here that shortly before completion of this paper, we received a preprint [MRV] by Massopust, Ruch, and Van Fleet, that also includes consideration of what we have called constrained convergence. However, our results are distinct and our focus is different. Among many interesting results, including ones relating MREs to multiresolution analysis, [MRV] contains conditions on the matrices c_k which result in "short support" for the scaling function.

Note Added in Proof. Following submission of this paper, we learned that some of our results on what we term unconstrained convergence appear also in Hervé [He]. However, Hervé considers only diagonalizable Δ and does not consider any forms of constrained convergence. His paper also contains many interesting results on multiresolution analysis for multiwavelets. In addition, we have just received the paper [CDP] by Cohen, Daubechies, and Plonka, which, in addition to presenting many interesting results analyzing the existence and regularity of multiwavelets, extends and applies some of our results on constrained convergence.

2. Notation and Observations

The spectral radius of a matrix A is denoted by $\rho(A)$. Matrix products are always expanded on the right, i.e., $\prod A_n = A_1 A_2 \cdots$. The function space $L^p(\mathbf{R})$ consists of all complex-valued functions f on the real line $\mathbf{R}$ satisfying $\int |f(x)|^p \, dx < \infty$. $\mathcal{S}(\mathbf{R})$ is the Schwartz class of all infinitely differentiable functions on $\mathbf{R}$ with rapid decay at infinity. $\mathcal{S}'(\mathbf{R})$ is its topological dual, the class of tempered distributions on $\mathbf{R}$. The Fourier transform of $f \in L^1(\mathbf{R})$ is $\hat{f}(\gamma) = \int f(x)\, e^{-2\pi i\gamma x}\, dx$. This maps $\mathcal{S}(\mathbf{R})$ onto itself and extends to $\mathcal{S}'(\mathbf{R})$ by duality.

Vector-valued function and distribution spaces are Cartesian products, e.g., $L^p(\mathbf{R})^r = \{f = (f_1, \dots, f_r)^{\mathrm{t}} : \text{each } f_i \in L^p(\mathbf{R})\}$. The Fourier transform of $f \in L^1(\mathbf{R})^r$ is defined componentwise as $\hat{f} = (\hat{f}_1, \dots, \hat{f}_r)^{\rm t}$. The support of $f \in \mathcal{S}'(\mathbf{R})^r$ is $\mathrm{supp}(f) = \bigcup_{i=1}^r \mathrm{supp}(f_i)$. The distributional derivative of $f = (f_1, \dots, f_r)^{\rm t} \in \mathcal{S}'(\mathbf{R})^r$ is $Df = (Df_1, \dots, Df_r)^{\rm t}$. If f is an integrable function, then its pointwise almost everywhere derivative, if it exists, is $f' = (f_1', \dots, f_r')^{\rm t}$.

Distributional solutions to the MRE are defined by duality; $f \in \mathcal{S}'(\mathbf{R})^r$ is a solution to equation (1) if

$$\langle f, \varphi \rangle \;=\; \sum_{k=0}^{N} c_k \, \langle f(2x-k), \varphi \rangle \;=\; \frac{1}{2} \sum_{k=0}^{N} c_k \left\langle f, \, \varphi\left(\tfrac{x+k}{2}\right) \right\rangle \quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R})^r.$$

Such a distributional solution f is termed a *scaling distribution*. If f is realizable as a function, then it is a *scaling function*. Any finite linear combination of scaling distributions is again a scaling distribution for the same MRE. If a scaling distribution f is compactly supported then it must be supported in $[0, N]$ since the MRE implies that $\mathrm{supp}(f) \subset \bigcup_{k=0}^N \mathrm{supp}(f(2x-k))$.

Any basis may be chosen in which to write the matrices $c_0, \dots, c_N$ and scaling distribution f, for if B is an invertible matrix, then f is a solution of the MRE determined by $c_0, \dots, c_N$ if and only if $B^{-1}f$ is a solution of the MRE determined by $B^{-1}c_0B, \dots, B^{-1}c_NB$.

One trivial way of obtaining MREs from scalar refinement equations is to let each c_k be diagonal. In this case the MRE is equivalent to r independent scalar refinement equations. More generally, a simplification occurs whenever $c_0, \dots, c_N$ share a common invariant subspace. In this case, $c_0, \dots, c_N$ can be simultaneously block triangularized by an appropriate change of basis. In particular, suppose $c_i = \begin{pmatrix} a_i & 0 \\ * & b_i \end{pmatrix}$, where each a_i is $s \times s$ and b_i is $(r-s) \times (r-s)$. Then, if $f = (f_1, \dots, f_r)^{\rm t}$ is a solution of the MRE determined by $c_0, \dots, c_N$, then $(f_1, \dots, f_s)^{\rm t}$ is a solution of the MRE determined by $a_0, \dots, a_N$. Moreover, $(g_1, \dots, g_{r-s})^{\rm t}$ is a solution of the MRE determined by $b_0, \dots, b_N$ if and only if $(0, \dots, 0, g_1, \dots, g_{r-s})^{\rm t}$ is a solution of the MRE determined by $c_0, \dots, c_N$. This latter technique need not capture all nontrivial solutions of the MRE determined by $c_0, \dots, c_N$. An example of this is given in Example 3.6a. On the other hand, the solution f' in Example 3.6b can be found in this way. Note that the matrices of Examples 1.1 and 3.5 share no common invariant subspaces.

3. Statement of Results

In this section we give a precise statement of our results, followed by examples illustrating the constrained and unconstrained cases. The proofs of these results follow in later sections.

First, we state some relationships between derivatives and the value of $\hat{f}(0)$. Derivatives, for example, are computed componentwise.

Theorem 3.1.

Assume f is a compactly supported scaling distribution. Then:

- **a.** $\mathrm{supp}(f) \subset [0, N]$.
- **b.** *If $\hat{f}(0) \neq 0$, then $\hat{f}(0)$ is a 1-eigenvector for Δ.*
- **c.** *If $\hat{f}(0) = 0$, then there is a compactly supported distributional solution F to the MRE determined by $\frac{1}{2}c_0, \dots, \frac{1}{2}c_N$ such that $DF = f$.*
- **d.** *The distributional derivative Df of f is a compactly supported solution to the MRE determined by the matrices $2c_0, \dots, 2c_N$ and satisfies $(Df)^\wedge(0) = 0$.*
- **e.** *If f is an integrable function which is differentiable pointwise almost everywhere then its derivative f' is a compactly supported solution to the MRE determined by the matrices $2c_0, \dots, 2c_N$. It need not be the case that $(f')^\wedge(0) = 0$.*
- **f.** *If f is an integrable function, it can have at most $r(N-1)-1$ continuous derivatives. In particular, if $f \in C^k(\mathbf{R})$, then $(f^{(k)}(1), \dots, f^{(k)}(N-1))^{\rm t}$ must be a 2^{-k}-eigenvector of the*

$r(N-1) \times r(N-1)$ matrix

$$A = \begin{pmatrix} c_1 & c_0 & 0 & \cdots & 0 & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_N & c_{N-1} \end{pmatrix}. \tag{10}$$

Theorem 3.1f provides the key to a fast recursive algorithm for plotting a continuous scaling function f; that is, the values of f at the integers are an eigenvector of A. Once these values are known, the refinement equation immediately specifies the values at the half-integers, then the quarter-integers, and so on.

The following result shows that, as in the scalar case, all compactly supported scaling functions f are nth derivatives of fundamental solutions for a modified MRE.

Theorem 3.2.

Assume f is a compactly supported scaling distribution. If $\hat{f}(0) = 0$ then 2^n is an eigenvalue of Δ with $n > 0$. Moreover, f is the nth distributional derivative of a compactly supported scaling distribution F for the MRE determined by the matrices $2^{-n}c_0, \ldots, 2^{-n}c_N$, and $\hat{F}(0) \neq 0$.

The fundamental solutions, i.e., those satisfying $\hat{f}(0) \neq 0$, are therefore of the greatest interest. The following theorem characterizes fundamental unconstrained solutions of MREs. For scalar refinement equations, these occur when the number $\Delta = M(0)$ is 1. Theorem 3.3 states that the analogous condition for MREs is that the infinite matrix product Δ^∞ exist and be nontrivial. This occurs exactly when we can make a similarity transformation so that $\Delta \sim \begin{pmatrix} I_s & 0 \\ 0 & J \end{pmatrix}$, where I_s is an $s \times s$ identity matrix with $1 \leq s \leq r$ and J has spectral radius $\rho(J) < 1$. In this case $\Delta^\infty \sim \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$.

Theorem 3.3.

The infinite product defining $P(\gamma)$ in (7) converges (uniformly on compact sets) if and only if Δ^∞ exists and is nontrivial. In this case, the mapping $v \mapsto f$ induced by (8) is a linear map of $\mathbf{C}^r$ onto the set of all compactly supported distributional solutions to the MRE. Its kernel is the kernel of Δ^∞. It is a bijection when restricted to the 1-eigenspace of Δ. In particular, each nontrivial compactly supported solution satisfies $\hat{f}(0) \neq 0$, and there exist exactly s independent solutions of the MRE, where s is the multiplicity of the eigenvalue 1 for Δ.

If the eigenvalues of Δ are not too large, then we can obtain fundamental constrained solutions via (9).

Theorem 3.4.

Assume that Δ^∞ does not exist but that there is a single eigenvalue λ for Δ such that $|\lambda| = \rho(\Delta) < 2$, with this eigenvalue nondegenerate. Then the limit in (9) converges uniformly on compact sets for each 1-eigenvector v of Δ. The mapping $v \mapsto f$ induced by (9) is a linear bijection of the 1-eigenspace of Δ onto the set of all compactly supported distributional solutions to the MRE. In particular, each nontrivial compactly supported solution satisfies $\hat{f}(0) \neq 0$, and there exist exactly s independent solutions of the MRE, where s is the multiplicity of the eigenvalue 1 for Δ. Finally, the limit in (9) diverges if v is a λ-eigenvector of Δ.

We note that [MRV] contains an example of a constrained solution to a MRE that does not satisfy the hypotheses of Theorem 3.4; the 2×2 matrix Δ in their example has eigenvalues 1 and -1.

We now present some examples illustrating Theorems 3.1 to 3.4.

Example 3.5. Let c_0, c_1, c_2, c_3 be as in Example 1.1. Set $M(\gamma) = \sum_{k=0}^{3} c_k\, e^{-2\pi i k\gamma}$ and $\Delta = M(0)$ throughout this example.

a. The matrix Δ has eigenvalues 1 and $-1/5$. Therefore, up to scale, there is a unique fundamental unconstrained compactly supported distributional solution f. The infinite product in (7) converges uniformly on compact sets, and (8) defines the Fourier transform of f. In fact, this f is the integrable solution discussed in Example 1.1 and shown in Figure 1.1.

b. Now consider the MRE determined by the matrices $2c_0$, $2c_1$, $2c_2$, $2c_3$. The distributional derivative Df of the solution f discussed in part (a) must be a solution of this new MRE, and must satisfy $(Df)^\wedge(0) = 0$. Note that 2Δ has eigenvalues 2 and $-2/5$. Since this f is Lipschitz, its pointwise almost-everywhere derivative coincides with its distributional derivative.

c. Finally, consider the MRE determined by the matrices $-5c_0$, $-5c_1$, $-5c_2$, $-5c_3$. The matrix -5Δ has eigenvalues 1 and -5. A fundamental solution to this MRE does exist, namely, $g = (\sqrt{2}\,\chi_{[0,1)}, -\chi_{[0,2)})^{\mathrm{t}}$. We have $\hat{g}(0) \neq 0$, and g is not merely a derivative of another solution since -5 is not an integer power of 2. Yet we conjecture that the limit in equation (9) does not converge for any nonzero v, even when v is a 1-eigenvector of Δ. We present numerical evidence for this claim, and theoretical considerations supporting it, in §7. □

Example 3.6. a. This MRE is due to Strang and Strela [SS2]. Let

$$c_0 = \begin{pmatrix} 1 & 0 \\ \sqrt{3}/2 & 0 \end{pmatrix}, \qquad c_1 = \begin{pmatrix} 1 & 0 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}, \qquad c_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

The eigenvalues of Δ are 1 and 1/2. The corresponding MRE has the unconstrained fundamental solution $f = (f_1, f_2)^{\mathrm{t}}$, with f_1 and f_2 the piecewise linear functions shown in Figure 3.1. The explicit formula is $f_1 = \chi_{[0,1)}$ and

$$f_2(x) = \frac{\sqrt{3}}{2} \begin{cases} -2 + 3\cdot 2^{-n}, & x \in (1-2^{-n}, 1-2^{-n-1}),\ n \geq 0, \\ 8 - 3\cdot 2^{-n} - 4x, & x \in (2-2^{-n}, 2-2^{-n-1}),\ n \geq 0. \end{cases}$$

It is proved in [SS2] that $\{f_i(x-k)\}_{k\in\mathbf{Z},\, i=1,2}$ is orthonormal, so there exists a corresponding orthonormal wavelet $g = (g_1, g_2)^{\mathrm{t}}$.

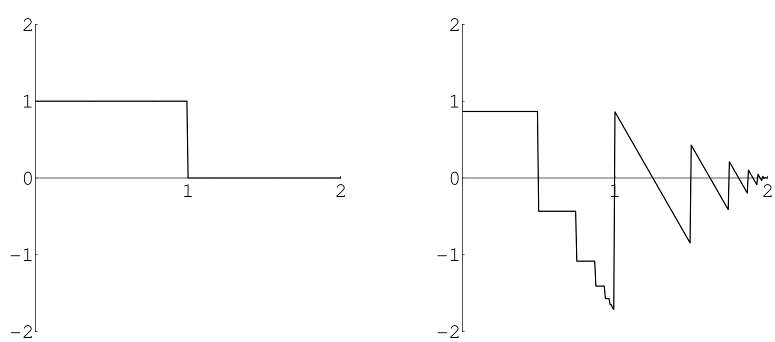

FIGURE 3.1. Scaling functions f_1 and f_2 from Example 3.6.

b. Now consider the MRE determined by the matrices $2c_0$, $2c_1$, $2c_2$. With f as in part a, the distributional derivative Df is necessarily a compactly supported solution to this MRE and satisfies $(Df)^\wedge(0) = 0$. This distributional derivative contains point masses in both components. However, there exists another, distinct, compactly supported solution, namely, the pointwise almost-everywhere derivative $f' = (0, -4\chi_{[1,2)})^{\mathrm{t}}$. This integrable scaling function satisfies $(f')^\wedge(0) = (0, -4)^{\mathrm{t}} \neq 0$. Note that $\rho(2\Delta) = 2$, so Theorem 3.4 does not apply to this MRE. Yet $(f')^\wedge$ can still be realized by the constrained algorithm given by (9) using the vector $v = (0, -4)^{\mathrm{t}}$, because the matrices $M(\gamma)$ share a common invariant subspace for all γ. □

4. Derivatives

In this section we prove Theorems 3.1 and 3.2.

If f is a scaling distribution and $\hat{f}$ is a continuous function, then we may consider (3) as an ordinary pointwise equation. Taking $\gamma = 0$, we then obtain

$$\hat{f}(0) \;=\; \Delta\,\hat{f}(0). \tag{11}$$

Therefore either $\hat{f}(0) = 0$ or $\hat{f}(0)$ is a 1-eigenvector for Δ.

The distributional derivative Df of a scaling distribution f is clearly a scaling distribution itself for the MRE determined by the matrices $2c_0, \dots, 2c_N$. If f is compactly supported, then $\hat{f}$ is analytic, so $(Df)^\wedge(\gamma) = 2\pi i \gamma \hat{f}(\gamma)$, and $(Df)^\wedge(0) = 0$.

If f is an integrable function and its pointwise almost-everywhere derivative f' exists, then f' will also be a solution of the MRE determined by $2c_0, \dots, 2c_N$. However, as shown in Example 3.6b, it need not be the case that $(f')^\wedge(0) = 0$, even if f is compactly supported.

In the converse direction, the following proposition establishes that if a compactly supported scaling distribution f satisfies $\hat{f}(0) = 0$, then it is the distributional derivative of a compactly supported scaling distribution for the MRE determined by $\frac{1}{2}c_0, \dots, \frac{1}{2}c_N$.

Proposition 4.1.

Assume $f \in \mathcal{S}'(\mathbf{R})^r$ is a compactly supported scaling distribution for the MRE determined by $c_0, \dots, c_N$. If $\hat{f}(0) = 0$, then there is a compactly supported scaling distribution F for the MRE determined by $\frac{1}{2}c_0, \dots, \frac{1}{2}c_N$, such that $DF = f$.

Proof. By the Paley–Wiener theorem for distributions, $\hat{f}$ extends to the complex plane with each $\hat{f}_i$ analytic and of exponential type, i.e., there exist B, C, and K such that $|\hat{f}_i(\gamma)| \leq C\,(1 + |\gamma|)^K\, e^{B\,|\mathrm{Im}(\gamma)|}$. Since $\hat{f}(0) = 0$, expanding $\hat{f}$ in a Taylor series gives

$$\hat{f}(\gamma) \;=\; \gamma\,\hat{f}'(0) \;+\; \frac{\gamma^2\,\hat{f}''(0)}{2} \;+\; \cdots, \qquad \gamma \in \mathbf{C}.$$

Hence $G(\gamma) = \hat{f}(\gamma)/(2\pi i \gamma)$ is analytic and of exponential type, with the same B and K but possibly different C. Therefore $G = \hat{F}$ with $F \in \mathcal{S}'(\mathbf{R})^r$ compactly supported, and it is clear that $DF = f$. □

If f is an integrable scaling function, then Proposition 4.1 has a simple nondistributional proof; f is compactly supported and $\hat{f}(0) = 0$, so the primitive $F(x) = \int_{-\infty}^{x} f(t)\,dt$ is an integrable function that satisfies the MRE determined by $\frac{1}{2}c_0, \dots, \frac{1}{2}c_N$.

Iterating Proposition 4.1, either f must be the nth derivative of some fundamental solution or we can form primitives forever. The following proposition ensures that this process must terminate.

Proposition 4.2.

If $\rho(\Delta) < 1$, then there are no nontrivial scaling distributions whose Fourier transforms are continuous functions.

Proof. If $\rho(\Delta) < 1$, then we can find a matrix norm $\|\cdot\|$ such that $\|M(0)\| = \|\Delta\| < 1$. Since M is continuous, there exists a $\delta > 0$ such that $\|M(\gamma)\| \leq R < 1$ for $|\gamma| < \delta$. Hence $\left\|\prod_{j=1}^{n} M(2^{-j}\gamma)\right\| \leq R^n$ for $|\gamma| < \delta$. Therefore $P_n(\gamma) = \prod_{j=1}^{n} M(2^{-j}\gamma) \to 0$ as $n \to \infty$ for $|\gamma| < \delta$, and the fact that

$P_{n+1}(2\gamma) = M(\gamma)\, P_n(\gamma)$ implies that this extends to all $\gamma \in \mathbf{R}$. From (5) and the continuity of $\hat{f}$ we therefore obtain $\hat{f} \equiv 0$. $\square$

Thus, any scaling distribution satisfying $\hat{f}(0) = 0$ must be the nth derivative of a scaling distribution F for the MRE determined by $2^{-n}c_0, \ldots, 2^{-n}c_N$, with F satisfying $\hat{F}(0) \neq 0$. By (11), $\hat{F}(0)$ is a 1-eigenvector of $\frac{1}{2}\sum 2^{-n}\, c_k$. Hence 2^n is an eigenvector for Δ. This proves Theorem 3.2.

The proof of Theorem 3.1 will be complete once we show that an integrable scaling function f can have at most $r(N-1)-1$ continuous derivatives. Let A be the $r(N-1) \times r(N-1)$ matrix defined in $r \times r$ blocks by $A_{ij} = c_{2i-j}$, i.e., A is as given in (10). If f is a continuous, compactly supported solution of the MRE, then $f(0) = f(N) = 0$ since $\mathrm{supp}(f) \subset [0, N]$. It follows that $(f(1), \ldots, f(N-1))^{\mathrm{t}}$ is a 1-eigenvector for A. This vector cannot be zero, for if it was then we could iterate the MRE to show that f vanishes at every dyadic rational point and, therefore, vanishes identically since it is continuous.

Now suppose that f is differentiable with a continuous derivative. Then f' satisfies the MRE determined by $2c_0, \ldots, 2c_N$. As above, $(f'(1), \ldots, f'(N-1))^{\mathrm{t}}$ is then a 1-eigenvector for $2A$. Thus both 1 and 1/2 are eigenvalues of A. Continuing in this way, if $f \in C^k(\mathbf{R})^r$ and $f^{(k)}$ is continuous, then $1, 1/2, \ldots, 1/2^k$ must all be eigenvalues of A. But A is a finite matrix, so we can only have $k+1 \leq r(N-1)$.

5. Unconstrained Solutions

In this section we prove Theorem 3.3, which characterizes the fundamental unconstrained solutions of MREs. We first require the following elementary fact.

Lemma 5.1.
Given a sequence $\{a_n\}_{n=0}^{\infty}$ of scalars and given $\lambda \in \mathbf{C}$, define $s_n = a_n + \lambda\, a_{n-1} + \cdots + \lambda^n\, a_0$. If $\lim_{n\to\infty} a_n = 0$ and $|\lambda| < 1$, then $\lim_{n\to\infty} s_n = 0$.

Proof. We compute

$$|a_n + \cdots + \lambda^{n/2}\, a_{n/2}| \;\leq\; \left(\sum_{k=0}^{\infty} |\lambda|^k\right) \sup_{n/2 \leq k \leq n} |a_k|$$

and

$$|\lambda^{n/2}\, a_{n/2} + \cdots + \lambda^n\, a_0| \;\leq\; \frac{n}{2}\, |\lambda|^{n/2} \sup_{0 \leq k \leq n/2} |a_k|. \qquad \square$$

Next, we characterize exactly when the infinite product in (7) converges to a nontrivial limit. Certainly this requires that $P(0) = \Delta^{\infty}$ exist, so $\rho(\Delta) \leq 1$. If $\rho(\Delta) < 1$, then the proof of Proposition 4.2 implies $P(\gamma) \equiv 0$. Therefore $\rho(\Delta) = 1$ and Δ^{∞} is nontrivial. The following proposition establishes that if Δ^{∞} exists and is nontrivial, then $P(\gamma)$ converges and is nontrivial.

Proposition 5.2.
If Δ^{∞} exists and is nontrivial, then the infinite matrix product defining $P(\gamma)$ in (7) converges uniformly on compact sets to a continuous matrix-valued function with at most polynomial growth at infinity.

Proof. Since 1 is a nondegenerate eigenvalue of Δ with all other eigenvalues less than 1 in absolute value, we can find a matrix norm $\|\cdot\|$ such that $\|\Delta\| = 1$. Therefore $\|\Delta + A\| \leq 1 + \|A\| \leq e^{\|A\|}$ for any A, and so $\|\prod_{j=1}^{n}(\Delta + A_j)\| \leq \exp\bigl(\sum_{j=1}^{n}\|A_j\|\bigr)$ for any matrices A_j. Now

$$\|M(\gamma) - \Delta\| \;=\; \Bigl\|\frac{1}{2}\sum_{k=0}^{N}(e^{-2\pi i k\gamma} - 1)\, c_k\Bigr\| \;\leq\; \frac{1}{2}\bigl(\max\|c_k\|\bigr)\sum_{k=0}^{N} 2\,\bigl|\sin \pi k\gamma\bigr| \;\leq\; C\,|\gamma|, \qquad (12)$$

so

$$\sum_{j=1}^{n} \|M(2^{-j}\gamma) - \Delta\| \;\leq\; C \sum_{j=1}^{\infty} 2^{-j}\,|\gamma| \;=\; C\,|\gamma|.$$

This implies $\|P_n(\gamma)\| \leq e^{C|\gamma|}$. In particular, if K is compact, then

$$B_K \;=\; \sup_n \sup_{\gamma \in K} \|P_n(\gamma)\| \;<\; \infty.$$

Now fix any eigenvector λ for Δ. Let v be any corresponding λ-eigenvector, normalized to unit length. For $\gamma \in K$, we then have

$$\begin{aligned} \|P_n(\gamma)v - \lambda\,P_{n-1}(\gamma)v\| \;&=\; \|P_{n-1}(\gamma)\,(M(2^{-n}\gamma)v - \Delta v)\| \\ &\leq\; \|P_{n-1}(\gamma)\|\,\|M(2^{-n}\gamma) - \Delta\| \;\leq\; B_K\,C\,\frac{|\gamma|}{2^n}. \end{aligned} \tag{13}$$

Note that (13) implies immediately that $P_n(\gamma)v$ converges uniformly on K if $\lambda = 1$. Consider then the other eigenvalues, which all satisfy $|\lambda| < 1$. We compute

$$\begin{aligned} \|P_n(\gamma)v\| \;&\leq\; \|P_n(\gamma)v - \lambda\,P_{n-1}(\gamma)v\| \;+\; \|\lambda\,P_{n-1}(\gamma)v - \lambda^2\,P_{n-2}(\gamma)v\| \\ &\qquad +\;\cdots\;+\; \|\lambda^{n-1}\,P_1(\gamma)v - \lambda^n\,P_0(\gamma)v\| \;+\; \|\lambda^n\,P_0(\gamma)v\| \\ &\leq\; B_K\,C\,|\gamma| \left(\frac{1}{2^n} \;+\; \frac{|\lambda|}{2^{n-1}} \;+\;\cdots\;+\; \frac{|\lambda|^{n-1}}{2} \;+\; |\lambda|^n \right). \end{aligned}$$

By Lemma 4.1 or direct computation, we conclude that $\lim_{n\to\infty} P_n(\gamma)v = 0$ uniformly on K when $|\lambda| < 1$.

If Δ is diagonalizable, then there is a basis $\{v_1, \dots, v_r\}$ for $\mathbf{C}^r$ consisting of eigenvectors of Δ. We have shown that $P_n(\gamma)v_k$ converges uniformly on K for each of these v_k. Therefore, we can conclude that $P_n(\gamma)$ itself converges uniformly on compact sets when Δ is diagonalizable.

For nondiagonalizable Δ, we proceed by considering the Jordan decomposition of Δ. Since the eigenvalue 1 for Δ is nondegenerate, we need only consider those eigenvalues λ satisfying $|\lambda| < 1$. Let $U = \{u \in \mathbf{C}^r : (\Delta - \lambda)^k u = 0 \text{ for some } k\}$. There exists a smallest integer $m > 0$ such that $(\Delta - \lambda)^m u = 0$ for all $u \in U$. By standard Jordan techniques, there exists a unit basis $\{u_1, \dots, u_m\}$ for U such that

$$\Delta u_1 \;=\; \lambda\,u_1 \qquad \text{and} \qquad \Delta u_k \;=\; \lambda\,u_k \;+\; u_{k-1}, \quad k = 2, \dots, m.$$

Since u_1 is a λ-eigenvector for Δ, we know from above that $P_n(\gamma)u_1$ converges uniformly to zero on K. Assume that $P_n(\gamma)u_{k-1}$ converges uniformly to zero on K for some $k > 1$. Then

$$\begin{aligned} \|P_n(\gamma)u_k - \lambda\,P_{n-1}(\gamma)u_k - P_{n-1}(\gamma)u_{k-1}\| \;&\leq\; \|P_{n-1}(\gamma)\|\,\|M(2^{-n}\gamma)u_k - \lambda\,u_k - u_{k-1}\| \\ &\leq\; B_K\,\|M(2^{-n}\gamma)u_k - \Delta u_k\| \;\leq\; B_K\,C\,\frac{|\gamma|}{2^n}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|P_n(\gamma)u_k\| \;&\leq\; \|P_n(\gamma)u_k - \lambda\,P_{n-1}(\gamma)u_k - P_{n-1}(\gamma)u_{k-1}\| \\ &\qquad +\; \|\lambda\,P_{n-1}(\gamma)u_k - \lambda^2\,P_{n-2}(\gamma)u_k - \lambda\,P_{n-2}(\gamma)u_{k-1}\| \\ &\qquad +\;\cdots\;+\; \|\lambda^{n-1}\,P_1(\gamma)u_k - \lambda^n\,P_0(\gamma)u_k - \lambda^{n-1}\,P_0(\gamma)u_{k-1}\| \\ &\qquad +\; \|\lambda^n\,P_0(\gamma)u_k\| \\ &\qquad +\; \|P_{n-1}(\gamma)u_{k-1}\| \;+\; \|\lambda\,P_{n-2}(\gamma)u_{k-1}\| \;+\;\cdots\;+\; \|\lambda^{n-1}\,P_0(\gamma)u_{k-1}\| \\ &\leq\; B_K\,C\,|\gamma| \left(\frac{1}{2^n} \;+\; \frac{|\lambda|}{2^{n-1}} \;+\;\cdots\;+\; \frac{|\lambda|^{n-1}}{2} \;+\; |\lambda|^n \right) \\ &\qquad +\; a_{n-1}(\gamma) \;+\; |\lambda|\,a_{n-2}(\gamma) \;+\;\cdots\;+\; |\lambda|^{n-1}\,a_0(\gamma), \end{aligned} \tag{14}$$

where $a_n(\gamma) = \|P_n(\gamma)u_{k-1}\|$. We already know that $\lim_{n\to\infty} 2^{-n} + 2^{-(n-1)}|\lambda| + \cdots + |\lambda|^n = 0$. By the induction hypothesis, $\lim_{n\to\infty} a_n(\gamma) = 0$ uniformly on K. From Lemma 4.1, it follows that $a_{n-1}(\gamma) + |\lambda|\, a_{n-2}(\gamma) + \cdots + |\lambda|^{n-1}\, a_0(\gamma)$ also converges to zero, uniformly on K. Therefore, (14) implies that $P_n(\gamma)u_k$ converges to zero uniformly on K, completing the induction.

We have now shown that $P_n(\gamma)v$ converges uniformly on K for each vector v in a basis for $\mathbf{C}^r$. It follows that $P_n(\gamma)$ itself converges uniformly on compact sets. Since $\|P_n(\gamma)\| \le e^{C|\gamma|}$, the same estimate must hold for the limit $P(\gamma)$. However, since $P(2\gamma) = M(\gamma)\, P(\gamma)$, the growth of P at infinity is in fact only polynomial, as

$$\sup_{|\gamma|\le 2^n} \|P(\gamma)\| \;\le\; \Bigl(\sup_{|\gamma|\le 1} \|P(\gamma)\|\Bigr) \cdot \Bigl(\sup_{\gamma\in\mathbf{R}} \|M(\gamma)\|\Bigr)^n \;=\; B\, R^n,$$

which implies $\|P(\gamma)\| \le BR\,(1+|\gamma|^2)^{\alpha/2}$ with $\alpha = \log_2 R \ge 0$. □

We can now prove Theorem 3.3.

Assume v is any vector in $\mathbf{C}^r$. Then by Proposition 5.2, $P(\gamma)v$ is a continuous vector-valued function with polynomial growth at infinity. Hence $P(\gamma)v$ is a tempered distribution, an element of $\mathcal{S}'(\mathbf{R})^r$. Let f be its inverse Fourier transform. Then f is a tempered distribution whose Fourier transform is a continuous function that satisfies $\hat{f}(2\gamma) = M(\gamma)\,\hat{f}(\gamma)$. Therefore, f is a scaling distribution.

To see that f is compactly supported, we modify a technique introduced by Deslauriers and Dubuc [DD] for one specific scalar refinement equation. Note that $P_n(\gamma)v = \bigl(\prod_{j=1}^n M(2^{-j}\gamma)\bigr)\, v$ is a vector-valued function whose entries are linear combinations of the trigonometric polynomials $e^{-2\pi i k\gamma/2^n}$ for $k = 0, \dots, 2^nN-1$. Thus $P_n(\gamma)v \in \mathcal{S}'(\mathbf{R})^r$, and its inverse Fourier transform μ_n is a vector-valued distribution whose entries are linear combinations of the point masses $\delta_{k/2^n}$ for $k = 0, \dots, 2^nN-1$. Since $\hat{\mu}_n(\gamma) \to P(\gamma)v = \hat{f}(\gamma)$ uniformly on compact sets, we have that $\mu_n \to f$ weakly. However, each μ_n is supported in $[0, N]$, so f must be supported in this same interval.

Suppose now that f is a scaling distribution whose Fourier transform is a continuous function. Then since the infinite product defining $P(\gamma)$ converges, we have

$$\hat{f}(\gamma) \;=\; \lim_{n\to\infty} P_n(\gamma)\,\hat{f}(2^{-n}\gamma) \;=\; P(\gamma)\,\hat{f}(0). \tag{15}$$

Hence $\hat{f}(0)$ is a 1-eigenvector of $P(0) = \Delta^\infty$. Then $\Delta\,\hat{f}(0) = \Delta\,\Delta^\infty\,\hat{f}(0) = \hat{f}(0)$, so $\hat{f}(0)$ is a 1-eigenvector of Δ.

We have shown that the mapping $v \mapsto f$ induced by (8) produces all compactly supported scaling distributions and, in fact, that all are obtained by using only 1-eigenvectors of Δ. It remains to show that the kernel of this mapping is the kernel of Δ^∞. Certainly, if $\hat{f}(\gamma) = P(\gamma)v$ is identically zero, then $\Delta^\infty v = P(0)v = \hat{f}(0) = 0$. Conversely, if we begin with $\Delta^\infty v = 0$ and set $\hat{f}(\gamma) = P(\gamma)v$, then $\hat{f}(0) = 0$. But since equation (15) must hold, this implies $f \equiv 0$.

This completes the proof of Theorem 3.3. In fact, it proves slightly more; when Δ^∞ exists and is nontrivial, all distributional solutions whose Fourier transforms are continuous functions must be compactly supported. This can fail if Δ^∞ does not exist, even in the scalar case [DL].

6. Constrained Solutions

In this section we prove Theorem 3.4. Assume that Δ^∞ does not exist but that there is a single eigenvalue λ such that $|\lambda| = \rho(\Delta) < 2$ and that this eigenvalue λ is nondegenerate with multiplicity t. Then we can find a similarity transformation so that $\Delta \sim \begin{pmatrix} \lambda I_t & 0 \\ 0 & J \end{pmatrix}$, where $\rho(J) < \lambda$. Set

$$\tilde{\Delta} \;=\; \frac{1}{\lambda}\,\Delta \qquad \text{and} \qquad \tilde{M}(\gamma) \;=\; \frac{1}{\lambda}\, M(\gamma).$$

Then $\tilde{\Delta}^\infty$ exists and is nontrivial. Proposition 5.2 therefore implies that

$$\tilde{P}(\gamma) \;=\; \prod_{j=1}^{\infty} \tilde{M}(2^{-j}\gamma) \;=\; \lim_{n\to\infty} \tilde{P}_n(\gamma)$$

converges uniformly on compactly supported sets. Let $\|\cdot\|$ be any vector norm on $\mathbf{C}^r$, and let $\|\cdot\|$ also represent the corresponding matrix norm. Then, as in (12), there is a constant C so that $\|\tilde{M}(\gamma) - \tilde{\Delta}\| \le C\,|\gamma|$ for all γ.

Now fix any compact set K. Since $\tilde{P}_n(\gamma)$ converges uniformly on compact sets we have $B_K = \sup_n \sup_{\gamma\in K} \|\tilde{P}_n(\gamma)\| < \infty$. The partial products P_n and $\tilde{P}_n$ are related by $P_n(\gamma) = \prod_{j=1}^{n} M(2^{-j}\gamma) = \lambda^n\,\tilde{P}(\gamma)$. Note that $P_n(\gamma)$ does not itself converge as $n \to \infty$. In particular, if w is a λ-eigenvector for Δ, then $P_n(0)\,w = \Delta^n\,w = \lambda^n\,w$. However, if v is a 1-eigenvector of Δ with unit length and $\gamma \in K$, then we have

$$\begin{aligned}
\|P_n(\gamma)\,v - P_{n-1}(\gamma)\,v\| \;&=\; \|P_{n-1}(\gamma)\,(M(2^{-n}\gamma)v - v)\| \\
&\le\; \|P_{n-1}(\gamma)\|\;\|M(2^{-n}\gamma)\,v - \Delta\,v\| \\
&=\; |\lambda|^{n-1}\;\|\tilde{P}_{n-1}(\gamma)\|\;|\lambda|\;\|\tilde{M}(2^{-n}\gamma)\,v - \tilde{\Delta}\,v\| \\
&\le\; B_K\,|\lambda|^n\;\|\tilde{M}(2^{-n}\gamma) - \tilde{\Delta}\| \\
&\le\; C\,B_K\,|\lambda|^n\;\frac{|\gamma|}{2^n}.
\end{aligned}$$

Since $|\lambda| < 2$, we conclude that $P_n(\gamma)\,v$ converges uniformly on compact sets to some continuous limit $G(\gamma)$. Clearly, $G(2\gamma) = M(\gamma)\,G(\gamma)$. This implies that $G(\gamma)$ has polynomial growth at infinity and, therefore, is the Fourier transform of a scaling distribution f. The same argument as in the proof of Theorem 3.3 shows that f is compactly supported, with $\operatorname{supp}(f) \subset [0,N]$.

Since $\hat{f}(0) = v$, the number of independent scaling distributions produced by this method is the multiplicity of the eigenvalue 1 for Δ. To see that we have found all compactly supported fundamental scaling distributions, suppose that g is any compactly supported fundamental scaling distribution. Let $v = \hat{g}(0)$, so v is a 1-eigenvalue for Δ. Now let f be the scaling distribution produced via (9), so $\hat{f}(0) = v$ as well. Then $h = f - g$ is a compactly supported scaling distribution satisfying $\hat{h}(0) = 0$. So, by Proposition 4.1, h has a compactly supported primitive H that is a solution of the MRE determined by $\frac{1}{2}c_0, \dots, \frac{1}{2}c_N$. However, $\rho(\frac{1}{2}\Delta) < 1$, so Proposition 4.2 implies $H = 0$, and therefore $g = f$. This completes the proof of Theorem 3.4.

7. Superconstrained Solutions

In this section we present numerical evidence that if Δ has large enough eigenvalues then constrained convergence of the infinite product can fail even though a fundamental solution to the MRE does exist. We conjecture that such "superconstrained" solutions occur if $\rho(\Delta) \ge 2$ and $c_0, \dots, c_N$ share no common invariant subspaces. This latter hypothesis can probably be weakened but not removed, since it is always possible to construct MREs that trivially include large eigenvalues for Δ. For example, if $c_0, \dots, c_N$ are all diagonal then the MRE is equivalent to r independent scalar refinement equations. Setting all but one component of a solution f to zero then reduces the MRE to a single scalar refinement equation, ignoring all eigenvalues of Δ except one.

Consider again the situation of Example 3.5c. This MRE is determined by matrices $-5c_0$, $-5c_1$, $-5c_2$, $-5c_3$, where c_0, c_1, c_2, c_3 are the matrices used in the construction in [GHM]. For simplicity of notation, let M be the trigonometric polynomial determined by the matrices $-5c_k$ and let $\Delta = M(0)$. Recall that Δ has eigenvalues -5 and 1, and that a fundamental, integrable solution g to this MRE does exist, namely, $g = (\sqrt{2}\,\chi_{[0,1)}, -\chi_{[0,2)})^{\mathrm{t}}$.

Now set $P_n(\gamma) = \prod_{j=1}^{n} M(2^{-j}\gamma)$. Certainly $P_n(\gamma)u$ diverges if u is a -5-eigenvector of Δ. Consider then a 1-eigenvector v. Up to scale, $v = \hat{g}(0) = (\sqrt{2}, -2)^{\mathrm{t}}$. Let $\|\cdot\|$ be the Euclidean norm on $\mathbf{C}^2$. Set $\gamma = 1$ and define $r_\gamma(n) = \|P_{n+1}(\gamma)v\|/\|P_n(\gamma)v\|$. Then we have calculated numerically the values

n:	1	2	5	10	25	50	100
$r_\gamma(n)$:	1.7677	2.3097	2.4969	2.5000	2.5000	2.5000	2.5000

Thus it appears that $\|P_n(\gamma)v\| \in o((5/2)^n)$ when $\gamma = 1$, which would imply that $P_n(\gamma)v$ does not converge. Numerically, the same phenomenon ($r_\gamma(n) \to 2.5$) occurs for many other choices of γ.

There are theoretical reasons why such a phenomena should occur. Consider that M is 1-periodic and that the mapping $\gamma \mapsto 2\gamma \bmod 1$ is an ergodic mapping of $[0,1)$ onto itself. Moreover, $P_n(2^n\gamma) = M(2^{n-1}\gamma) \cdots M(\gamma)$. Therefore, we can apply the multiplicative ergodic theorem of Oseledec [ER]; there exist characteristic exponents $\mu_1 > \cdots > \mu_r$ so that for almost every γ there exist corresponding subspaces $\mathbf{C}^r = E_1^{(\gamma)} \supset \cdots \supset E_r^{(\gamma)}$ such that

$$\lim_{n\to\infty} \|P_n(2^n\gamma)v\|^{1/n} = e^{\mu_k} \qquad \text{if } v \in E_k^{(\gamma)} \setminus E_{k+1}^{(\gamma)}. \tag{16}$$

If, say, $E_k^{(\gamma)}$ was independent of γ and the convergence in (16) was uniform in a neighborhood U of zero, then we could conclude that $\lim_{n\to\infty} \|P_n(\gamma)v\|^{1/n} = e^{\mu_k}$ for $\gamma \in U$ and $v \in E_k^{(\gamma)} \setminus E_{k+1}^{(\gamma)}$.

Therefore, we conjecture that $e^{\mu_1} = 5/2$ for this example and that, in general, $e^{\mu_1} = \rho(\Delta)/2$. The difficulty in verifying this theoretically is the usual problem with implementing the multiplicative ergodic theorem; the characteristic exponents and corresponding subspaces are difficult to compute.

8. Noncompactly Supported Solutions

We conclude with some brief comments on the broader class of solutions to refinement equations that are not compactly supported.

For simplicity, assume that Δ^∞ exists and is nontrivial. Let f be any scaling distribution whose Fourier transform is a continuous function. Let $G(\gamma)$ be a *scalar-valued* multiplicatively periodic function, i.e., $G(2\gamma) = G(\gamma)$ for almost all γ. If $G(\gamma)\,\hat{f}(\gamma)$ is a tempered distribution, then g defined by $\hat{g}(\gamma) = G(\gamma)\,\hat{f}(\gamma)$ is also a scaling distribution for the same MRE. For example, this is so if f is an integrable function and G is a bounded, multiplicatively periodic function. In particular, the Hilbert transform of any scaling function is again a scaling function. Note that $\hat{f}(0) \neq 0$ and that G cannot be continuous at $\gamma = 0$ unless G is identically constant. Therefore, if G is not identically constant, then such a solution g cannot be compactly supported.

For scalar refinement equations, it is easy to see that all distributional solutions whose Fourier transforms are realizable as functions are obtained in this manner. We know that a fundamental solution f exists, defined by $\hat{f}(\gamma) = P(\gamma)v$, where v is now just a scalar. If g is any other scaling distribution whose Fourier transform is a function, then $G(\gamma) = \hat{g}(\gamma)/\hat{f}(\gamma)$ exists almost everywhere since f is compactly supported. As both f and g are solutions of the same refinement equation and $M(\gamma) \neq 0$ almost everywhere, we must have G multiplicatively periodic. It would be interesting to determine whether an analogous characterization holds for MREs.

Finally, it is interesting to note that there exist scaling distributions whose Fourier transforms are not realizable as functions. For example, consider the simplest scalar refinement equation $f(x) = 2f(2x)$. The fundamental compactly supported solution is $f = \delta$. A noncompactly supported distributional solution is $g = \mathrm{pv}(1/x)$; note that $\hat{g}(\gamma) = \pi i$ if $\gamma > 0$ and $-\pi i$ if $\gamma < 0$. A noncompactly supported distributional solution whose Fourier transform is not a function is $h = \mathrm{pv}\big(\sum_{n=-\infty}^{\infty} (\delta_{2^n} - \delta_{-2^n})\big)$, meaning $\langle h, \varphi \rangle = \lim_{N\to\infty} \sum_{n=-N}^{N} (\varphi(2^n) - \varphi(-2^n))$ for $\varphi \in \mathcal{S}(\mathbf{R})$. If $\hat{h}$ was a function, then it would be $\hat{h}(\gamma) = 2i \lim_{N\to\infty} \sum_{n=-N}^{N} \sin 2^n\gamma$. However, $\sum_{n=-\infty}^{0} \sin 2^n\gamma$ converges uniformly on any compact interval, while $\sum_{n=1}^{\infty} \sin 2^n \gamma$ diverges almost everywhere since $\{2^n\}_{n=1}^{\infty}$ is lacunary [Z, p. 203].

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