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WAVELETS AND FRAMES

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Abstract. This paper presents basic results about frames in Hilbert spaces, and gives examples of two types of frames for $L^2(\mathbf{R})$. These are the Weyl-Heisenberg frames, which are translations and modulations of a single "mother wavelet", and the affine frames, which are translations and dilations of a mother wavelet.

0. Introduction. It is a well-known fact that every separable Hilbert space, and in particular $L^2(\mathbf{R})$, possesses an orthonormal basis. One of the major properties of such sequences is that they provide a "decomposition" of the space. That is, if $\{e_n\}$ is an orthonormal basis for a Hilbert space H then every $x \in H$ can be written $x = \sum_n \langle x, e_n \rangle e_n$. Unfortunately, orthonormal bases can often be difficult to find or inconvenient to work with. For example, one orthonormal basis for $L^2(\mathbf{R})$ is the sequence $\{\varphi_{mn}\}$, where

$$\varphi_{mn}(x) \;=\; e^{2\pi i(x-n)m}\,\chi_{[n,n+1)}(x).$$

However, these functions are discontinuous, which can make the representation of even smooth functions in $L^2(\mathbf{R})$ unpleasant.

Frames are an alternative to orthonormal bases. By giving up the requirements of orthogonality and uniqueness of decomposition we allow much more freedom in our choice of "basis vectors", while still retaining the ability to decompose the space. In Section 2 we define the notion of a frame and show that if $\{x_n\}$ is a frame then every $x \in H$ can be written $x = \sum_n c_n x_n$ in a good way, i.e., the scalars are computable, the series converges unconditionally, etc. The remainder of the paper is devoted to finding frames for $L^2(\mathbf{R})$. These fall into two general categories, which we call Weyl-Heisenberg and affine frames.

W-H frames, studied in Section 3, are similar in structure to the orthonormal basis $\{\varphi_{mn}\}$ given above. In particular, note that φ_{mn} can be written $\varphi_{mn} = T_n E_m \varphi$, where $\varphi = \chi_{[0,1)}$, E_m denotes multiplication by $e^{2\pi imx}$, and T_n denotes translation by n. In the same way, W-H frames are composed of discrete modulates and translates of a single function, called the "mother wavelet". Unlike orthonormal bases, we show that it is possible to find W-H frames whose mother wavelet is smooth. Moreover, in Section 4 we indicate that if a W-H frame should happen to form a basis then the mother wavelet is either not smooth or does not decay quickly. Section 4 is devoted to the Zak transform, also known as the Weil-Brezin map [AT], which is an important tool in the analysis of W-H frames.

In Section 5 we turn to a fundamentally different method of constructing frames for $L^2(\mathbf{R})$. These "affine frames" are obtained by taking discrete translates and dilates of the mother wavelet. A well-known example is the standard Haar system, whose mother wavelet is not smooth. After stating some of the basic results about

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affine frames, we briefly mention the Meyer wavelet. This is a C^∞ function with compactly supported Fourier transform, and the affine frame it generates forms an orthonormal basis for $L^2(\mathbf{R})$. Comparing this with our above remarks on W-H frames and bases, we see that there is a great deal of difference between the two types of frames.

Although we omit or only sketch most of the proofs in this article, they can all be found in [HW], along with a great deal of additional material, such as the relationship of W-H and affine frames to continuous representations of $L^2(\mathbf{R})$. I would like to thank David Walnut, my colleague at the University of Maryland and Mitre Corp., for permission to use some of the material from [HW] in this article. Another supurb reference on the subject of W-H and affine frames is [D1], from which many of the results quoted in this paper were taken.

1. Notation. We write $\mathbf{R}$ for the real line thought of as the time axis, and $\hat{R}$ for its dual group, the real line throught of as the frequency axis. All sequences and series with undefined limits are to be taken over $\mathbf{Z}$, the set of integers. $L^2(\mathbf{R})$ is the Hilbert space of all complex-valued, square-integrable functions f on $\mathbf{R}$, normed by $\|f\|_2 = \left(\int |f(x)|^2\,dx\right)^{1/2}$. The inner product of f, $g \in L^2(\mathbf{R})$ is $\langle f, g\rangle = \int f(x)\,\overline{g(x)}\,dx$. The Fourier transform of an integrable f is

$$\hat{f}(\gamma) = \int_{\mathbf{R}} f(x)\, e^{-2\pi i\gamma x}\, dx, \qquad \text{for } \gamma \in \hat{R}.$$

Given a function f we define

Translation:	$T_a f(x) = f(x-a)$,	for $a \in \mathbf{R}$;
Modulation:	$E_a f(x) = e^{2\pi i a x} f(x)$,	for $a \in \mathbf{R}$;
Dilation:	$D_a f(x) = \lvert a\rvert^{-1/2} f(x/a)$,	for $a \in \mathbf{R}\backslash\{0\}$.

Each of these is a unitary operator from Ł2 onto itself, i.e., a linear bijective isometry. We also use the symbol E_a by itself to refer to the **exponential function**, i.e., $E_a(x) = e^{2\pi i a x}$.

2. Frames in Hilbert Spaces.

DEFINITION 2.1 [DS]. A sequence $\{x_n\}$ in a Hilbert space H is a **frame** if there exist numbers $A, B > 0$ such that for all $x \in H$ we have

$$A\|x\|^2 \;\le\; \sum_n |\langle x, x_n\rangle|^2 \;\le\; B\|x\|^2.$$

The numbers A, B are called the **frame bounds**. The frame is **tight** if $A = B$. The frame is **exact** if it ceases to be a frame whenever any single element is deleted from the sequence.

Frames were first introduced in 1952 by Duffin and Schaeffer in connection with nonharmonic Fourier series [DS].

Since $\sum |\langle x, x_n\rangle|^2$ is a series of positive real numbers it converges absolutely, hence unconditionally. That is, every rearrangement of the sum also converges, and converges to the same value. Therefore every rearrangement of a frame is also a frame, and all sums involving frames actually converge unconditionally.

If follows immediately from the definition that all frames are complete That is, the only $x \in H$ orthogonal to every x_n is $x = 0$, or equivalently, the set of finite linear combinations of the x_n is dense in H. The following theorem shows that they also provide a decomposition of the space similar to that of orthonormal bases. Given operators $S, T: H \to H$ we write $S \le T$ if $\langle Sx, x\rangle \le \langle Tx, x\rangle$ for all $x \in H$, and we denote by I the identity map on H, i.e., $Ix = x$ for all $x \in H$.

THEOREM 2.2 [DS]. *Given a sequence $\{x_n\}$ in a Hilbert space H, the following two statements are equivalent:*

(1) *$\{x_n\}$ is a frame with bounds A, B.*
(2) *$Sx = \sum \langle x, x_n\rangle x_n$ is a bounded linear operator with $AI \le S \le BI$, called the* **frame operator** *for $\{x_n\}$.*

Proof. (2) $\Rightarrow$ (1). Follors from $\langle Ix, x\rangle = \|x\|^2$ and $\langle Sx, x\rangle = \sum |\langle x, x_n\rangle|^2$.

(1) $\Rightarrow$ (2). S is well-defined and continuous since

$$\begin{aligned}
\|Sx\|^2 &= \sup_{\|y\|=1} |\langle Sx, y\rangle|^2 \\
&= \sup_{\|y\|=1} \left|\sum_n \langle x, x_n\rangle\langle x_n, y\rangle\right|^2 \\
&\le \sup_{\|y\|=1} \left(\sum_n |\langle x, x_n\rangle|^2\right)\left(\sum_n |\langle x_n, y\rangle|^2\right) \\
&\le \sup_{\|y\|=1} B\|x\|^2 \cdot B\|y\|^2 \\
&= B^2\|x\|^2.
\end{aligned}$$

The relations $AI \le S \le BI$ then follow immediately from the definition of frames. $\square$

We say that a mapping $U: H \to H$ is **invertible**, or a **topological isomorphism**, if U is linear, bijective, continuous, and U^{-1} is continuous.

COROLLARY 2.3 [DS].

(1) *S is invertible and $B^{-1}I \le S^{-1} \le A^{-1}I$.*
(2) *$\{S^{-1}x_n\}$ is a frame with bounds $1/B, 1/A$, called the* **dual frame** *of $\{x_n\}$.*
(3) *Every $x \in H$ can be written*

$$x = \sum \langle x, S^{-1}x_n\rangle x_n = \sum \langle x, x_n\rangle S^{-1}x_n.$$

Proof. 1. Since $AI \le S \le BI$ we have $\|I - \frac{1}{B}S\| \le \|I - \frac{A}{B}S\| = \frac{B-A}{B} < 1$. Elementary Hilbert space results therefore imply that $\frac{1}{B}S$ is invertible, so S itself must be invertible. Since S^{-1} is a positive operator and commutes with both I

and S, we can multiply through by S^{-1} in the equation $AI \le S \le BI$ to obtain $B^{-1}I \le S^{-1} \le A^{-1}I$.

2. Since S^{-1} is positive it is self-adjoint. Therefore,

$$\begin{aligned}\sum_n \langle x, S^{-1}x_n\rangle S^{-1}x_n &= \sum_n \langle S^{-1}x, x_n\rangle S^{-1}x_n \\ &= S^{-1}\left(\sum_n \langle S^{-1}x, x_n\rangle x_n\right) \\ &= S^{-1}S(S^{-1}x) \\ &= S^{-1}x.\end{aligned}$$

The result then follows from the fact that $B^{-1}I \le S^{-1} \le A^{-1}I$ and Theorem 2.2 part 2.

3. Simply expand $x = S(S^{-1}x)$ and $x = S^{-1}(Sx)$. □

Corollary 2.4. *If $\{x_n\}$ is a tight frame, i.e., if $A = B$, then*

(1) *$S = AI$.*
(2) *$S^{-1} = A^{-1}I$.*
(3) *Every $x \in H$ can be written $x = A^{-1}\sum\langle x, x_n\rangle x_n$.*

Proposition 2.5 [DS]. *Given a frame $\{x_n\}$ and given $x \in H$ define $a_n = \langle x, S^{-1}x_n\rangle$, so $x = \sum a_n x_n$. If it is possible to find other scalars c_n such that $x = \sum c_n x_n$ then we must have*

$$\sum |c_n|^2 = \sum |a_n|^2 + \sum |a_n - c_n|^2.$$

Proof. Follows from $\sum |a_n|^2 = \langle x, S^{-1}x\rangle = \sum c_n \overline{a}_n$. □

Theorem 2.6 [DS]. *The removal of a vector from a frame leaves either a frame or an incomplete set. In fact,*

$$\begin{aligned}\langle x_m, S^{-1}x_m\rangle &\ne 1 \Rightarrow \{x_n\}_{n\ne m} \text{ is a frame;} \\ \langle x_m, S^{-1}x_m\rangle &= 1 \Rightarrow \{x_n\}_{n\ne m} \text{ is incomplete.}\end{aligned}$$

Proof. Fix m, and define $a_n = \langle x_m, S^{-1}x_n\rangle = \langle S^{-1}x_m, x_n\rangle$. We know that $x_m = \sum a_n x_n$, but we also have $x_m = \sum c_n x_n$ where $c_m = 1$ and $c_n = 0$ for $n \ne m$. Using Proposition 2.5 we find that

$$\sum_{n\ne m} |a_n|^2 = \frac{1 - |a_m|^2 - |a_m - 1|^2}{2} < \infty.$$

Suppose now that $a_m = 1$. Then $\sum_{n\ne m} |a_n|^2 = 0$, so $a_n = \langle S^{-1}x_m, x_n\rangle = 0$ for $n \ne m$. That is, $S^{-1}x_m$ is orthogonal to x_n for every $n \ne m$. But $S^{-1}x_m \ne 0$ since $\langle S^{-1}x_m, x_m\rangle = a_m = 1 \ne 0$. Therefore $\{x_n\}_{n\ne m}$ is incomplete in this case.

On the other hand, suppose $a_m \neq 1$. Then $x_m = \frac{1}{1-a_m} \sum_{n \neq m} a_n x_n$, so for $x \in H$ we have

$$|\langle x, x_m \rangle|^2 \;=\; \left| \frac{1}{1-a_m} \sum_{n \neq m} a_n \langle x, x_n \rangle \right|^2 \;\leq\; \frac{1}{|1-a_m|^2} \biggl(\sum_{n \neq m} |a_n|^2 \biggr) \Bigl(|\langle x, x_n \rangle|^2 \Bigr).$$

Therefore,

$$\sum_n |\langle x, x_n \rangle|^2 \;=\; |\langle x, x_m \rangle|^2 + \sum_{n \neq m} |\langle x, x_n \rangle|^2 \;\leq\; C \sum_{n \neq m} |\langle x, x_n \rangle|^2,$$

where $C = 1 + |1 - a_m|^{-2} \sum_{n \neq m} |a_n|^2$. It follows immediately that $\{x_n\}_{n \neq m}$ is a frame with bounds $A/C, B$. $\square$

Corollary 2.7 [DS]. *If $\{x_n\}$ is an exact frame then $\{x_n\}$ and $\{S^{-1}x_n\}$ are* **biorthogonal**, *i.e.,*

$$\langle x_m, S^{-1} x_n \rangle \;=\; \delta_{mn} \;=\; \begin{cases} 1, & \textit{if } m = n; \\ 0, & \textit{if } m \neq n. \end{cases}$$

Definition 2.8. A sequence $\{\varphi_n\}$ in a Hilbert space H is a **basis** for H if for every $x \in H$ there exist *unique* scalars c_n such that

$$x \;=\; \sum c_n \varphi_n.$$

The basis is **bounded** if $0 < \inf \|\varphi_n\| \leq \sup \|\varphi_n\| < \infty$. The basis is **unconditional** if the series $\sum c_n \varphi_n$ converges unconditionally, that is, every permutation of the series converges.

In finite-dimensional spaces, a series converges unconditionally if and only if it converges absolutely. In infinite-dimensional spaces, absolute convergence still implies unconditional convergence but the reverse need not be true. In Hilbert spaces, all bounded unconditional bases are **equivalent** to orthonormal bases. That is, if $\{\varphi_n\}$ is a bounded unconditional basis, then there is an orthonormal basis $\{e_n\}$ and a topological isomorphism $U \colon H \to H$ such that $\varphi_n = U e_n$ for all n (see [Y]).

It is easy to see that inexact frames are not bases. However, we do have the following characterization of exact frames.

Theorem 2.9 [DS; Y]**.** *A sequence $\{x_n\}$ in a Hilbert space H is an exact frame for H if and only if it is a bounded unconditional basis for H.*

Proof [H]. $\Rightarrow$. Assume $\{x_n\}$ is an exact frame with bounds A, B. Then $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthogonal, so for m fixed we have

$$A \, \|S^{-1} x_m\|^2 \;\leq\; \sum_n |\langle S^{-1} x_m, x_n \rangle|^2 \;=\; |\langle S^{-1} x_m, x_m \rangle|^2 \;\leq\; \|S^{-1} x_m\|^2 \|x_m\|^2.$$

Therefore $\|x_m\|^2 \geq A$. Also,

$$\|x_m\|^4 \;=\; |\langle x_m, x_m\rangle|^2 \;\leq\; \sum_n |\langle x_m, x_n\rangle|^2 \;\leq\; B\,\|x_m\|^2,$$

so $\|x_m\|^2 \leq B$. Thus the sequence $\{x_n\}$ is bounded in norm. By Corollary 2.3, $x = \sum \langle x, S^{-1}x_n\rangle x_n$ for all $x \in H$, and the biorthogonality of $\{x_n\}$ and $\{S^{-1}x_n\}$ implies that this representation is unique, so $\{x_n\}$ is a basis for H. Since every permutation of a frame is also a frame, we conclude that the basis is unconditional.

$\Leftarrow$. Assume $\{x_n\}$ is a bounded unconditional basis for H. Then there is an orthonormal basis $\{e_n\}$ and a topological isomorphism $U\colon H \to H$ such that $Ue_n = x_n$ for all n. Given $x \in H$ we therefore have

$$\sum_n |\langle x, x_n\rangle|^2 \;=\; \sum_n |\langle x, Ue_n\rangle|^2 \;=\; \sum_n |\langle U^*x, e_n\rangle|^2 \;=\; \|U^*x\|^2.$$

But

$$\frac{\|x\|}{\|{U^*}^{-1}\|} \;\leq\; \|U^*x\| \;\leq\; \|U^*\|\,\|x\|,$$

so $\{x_n\}$ forms a frame. It is clearly exact since the removal of any vector from a basis leaves an incomplete set. $\square$

3. Weyl-Heisenberg Frames.

Definition 3.1. Given $g \in$ Ł2 and $a, b > 0$, we say that (g, a, b) **generates a W-H frame** for $L^2(\mathbf{R})$ if $\{T_{na}E_{mb}g\}_{m,n\in\mathbf{Z}}$ is a frame for $L^2(\mathbf{R})$. The function g is referred to as the **mother wavelet**. Together, the numbers a, b are the **frame parameters**. Individually, a is the **shift parameter** and b is the **modulation parameter**.

Theorem 3.2 [DGM]. *Assume $g \in L^2(\mathbf{R})$ and $a, b > 0$ satisfy:*

(1) *There exist constants A, B such that*

$$0 \;<\; A \;=\; \operatorname*{ess\,inf}_{x\in\mathbf{R}} \sum_n |g(x-na)|^2 \;\leq\; \operatorname*{ess\,sup}_{x\in\mathbf{R}} \sum_n |g(x-na)|^2 \;=\; B \;<\; \infty;$$

(2) *g has compact support, with* $\operatorname{supp}(g) \subset I \subset \mathbf{R}$, *where I is some interval of length $1/b$.*

Then (g, a, b) generates a W-H frame for $L^2(\mathbf{R})$ with frame bounds $b^{-1}A, b^{-1}B$.

Proof. Fix n, and observe that the function $f \cdot \overline{T_{na}g}$ is supported in $I_n = I + na = \{x + na : x \in I\}$, which is an interval of length $1/b$. From the Plancherel formula for Fourier series, we therefore have

$$\begin{aligned}\sum_n \sum_m |\langle f, T_{na}E_{mb}g\rangle|^2 \;&=\; \sum_n \sum_m |\langle f \cdot \overline{T_{na}g}, E_{mb}\rangle|^2 \\ &=\; \sum_n b^{-1} \int_{I_n} |f(x)|^2\,|g(x-na)|^2\,dx \\ &=\; b^{-1} \int_{\mathbf{R}} |f(x)|^2 \sum_n |g(x-na)|^2\,dx,\end{aligned}$$

from which the result follows. $\square$

By taking the functions f to be analyzed which have compact support, one can prove:

PROPOSITION 3.3 [D1]. *Whether g has compact support or not, it is necessary that condition (1) of Theorem 3.2 hold in order that $\{T_{na}E_{mb}g\}$ make a frame.*

REMARK 3.4. It is easy to see that if g satisfies condition (2) of Theorem 3.2 and if $ab > 1$ then $\sum_n |g(x-na)|^2$ is not bounded below, so that g cannot generate a frame. In fact, the set $\{T_{na}E_{mb}g\}$ is not even complete, since $\cup_n \operatorname{supp}(T_{na}g)$ does not cover $\mathbf{R}$. This is a simple illustration of a more general phenomenon discussed in Section 4, namely that if $ab > 1$ then $\{T_{na}E_{mb}g\}$ can never be complete in $L^2(\mathbf{R})$ for *any* $g \in L^2(\mathbf{R})$.

PROPOSITION 3.5 [D1]. *If (g, a, b) generates a W-H frame for $L^2(\mathbf{R})$, then $(\hat{g}, b, a)$ generates a W-H frame for $L^2(\hat{R})$.*

Proof. Follows immediately from $(T_{na}E_{mb}g)^\wedge = E_{-na}T_{-mb}\hat{g}$. $\square$

REMARK 3.6. It follows from Propositions 3.3 and 3.5 that if (g, a, b) generates a frame for $L^2(\mathbf{R})$ then both g and $\hat{g}$ are bounded functions.

Theorem 3.2 used Fourier series arguments to derive conditions under which a compactly supported function g could be a mother wavelet. If g does not have compact support then this argument breaks down. It turns out, however, that one can perform a similar calculation if g is not compactly supported.

THEOREM 3.7 [D1]. *Assume $g \in L^2(\mathbf{R})$ and $a > 0$ satisfy:*

(1) *There exist constants A, B such that*

$$0 \;<\; A \;=\; \operatorname*{ess\,inf}_{x\in\mathbf{R}} \sum_n |g(x-na)|^2 \;\le\; \operatorname*{ess\,sup}_{x\in\mathbf{R}} \sum_n |g(x-na)|^2 \;=\; B \;<\; \infty;$$

(2) $\lim\limits_{b\to 0} \sum\limits_{k\neq 0} \beta(k/b) = 0$, *where*

$$\beta(s) \;=\; \operatorname*{ess\,sup}_{x\in\mathbf{R}} \sum_n |g(x-na)|\,|g(x-s-na)|.$$

Then there exists a number $b_0 > 0$ such that (g, a, b) generates a W-H frame for $L^2(\mathbf{R})$ for each $0 < b < b_0$.

REMARK 3.8. Condition (2) of Theorem 3.7 is purely a decay condition on g and is satisfied, for example, if

$$|g(x)| \;\le\; C(1 + |x|^2)^{-1}$$

for some $C < \infty$ and all $x \in \mathbf{R}$.

EXAMPLE 3.9. Given any $a, b > 0$ with $ab < 1$ it is easy to construct a smooth mother wavelet which generates a W-H frame for those parameters [D1; DGM]. For example, assume $1/2b \le a < 1/b$ and set $\lambda = 1/b - a$. Since $\lambda > 0$ we can define

$$g(x) \;=\; a^{-1/2} \begin{cases} 0, & x \le 0; \\ v(x/\lambda), & 0 \le x \le \lambda; \\ 1, & \lambda \le x \le a; \\ \left(1 - v(\frac{x-a}{\lambda}^2)\right)^{1/2}, & a \le x \le 1/b; \\ 0, & 1/b \le x; \end{cases}$$

where $v \in C^\infty(\mathbf{R})$ is any function such that $v(x) = 0$ if $x \le 0$, $v(x) = 1$ if $x \ge 1$, and $0 < v(x) < 1$ if $0 < x < 1$. It is easy to see then that $g \in C^\infty(\mathbf{R})$, $\|g\|_2 = 1$, $\mathrm{supp}(g) \subset [0, 1/b]$, and $\sum |g(x - na)|^2 \equiv 1/a$. It therefore follows from Theorem 3.2 that g generates a tight W-H frame with frame bound $A = B = \frac{1}{ab}$.

4. The Zak transform. The Zak transform, also known as the Weil-Brezin map, has been used explicitly and implicitly in many mathematical and signal processing articles. Its history may even extend as far back as Gauss. Zak studied this operator beginning in the 1960s, in connection with solid state physics [Z].

DEFINITION 4.1. The **Zak transform** of a function f is (formally)

$$Zf(t, \omega) \;=\; a^{1/2} \sum_{k \in \mathbf{Z}} f(ta - ka)\, e^{-2\pi i k \omega} \qquad \text{for } t \in \mathbf{R},\ \omega \in \hat{R},$$

where $a \in \mathbf{R} \backslash \{0\}$ is fixed.

Zf is defined pointwise at least for continuous functions with compact support. Formally, we have the **quasiperiodicity relations**

$$\begin{aligned} Zf(t+1, \omega) \;&=\; e^{-2\pi i \omega} Zf(t, \omega) \\ Zf(t, \omega + 1) \;&=\; Zf(t, \omega) \end{aligned}$$

Therefore, the values of $Zf(t, \omega)$ for $(t, \omega) \in \mathbf{R} \times \hat{R}$ are completely determined by its values in the unit square, i.e., for $(t, \omega) \in Q = [0, 1) \times [0, 1)$.

DEFINITION 4.2. We define $L^2(Q)$ to be the Hilbert space

$$L^2(Q) \;=\; \Big\{ F : \|F\|_2 = \Big(\int_0^1 \int_0^1 |F(t, \omega)|^2 \, d\omega \, dt \Big)^{1/2} < \infty \Big\}.$$

We denote by $E_{(m,n)}$ the two-dimensional exponential $E_{(m,n)}(t, \omega) = e^{2\pi i m t} e^{2\pi i n \omega}$. Recall that the set of two-dimensional exponentials $\{E_{(m,n)}\}_{m,n \in \mathbf{Z}}$ forms an orthonormal basis for $L^2(Q)$.

THEOREM 4.3 [J]. *The Zak transform is a unitary map of $L^2(\mathbf{R})$ onto $L^2(Q)$, i.e., Z is a linear bijective isometry.*

In general, if Zf is continuous on Q it need not be true that Zf is continuous on $\mathbf{R} \times \hat{R}$. For example, consider the function f such that Zf is identically 1 on Q. Moreover, one can prove the following.

Theorem 4.4 [J]. *If $f \in L^2(\mathbf{R})$ is such that Zf is continuous on $\mathbf{R} \times \hat{R}$ then Zf has a zero in Q.*

The unitary nature of the Zak transform allows us to translate conditions on frames for $L^2(\mathbf{R})$ into conditions for frames for $L^2(Q)$, where things are frequently easier to deal with.

Proposition 4.5. *Given functions $g_n \in L^2(\mathbf{R})$. Then $\{g_n\}$ is complete/a frame/ an exact frame/an orthonormal basis for $L^2(\mathbf{R})$ if and only if $\{Zg_n\}$ is complete/a frame/an exact frame/an orthonormal basis for $L^2(Q)$.*

The Zak transform is particularly good for analyzing W-H frames when $ab = 1$.

Theorem 4.6 [J]. *Assume $g \in L^2(\mathbf{R})$ and $a, b > 0$ with $ab = 1$. Then*

$$Z(T_{na}E_{mb}g) \;=\; E_{(m,n)}Zg.$$

Corollary 4.7 [D1; B]. *Given $g \in L^2(\mathbf{R})$ and $a, b > 0$ with $ab = 1$. Then $\{T_{na}E_{mb}g\}$ is a complete set in $L^2(\mathbf{R})$ if and only if $Zg \neq 0$ a.e. in Q.*

Corollary 4.8 [D1; B]. *Given $g \in L^2(\mathbf{R})$ and $a, b > 0$ with $ab = 1$. Then the following statements are equivalent, i.e., each implies the other.*

(1) $0 < A \leq |Zg(t,\omega)|^2 \leq B < \infty$ *a.e. in Q.*
(2) $\{T_{na}E_{mb}g\}_{m,n\in\mathbf{Z}}$ *is a frame for $L^2(\mathbf{R})$ with frame bounds A, B.*
(3) $\{T_{na}E_{mb}g\}_{m,n\in\mathbf{Z}}$ *is an exact frame for $L^2(\mathbf{R})$ with frame bounds A, B.*

Proof. $1 \Rightarrow 2$. Assume 1 holds. By Proposition 4.5 and Theorem 4.6, it suffices to show that $\{E_{(m,n)}Zg\}$ is a frame for $L^2(Q)$ with frame bounds A, B. So, choose any $F \in L^2(Q)$. As Zg is a bounded function, we have that $F \cdot \overline{Zg} \in L^2(Q)$. But $\{E_{(m,n)}\}$ is an orthonormal basis for $L^2(Q)$, so we have

$$\sum_{m,n} |\langle F, E_{(m,n)}Zg\rangle|^2 \;=\; \sum_{m,n} |\langle F \cdot \overline{Zg}, E_{(m,n)}\rangle|^2 \;=\; \|F \cdot \overline{Zg}\|_2^2,$$

from which the result follows.

$2 \Rightarrow 1$. Assume 2 holds. By Proposition 4.5 and Theorem 4.6, $\{E_{(m,n)}Zg\}$ is therefore a frame for $L^2(Q)$ with frame bounds A, B. But $\{E_{(m,n)}\}$ is an orthonormal basis for $L^2(Q)$, so $\sum |\langle F, E_{(m,n)}Zg\rangle|^2 = \|F \cdot \overline{Zg}\|_2^2$, as above. Thus $A\|F\|_2^2 \leq \|F \cdot \overline{Zg}\|_2^2 \leq B\|F\|_2^2$ for all $F \in L^2(Q)$, which implies easily that $A \leq |Zg|^2 \leq B$ a.e.

$2 \Rightarrow 3$. Assume 2 holds, so $\{E_{(m,n)}Zg\}$ is a frame for $L^2(Q)$. By $1 \Leftrightarrow 2$ we know Zg is bounded above and below. Therefore, the mapping U defined by $UF = F \cdot Zg$ is a topological isomorphism of $L^2(Q)$ onto itself. Recall from Section 2 that exact frames are bounded unconditional bases, and that bounded unconditional bases are equivalent to orthonormal bases. Since $\{E_{(m,n)}Zg\}$ is obtained from the orthonormal basis $\{E_{(m,n)}\}$ by the topological isomorphism U, we see that $\{E_{(m,n)}Zg\}$ is a bounded unconditional basis, hence an exact frame. $\square$

Corollary 4.9 [D1; B]. *Given $g \in L^2(\mathbf{R})$ and $a, b > 0$ with $ab = 1$. Then $\{T_{na}E_{mb}g\}$ is an orthonormal basis for $L^2(\mathbf{R})$ if and only if $|Zg(t,\omega) = 1$ a.e. in Q.*

The preceding results give us hope that we can find good orthonormal bases for $L^2(\mathbf{R})$, since all we need do is find some nice function whose Zak transform has absolute value 1. Let us look at some examples.

Example 4.10 [DGM; BGZ]. The Zak transform of the Gaussian function $g(x) = e^{-\pi x^2}$ is continuous and has a single zero in Q. Therefore the Weyl–Heisenberg states $\{T_{na}E_{mb}g\}_{m,n\in\mathbf{Z}}$ for $ab = 1$ are complete in $L^2(\mathbf{R})$ but do not form a frame. However, one can show that g does generate a frame for other values of ab, in particular, for $ab = 1/2$.

Example 4.11. Let $a = b = 1$ and $g = \chi_{[0,1)}$. The Zak transform of g is $Zg(t,\omega) \equiv 1$ for $(t,\omega) \in Q$. Therefore g generates a W-H frame with $A = B = 1$. In fact, this W-H frame is an orthonormal basis for $L^2(\mathbf{R})$ by Corollary 4.9.

This last example is a bit unpleasant since g is not smooth. This will introduce discontinuities even when analyzing functions which are smooth. The following theorem, due to Balian, Coifman, and Semmes, shows that something like this always happens when $ab = 1$, namely, either g is not smooth or it does not decay very fast.

Theorem 4.12 [D1]. *Given $g \in L^2(\mathbf{R})$ and $a, b > 0$ with $ab = 1$. If (g, a, b) generates a W-H frame, then either $xg(x) \notin L^2(\mathbf{R})$ or $g' \notin L^2(\mathbf{R})$.*

Thus, W-H frames with $ab = 1$ are bases for $L^2(\mathbf{R})$ but are not very nice. It can be shown that all W-H frames with $ab < 1$ are inexact, and we indicate below that it is impossible to construct a W-H frame when $ab > 1$. Thus $ab = 1$ is a sort of "critical value" for W-H frames. Daubechies explains this as a "Nyquist density", as occurs in information processing [D1].

For the case $ab = N > 1$, an easy calculation gives $Z(T_{na}E_{mb}g) = E_{(mN,n)}Zg$. But $\{E_{(mN,n)}\}$ is only a part of the orthonormal basis $\{E_{(m,n)}\}$, so it is easy to prove that $E_{(mN,n)}Zg$ is incomplete in $L^2(Q)$ no matter what $g \in L^2(\mathbf{R})$ is chosen. The proof can be adapted to cover the case $ab > 1$ and rational. For $ab > 1$ and irrational, the result follows by computing the coupling constant of a certain Von Neumann algebra [D1; R].

5. Affine Frames.

Definition 5.1. We define

$$\begin{aligned} H^2_+ &= \{f \in L^2(\mathbf{R}) : \mathrm{supp}(\hat{f}) \subset [0,\infty)\}, \\ H^2_- &= \{f \in L^2(\mathbf{R}) : \mathrm{supp}(\hat{f}) \subset (-\infty,0]\}. \end{aligned}$$

These are Hilbert spaces, with the same inner products as the $L^2(\mathbf{R})$ inner product, and with norms

$$\|f\|_{H^2_+} = \left(\int_0^\infty |\hat{f}(\gamma)|^2\,d\gamma\right)^{1/2} \quad \text{and} \quad \|f\|_{H^2_-} = \left(\int_{-\infty}^0 |\hat{f}(\gamma)|^2\,d\gamma\right)^{1/2}.$$

Moreover, $H^2_+(\mathbf{R})$ and $H^2_-(\mathbf{R})$ are closed subspaces of the Hilbert space $L^2(\mathbf{R})$ and each is the orthogonal complement of the other.

Definition 5.2. Given $g \in H^2_+(\mathbf{R})$, $a > 1$, and $b > 0$, we say that (g, a, b) **generates an affine frame** for $H^2_+(\mathbf{R})$ if $\{D_{a^n} T_{mb} g\}_{m,n \in \mathbf{Z}}$ is a frame for $H^2_+(\mathbf{R})$. The function g is referred to as the **mother wavelet**. The numbers a, b together are the **frame parameters**, a being the **dilation parameter**, and b the **shift parameter**.

We make similar definitions for affine frames for $H^2_-(\mathbf{R})$ and $L^2(\mathbf{R})$, and remark that it is sometimes necessary to take two mother wavelets in order to form a frame for $L^2(\mathbf{R})$ (cf. Theorem 5.4).

Theorem 5.3 [DGM]**.** *Let $g \in L^2(\mathbf{R})$ be such that* $\operatorname{supp}(\hat{g}) \subset [l, L]$*, where* $0 \le l < L < \infty$*, and let $a > 1$ and $b > 0$ be such that:*

(1) *There exist constants A, B such that*

$$0 \ < \ A \ = \ \operatorname*{ess\,inf}_{\gamma \ge 0} \sum_n |\hat{g}(a^n \gamma)|^2 \ \le \ \operatorname*{ess\,sup}_{\gamma \ge 0} \sum_n |\hat{g}(a^n \gamma)|^2 \ = \ B \ < \ \infty;$$

(2) $(L - l) \le 1/b$.

Then $\{D_{a^n} T_{mb} g\}$ is a frame for $H^2_+(\mathbf{R})$ with bounds $b^{-1}A, b^{-1}B$.

Proof. Fix $n \in \mathbf{Z}$. Then by condition (2), the function $D_{a^n} \hat{f} \cdot \bar{\hat{g}}$ is supported in $I = [l, l + 1/b]$, which is an interval of length $1/b$. The Plancherel formula for Fourier series therefore implies that

$$\begin{aligned}
\sum_n \sum_m |\langle f, D_{a^n} T_{mb} g \rangle|^2 \ &= \ \sum_n \sum_m |\langle D_{a^n} \hat{f} \cdot \bar{\hat{g}}, E_{-mb} \rangle|^2 \\
&= \ \sum_n b^{-1} \int_I |D_{a^n} \hat{f}(\gamma) \cdot \overline{\hat{g}(\gamma)}|^2 \, d\gamma \\
&= \ \sum_n b^{-1} \int_0^\infty |\hat{f}(\gamma)|^2 \, |\hat{g}(a^n \gamma)|^2 \, d\gamma \\
&= \ b^{-1} \int_0^\infty |\hat{f}(\gamma)|^2 \cdot \sum_n |\hat{g}(a^n \gamma)|^2 \, d\gamma,
\end{aligned}$$

from which the result follows. □

A similar theorem can be formulated for $H^2_-(\mathbf{R})$. It is easy to see that if we combine a frmae for $H^2_+(\mathbf{R})$ with one for $H^2_-(\mathbf{R})$ then we obtain a frame for $L^2(\mathbf{R})$.

Theorem 5.4. *Let $g_1, g_2 \in L^2(\mathbf{R})$ satisfy* $\operatorname{supp}(\hat{g}_1) \subset [-L, -l]$ *and* $\operatorname{supp}(\hat{g}_2) \subset [l, L]$*, where* $0 \le l < L < \infty$*, and let $a > 1$, $b > 0$ be such that:*

(1) *There exist constants A, B such that*

$$\begin{aligned}
0 \ < \ A \ = \ &\min\Big\{ \operatorname*{ess\,inf}_{\gamma \le 0} \sum_n |\hat{g}_1(a^n \gamma)|^2, \operatorname*{ess\,inf}_{\gamma \ge 0} \sum_n |\hat{g}_2(a^n \gamma)|^2 \Big\} \\
&\le \ \max\Big\{ \operatorname*{ess\,sup}_{\gamma \le 0} \sum_n |\hat{g}_1(a^n \gamma)|^2, \operatorname*{ess\,sup}_{\gamma \ge 0} \sum_n |\hat{g}_2(a^n \gamma)|^2 \Big\} \ = \ B \ < \ \infty;
\end{aligned}$$

(2) $(L - l) \le 1/b$.

Then the collection of functions $\{D_{a^n}T_{mb}g_1, D_{a^n}T_{mb}g_2\}$ is a frame for $L^2(\mathbf{R})$ with bounds $b^{-1}A, b^{-1}B$.

In analogy with Theorem 3.7, the following theorem gives a condition on g so that (g, a, b) generates an affine frame for $L^2(\mathbf{R})$ for some frame parameters. In particular, g need not have a compactly supported Fourier transform. Note that unlike Thoerem 5.4, this theorem requires only one mother wavelet.

THEOREM 5.5 [D1]. *Assume $g \in L^2(\mathbf{R})$ and $a > 1$ satisfy:*

(1) *There exist numbers A, B such that*

$$0 \;<\; A \;=\; \operatorname*{ess\,inf}_{\gamma\in\mathbf{R}} \sum_n |\hat{g}(a^n\gamma)|^2 \;\le\; \operatorname*{ess\,sup}_{\gamma\in\mathbf{R}} \sum_n |\hat{g}(a^n\gamma)|^2 \;=\; B \;<\; \infty;$$

(2) $\lim\limits_{b\to 0} \sum\limits_{k\neq 0} \beta(k/b)^{1/2}\beta(-k/b)^{1/2} = 0,$ *where*

$$\beta(s) \;=\; \operatorname*{ess\,sup}_{|\gamma|\in[1,a]} \sum_n |\hat{g}(a^n\gamma)|\,|\hat{g}(a^n\gamma - s)|.$$

Then there exists a number $b_0 > 0$ such that $\{D_{a^n}T_{mb}g\}$ is a frame for $L^2(\mathbf{R})$ for each $0 < b < b_0$.

When $a = 2$ an improved version of Theorem 5.5 holds in which the function β is replaced by a new function β_1 which takes into account possible cancellations which may arise from the phase portion of $\hat{g}$ and which are lost in the function β. This improved theorem is especially useful in analyzing the Meyer wavelet (described below). The theorem is due to Tchamitchian.

THEOREM 5.6 [D1]. *Let $g \in L^2(\mathbf{R})$ and $a = 2$ satisfy the hypotheses of Theorem 5.5, and assume $b > 0$. If $\{D_{2^n}T_{mb}g\}$ is a frame for $L^2(\mathbf{R})$ with frame bounds A', B' then*

$$A' \;\ge\; b^{-1}\left(A \;-\; 2\sum_{l=0}^{\infty} \beta_1(\tfrac{2l+1}{b})^{1/2}\beta_1(-\tfrac{2l+1}{b})^{1/2}\right)$$

and

$$B' \;\le\; b^{-1}\left(B \;+\; 2\sum_{l=0}^{\infty} \beta_1(\tfrac{2l+1}{b})^{1/2}\beta_1(-\tfrac{2l+1}{b})^{1/2}\right),$$

where

$$\beta_1(s) \;=\; \operatorname*{ess\,sup}_{\gamma\in\hat{R}} \sum_m \left|\sum_{j\ge 0} \hat{g}(2^{m+j}\gamma)\;\bar{\hat{g}}\big(2^j(2^m\gamma + s)\big)\right|$$

and A, B are as in Theorem 5.5.

EXAMPLE 5.7. Let $g_1 = \chi_{(-2,-1]}$ and $g_2 = \chi_{[1,2)}$, and take $a = 2$ and $b = 1$. Then $\{D_{2^n}T_m g_1, D_{2^n}T_m g_2\}$ is a tight affine frame for $L^2(\mathbf{R})$, and in fact is an orthonormal basis for $L^2(\mathbf{R})$. Note, however, that the elements of this orthonormal basis are not smooth. In the next example we discuss the Meyer wavelet, which is a C^∞ function that generates an affine orthonormal basis for $L^2(\mathbf{R})$.

As we saw in Section 4, a W-H frame forms a basis for $L^2(\mathbf{R})$ if and only if $ab = 1$. Moreover, W–H frames for this critical value are composed of functions which are either not smooth or do not decay quickly. Y. Meyer showed that a very different situation holds for the affine case when he exhibited a C^∞ function with compactly supported Fourier transform which generates an affine orthonormal basis for $L^2(\mathbf{R})$.

DEFINITION 5.8 [D2]. The **Meyer wavelet** is the function $\psi \in L^2(\mathbf{R})$ defined by:

$$\hat{\psi}(\gamma) \;=\; e^{i\gamma/2}\omega(|\gamma|),$$

where

$$\omega(\gamma) \;=\; \begin{cases} 0, & \gamma \le 1/3; \\ \sin\frac{\pi}{2}v(3\gamma-1), & 1/3 \le \gamma \le 2/3; \\ \cos\frac{\pi}{2}v(\frac{3\gamma}{2}-1), & 2/3 \le \gamma \le 4/3; \\ 0, & \gamma \ge 4/3; \end{cases}$$

and $v \in C^\infty(\hat{R})$ is such that $v(\gamma) = 0$ for $\gamma \le 0$ and $\gamma \ge 1$, $0 \le v(\gamma) \le 1$ for $\gamma \in [0,1]$, and $v(\gamma) + v(1-\gamma) = 1$ for $\gamma \in [0,1]$.

The last condition on v is crucial in showing that ψ generates an orthonormal basis for $L^2(\mathbf{R})$. Note that $\hat{\psi}$ is compactly supported and lies in both the negative and positive portions of the real line. Therefore, we should only need dilations and translations of ψ in order to form a frame for $L^2(\mathbf{R})$, not two functions ψ_1, ψ_2 as in Theorem 5.4 or Example 5.7.

LEMMA 5.9. *Let ψ be as above, and let β_1 be as in Theorem 5.6. Then*

(1) $\|\psi\|_2 = 1$.
(2) $\sum_n |\hat{\psi}(2^n\gamma)|^2 \equiv 1$.
(3) $\beta_1(k) = 0$ *for every odd* $k \in \mathbf{Z}$.

From Theorem 5.6 we therefore have that ψ generates a tight affine frame for $L^2(\mathbf{R})$ for the parameters $a = 2, b = 1$. As pointed out before, ψ actually generates an orthonormal basis for $L^2(\mathbf{R})$. Meyer, Lemarie, Daubechies, et al., have developed the concept of "multiscale analysis" to understand more fully the phenomena exhibited by the Meyer wavelet [D2; LM].

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