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Polynomial Reproduction by Refinable Functions

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ABSTRACT In this paper we give an expository review of the problem of reproducing polynomials from integer translates of a compactly supported refinable function vector. This property, which is called the accuracy of the function vector, is important for the construction of wavelet bases and is related to order of approximation, smoothness and other properties.

1 Introduction

In this paper, we will study the question of when a compactly supported function can exactly reproduce polynomials as linear combinations of its integer translates. We will show that to each compactly supported function f we can associate a maximum non-negative integer n such that f reproduces all the polynomials of degree less than n. This number n is the *accuracy* of f.

Accuracy has played an important role in both approximation theory and in wavelet theory. In approximation theory, it is closely related to the approximation properties of shift invariant spaces, often generated by splines or finite elements. In wavelet theory, one of the most successful and systematic ways of constructing smooth, compactly supported, orthonormal wavelet bases for $\mathcal{L}^2(\mathbf{R})$ is based on the factorization of a 2π-periodic *symbol* which determines a *scaling function* [Dau92]. This factorization of the symbol is closely related to the accuracy of the scaling function. Each

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scaling function satisfies a *dilation equation* of the type

$$f(x) \;=\; \sum_k c_k\, f(2x-k). \tag{6.1}$$

If the scaling function has accuracy n, then the corresponding wavelet will have n zero moments. This implies that the wavelet transform of the smooth part of a signal will yield only small coefficients, which leads to good compression ratios in applications involving signal compression. It is also the key to characterizing spaces of smooth functions via wavelet transforms.

Equation (6.1) also plays a key role in the context of subdivision schemes [CDM91], where it is known as a *refinement equation*, and the solution f is a *refinable function*. Accuracy is necessary for a refinable function to be smooth, although it is not sufficient.

Generalizations of the refinement equation (6.1) allow functions with domain $\mathbf{R}^d$ and a general dilation matrix A in place of the dilation factor 2, or allow multiple functions $f_1, \dots, f_r$ to each be written as linear combinations of translated and dilated versions of all of the f_i. Refinable functions or their generalizations to higher dimensions or multiple functions may also be viewed as a particular case of self-similar functions as studied in fractal geometry, e.g., [Baj57], [Dub85], [Bar86], [CM98].

The problem of determining when a solution to equation (6.1) exists and of determining the smoothness properties of the resulting solution has been studied by many researchers. Usually, the goal is to characterize these properties in terms of conditions on the coefficients c_k. A short and incomplete list of references, might include [CH94], [Dau88], [DL91], [DL92], [Dub86], [DGL91], [Eir92], [LW95], [MP89], [Rio92], [Vil92], [Wan95], and others.

Our objective in this paper is to give an accessible and self-contained account of the characterization of the accuracy of refinable functions in a way that introduces techniques used in the study of generalized forms of the refinement equation. We do not intend to give a complete survey of all results on accuracy; rather, we focus on that part of accuracy that is most relevant to wavelet theory. The history of this problem is long and convoluted; we will mention only a few papers that bear directly on our discussion, and provide in the references some of the additional papers that are related to this problem.

The classical study of the order of approximation by integer translates of a single function begins with the *Strang–Fix conditions*, first derived in [Sch46] and [SF73]. For a refinable function satisfying some extra hypotheses, the Strang–Fix conditions reduce to a finite set of equations on the coefficients c_k, called the *sum rules*. In approximation theory, it is important to study order of approximation assuming only very weak extra conditions on f; we refer to [dBVR94a] and related references for this type of approach. In wavelet theory, it is natural to assume much stronger conditions on f, such as compact support, and linear independence or even orthogonality of the translates $\{f(x+k)\}$. We therefore restrict our attention in this paper to the case of compactly supported functions with linearly independent translates. These hypotheses also allow us in some cases to present sharper results or simpler proofs, and allow us to avoid convergence problems. However, many of the results are still valid without the compactness constraint if we impose instead a certain decay of the refinable function.

One generalization of the refinement equation is to allow multiple functions. In wavelet theory, this leads to *multiwavelet bases*, which can simultaneously combine several desirable properties that cannot be simultaneously realized by classical wavelet bases, such as symmetry and compact support [DGHM96]. Plonka [Plo97] and Heil, Strang, and Strela [SS94], [HSS96], independently characterized the conditions for accuracy in the one-dimensional, multi-function setting, with later insights by Jia, Riemenschneider, and Zhou [JRZ96].

In one dimension with a single function, the sum rules are equivalent to a factorization of the 2π-periodic symbol defined by $m_0(\omega) = \sum_{k=0}^{N} c_k e^{-i\omega k}$. However, such factorizations need not exist in higher dimensions. Instead, in the construction of smooth wavelets or multiwavelets in higher dimensions, it may be preferable to start directly from the sum rules to construct an appropriate choice of coefficients $\{c_k\}$ which lead to a scaling function which is both sufficiently smooth and is orthogonal to its lattice translates. For example, Belogay and Wang [BW97] used such an approach to induce a partial factorization the symbol resulting from certain choices of coefficients, thereby leading to the construction of two-dimensional, non-separable, orthogonal wavelets using a special dilation matrix of determinant 2. On the other hand, many time-domain conditions for accuracy do carry over to the higher-dimensional, single-function case, although with much greater technical difficulties. The papers [CHM96], [CHM97] extend the characterization of accuracy to the case of higher dimensions combined with multiple functions. This work was applied to construct non-separable multiwavelets in $\mathbf{R}^2$ [CHM98].

For clarity, we have chosen to describe in complete detail only the simplest case of one function in one variable. This appears in Section 2. Many of the statements of the results translate easily to the generalized cases, but the proofs are usually much more technical. We therefore have often presented proofs of the one-dimensional, single-function results which illustrate the ideas behind the proofs of the more general cases without the obscuring technical details. However, since some results are not valid in the general case, or need extra hypothesis, or are simply unknown, we summarize the general cases in Sections 3 and 4, stating the results but omitting the proofs, referring instead to [CHM96], [CHM97] for complete details. Specifically, in Section 3 we discuss some of the difficulties that appear when considering an arbitrary dilation matrix in higher dimensions, and in Section 4 we present the statements of these results in the general context of higher dimensions and multiple functions. This setting of course includes most of the previous results as particular cases. Finally, in Section 5 we briefly discuss some of the relationships that appear between accuracy and order of approximation or smoothness.

We emphasize that, because of the philosophy of the paper, we have often combined into one statement results that summarize the work of many independent authors. Moreover, many of the ideas presented in this paper have arisen in multiple contexts and have several independent proofs by numerous authors. Thus it is nearly impossible to give complete individual credit for the results. Instead, we have attempted to give a simple, self-contained introduction to the problem of accuracy. We present our own personal viewpoint and our own personal approaches to the proofs, but include in our bibliography numerous references that include related results.

2 One Dimension with a Single Function

Throughout this section we will only consider functions that are defined on the real line $\mathbf{R}$ and take values in the complex plane $\mathbf{C}$. Sequences and series with unspecified limits are understood to be indexed by the set of integers $\mathbf{Z}$.

Definition 1 Let f be a compactly supported function. Then a function g is *reproducible by integer translates of* f (or simply *reproducible by* f) if there exist complex scalars $\{\alpha_k\}_{k\in\mathbf{Z}}$ such that

$$g(x) \;=\; \sum_k \alpha_k\, f(x+k). \tag{6.2}$$

If we define the *shift-invariant space generated by* f to be

$$\mathcal{S}(f) \;=\; \left\{g\colon \mathbf{R}\to\mathbf{C} \;:\; \exists\, \alpha_k\in\mathbf{C} \text{ such that } g(x)=\sum_k \alpha_k\, f(x+k)\right\},$$

then g is reproducible by f if and only if $g\in\mathcal{S}(f)$.

We say that translates of f are *linearly independent* if for every choice of scalars $\alpha_k\in\mathbf{C}$ we have

$$\sum_k \alpha_k\, f(x+k)=0 \qquad\Longleftrightarrow\qquad \alpha_k=0 \;\text{ for every } k.$$

Throughout this paper, we will usually work with functions which satisfy the following "standard hypotheses":

$$\begin{array}{l} \text{a) } f \text{ is compactly supported, and} \\ \text{b) translates of } f \text{ are linearly independent.} \end{array} \tag{6.3}$$

We remark that if $f\in\mathcal{L}^2(\mathbf{R})$ is compactly supported and has independent translates, then the translates of f are a Riesz basis for the subspace of $\mathcal{L}^2(\mathbf{R})$ that they span [JW93], [Jia95]. As a consequence, there exist constants A, $B > 0$ such that:

$$\forall\, \omega \in \mathbf{R}, \quad A \;\le\; \sum_k |\hat{f}(\omega + 2k\pi)|^2 \;\le\; B, \tag{6.4}$$

where $\hat{f}(\omega) = \int f(x)\, e^{-ix\omega}\, dx$ is the Fourier transform of f.

2.1 Accuracy

In this section we study the question of the reproducibility of polynomials from integer translates of a general compactly supported (but not necessarily refinable) function f. That is, given f, we seek to determine what conditions must be imposed on f in order that all polynomials up to a given degree can be exactly reproduced from integer translates of f.

Let Π_n be the space of all polynomials of degree less or equal than n, i.e.,

$$\Pi_n \;=\; \left\{ p(x) = \sum_{k=0}^{n} a_k x^k : a_k \in \mathbf{C} \right\}.$$

We show in Proposition 1 below that if a polynomial p is reproducible by f, then every polynomial of smaller degree is also reproducible by f. Therefore, a natural question is whether there is a maximum number n such that $\Pi_n \subset \mathcal{S}(f)$, or whether there exists an f that reproduces every polynomial. We show in Proposition 4 that for compactly supported functions there is always a maximum n, and that this n depends on the diameter of the support of f.

These remarks motivate the following definition.

Definition 2 Let f be a compactly supported function. Then f has *accuracy* n if n is the maximum integer such that $\Pi_{n-1} \subset \mathcal{S}(f)$.

We will also discuss an interesting property related to the scalars used to reproduce a given polynomial p from translates of f. Suppose that f satisfies (6.3). If a polynomial p can be written as $p(x) = \sum \alpha_k\, f(x+k)$, then we show in Proposition 2 that $\alpha_k = u_p(k)$ where u_p is a polynomial of the same degree as p. Hence, $p(x) = \sum u_p(k)\, f(x+k)$. Furthermore, we show in Proposition 3 that $\frac{dp}{dx}(x) = \sum \frac{du_p}{dx}(k)\, f(x+k)$.

As a consequence of these remarks, if f has accuracy n then the map $\mathbf{u}\colon \Pi_{n-1} \to \Pi_{n-1}$ defined by $\mathbf{u}(p) = u_p$ satisfies:

i) $\mathbf{u}$ is linear,

ii) $\mathbf{u}$ preserves degree, and

iii) $\mathbf{u}$ commutes with $\frac{d}{dx}$.

We now present the proofs of the above remarks.

Proposition 1 *Let f be a compactly supported function. If $p \in \mathcal{S}(f)$ is a polynomial of degree m, then $\Pi_m \subset \mathcal{S}(f)$.*

Proof:
Since polynomials of different degrees are linearly independent and since $\mathcal{S}(f)$ is a linear space, it is enough to prove the following statement:

> If q is a polynomial of degree $s > 0$ such that $q \in \mathcal{S}(f)$, then there exists a polynomial u of degree $s-1$ such that $u \in \mathcal{S}(f)$.

Let q be any polynomial of degree $s > 0$ such that $q \in \mathcal{S}(f)$. Without loss of generality, assume that $q(x) = x^s + \sum_{r=0}^{s-1} b_r x^r$. Since q is reproducible by f, there exists scalars α_k^s such that

$$q(x) \;=\; \sum_k \alpha_k^s\, f(x+k).$$

Then we have that

$$q(x+1) \;=\; \sum_k \alpha_k^s\, f(x+1+k) \;=\; \sum_k \alpha_{k-1}^s\, f(x+k),$$

and also that

$$\begin{aligned} q(x+1) \;&=\; (x+1)^s \;+\; \sum_{r=0}^{s-1} b_r (x+1)^r \\ &=\; x^s \;+\; (s+b_{s-1})x^{s-1} \;+\; \sum_{r=0}^{s-2} (b_r + b_r')x^r, \end{aligned}$$

where

$$b_r' \;=\; \binom{s}{r} \;+\; \sum_{t=r+1}^{s-1} b_t \binom{t}{r}.$$

Therefore, if we set

$$u(x) \;=\; q(x+1) - q(x) \;=\; sx^{s-1} \;+\; \sum_{r=1}^{s-2} b_r' x^r,$$

then u is a polynomial of degree $s-1$ which satisfies

$$u(x) \;=\; \sum_k (\alpha_{k-1}^s - \alpha_k^s)\, f(x+k).$$

Hence $u \in \mathcal{S}(f)$, which completes the proof. ∎

The preceeding result plays an important role in the proof of the following proposition.

Proposition 2 *Assume that f satisfies the standard hypotheses* (6.3)*. Let $p \in \mathcal{S}(f)$ be a polynomial of degree m, and write $p(x) = \sum \alpha_k\, f(x+k)$. Then there exists a polynomial $u_p(x)$ of degree m such that $\alpha_k = u_p(k)$.*

Proof:

Suppose first that $p(x) = x^m$ and $p \in \mathcal{S}(f)$. Then by Proposition 1, we must have $\Pi_m \subset \mathcal{S}(f)$. Hence, for each $0 \le r \le m$, there exist scalars α_k^r such that

$$x^r \;=\; \sum_k \alpha_k^r\, f(x+k). \tag{6.5}$$

Then, for each $\ell \in \mathbf{Z}$ we have by the binomial theorem that

$$\begin{aligned}
\sum_k \alpha_{k+\ell}^n\, f(x+k) \;&=\; (x-\ell)^m \\
&=\; \sum_{r=0}^{m} \binom{m}{r} (-\ell)^{m-r}\, x^r \\
&=\; \sum_{r=0}^{m} \binom{m}{r} (-\ell)^{m-r} \sum_k \alpha_k^r\, f(x+k) \\
&=\; \sum_k \left(\sum_{r=0}^{m} \binom{m}{r} (-\ell)^{m-r} \alpha_k^r \right) f(x+k).
\end{aligned}$$

Since the translates of f are independent, it follows that

$$\forall\, k \in \mathbf{Z}, \quad \alpha_{k+\ell}^m \;=\; \sum_{r=0}^{m} \binom{m}{r} (-\ell)^{m-r}\, \alpha_k^r.$$

Define $u_p(\ell)$ to be the $k=0$ case of this equation, i.e.,

$$u_p(\ell) \;=\; \alpha_\ell^m \;=\; \sum_{r=0}^{m} \binom{m}{r} (-\ell)^{m-r}\, \alpha_0^r \tag{6.6}$$

Then u_p is a polynomial in ℓ, and u_p has degree m if and only if $\alpha_0^0 \ne 0$. Now, by (6.6) for the case $m=0$, we have $\alpha_\ell^0 = \alpha_0^0$ for every ℓ, so $1 = \alpha_0^0 \sum f(x+k)$ by (6.5). The independence of the translates of f therefore implies that $\alpha_0^0 \ne 0$, and hence that u_p has degree m.

This proves the proposition for the case that $p(x) = x^m$. Now suppose that p is any polynomial in $\mathcal{S}(f)$ with degree m. Then by Proposition 1, we have $\Pi_m \subset \mathcal{S}(f)$. The result then follows by writing $p(x) = \sum_{s=0}^m a_s x^s$ and applying the results above to each individual term x^s. ■

Corollary 1 *Assume that f satisfies the standard hypotheses* (6.3)*. If f has accuracy n, then the map $\mathbf{u}\colon \Pi_{n-1} \to \Pi_{n-1}$ defined by $\mathbf{u}(p) = u_p$ is a linear bijection of Π_{n-1} onto itself.*

Proposition 3 *Assume that f is compactly supported, let $s \geq 0$, and suppose that there is a polynomial u such that*

$$x^s \;=\; \sum_k u(k)\, f(x+k).$$

Then for each $0 \leq t \leq s$, we have

$$x^t \;=\; C_t \sum_k u^{(s-t)}(k)\, f(x+k),$$

where

$$C_t \;=\; (-1)^{s-t}\, \frac{t!}{s!}$$

and $u^{(s-t)}$ is the $(s-t)$th derivative of u.

Proof:

Note first that $(x+\ell)^s = \sum_k u(k-\ell)\, f(x+k)$ for each $\ell \in \mathbf{Z}$. For each fixed x, define

$$g_x(y) \;=\; (x+y)^s \qquad \text{and} \qquad h_x(y) \;=\; \sum_k u(k-y)\, f(x+k).$$

Then g_x and h_x are both polynomials in the variable y. Moreover, $g_x(\ell) = h_x(\ell)$ for every integer ℓ, so we must have $g_x(y) = h_x(y)$ for every $y \in \mathbf{R}$. Thus, for every x and y we have

$$(x+y)^s \;=\; \sum_k u(k-y)\, f(x+k). \tag{6.7}$$

By taking the derivative with respect to y on both sides of (6.7) and then setting $y = 0$, we find that

$$s x^{s-1} \;=\; -\sum_k u'(k)\, f(x+k).$$

The proof then follows by repetition of this argument. ∎

Now let $\delta(K)$ denote the diameter of a compact set K, let $[a]$ denote the integer part of a real number a, and let Π denote the set of all polynomials with complex coefficients. We then have the following proposition.

Proposition 4 *Let f be a compactly supported function. Then the set of polynomials reproducible by f is a finite-dimensional subspace of* Π, *and*

$$\dim(\Pi \cap \mathcal{S}(f)) \leq [\delta(\operatorname{supp}(f))] + 1.$$

Proof:

If $f(x) = 0$ a.e., then the result is trivial. Otherwise, let I be the minimal interval containing the support of f, and let m be any natural number such that $|I| < m$. First we show that no polynomial of degree m can be reproduced by f.

We proceed by contradiction. Assume that there exists a polynomial of degree m which is reproducible by f. Then, by Proposition 1, the m linearly independent polynomials $(x - x_0), \ldots, (x - x_0)^m$ can all be reproduced by f. That is, there exist scalars α_k^s such that

$$(x - x_0)^s \;=\; \sum_k \alpha_k^s \, f(x + k), \qquad 1 \leq s \leq m. \tag{6.8}$$

Since $|I| < m$, there exists an $x_0 \in I$ and an integer ℓ_0 such that $f(x_0+\ell)$ can be nonzero for at most $\ell \in \{\ell_0 + 1, \ldots, \ell_0 + m\}$. Moreover, we can find a ball $B(x_0, r)$ such that if $x \in B(x_0, r)$ then $f(x + \ell)$ can be nonzero for at most $\ell \in \{\ell_0 + 1, \ldots, \ell_0 + m\}$. Hence, from (6.8),

$$\forall\, x \in B(x_0, r), \quad (x - x_0)^s \;=\; \sum_{k=1}^{m} \alpha_k^s \, f(x + \ell_0 + k).$$

Define vectors $V = (f(x_0 + \ell_0 + 1), \ldots, f(x_0 + \ell_0 + m)) \in \mathbf{C}^m$ and $\alpha^s = (\overline{\alpha_1^s}, \ldots, \overline{\alpha_m^s}) \in \mathbf{C}^m$. Then the dot product of V and α^s is

$$\langle V, \alpha^s \rangle \;=\; \sum_{k=1}^{m} \alpha_k^s \, f(x_0 + \ell_0 + m) \;=\; (x_0 - x_0)^s \;=\; 0, \qquad 1 \leq s \leq m.$$

Since $V \neq 0$, the m vectors $\{\alpha^1, \ldots, \alpha^m\}$ must therefore be linearly dependent in $\mathbf{C}^m$. Hence, there exist scalars λ_s, not all zero, such that $\sum_{s=1}^m \overline{\lambda_s}\, \alpha^s = 0$. Therefore, if we define a polynomial $P(x)$ by

$$P(x) \;=\; \sum_{s=1}^{m} \lambda_s (x - x_0)^s,$$

then for $x \in B(x_0, r)$ we have

$$\begin{aligned} P(x) \;&=\; \sum_{s=1}^{m} \lambda_s \sum_{k=1}^{m} \alpha_k^s \, f(x + \ell_0 + k) \\ &=\; \sum_{k=1}^{m} \left(\sum_{s=1}^{m} \lambda_s \alpha_k^s \right) f(x + \ell_0 + k) \;=\; 0. \end{aligned}$$

Hence we must have $P \equiv 0$, from which it follows that $(x-x_0), \ldots, (x-x_0)^m$ are linearly dependent polynomials, which is a contradiction.

Now, if we choose m such that $m-1 \le |I| < m$, then

$$\dim(\Pi \cap \mathcal{S}(f)) \le m = [|I|] + 1 = [\delta(\mathrm{supp}(f))] + 1.$$

■

2.2 A Fourier Characterization of Accuracy

In this section, we will study how polynomial reproducibility is reflected in the Fourier transform of f. The resulting well-known conditions for accuracy are usually called the *Strang–Fix* conditions, first discussed in [Sch46] and [SF73].

Definition 3 A compactly supported function $f \in \mathcal{L}^2(\mathbf{R})$ satisfies the *Strang–Fix conditions of order* n if

$$\hat{f}(0) \neq 0 \quad \text{and} \quad \hat{f}^{(s)}(2k\pi) = 0, \ \forall\, k \in \mathbf{Z} - \{0\}, \ \ 0 \le s \le n-1. \tag{6.9}$$

The next theorem shows the equivalence between the Strang–Fix conditions and accuracy. The link between the time and frequency domains is provided by the Poisson Summation Formula. The proof here is adapted from [SF73] and is included because of its clarity and elegance.

Theorem 1 *Assume that* $f \in C^1(\mathbf{R})$ *satisfies the standard hypotheses* (6.3). *Then the following statements are equivalent:*

i) f *satisfies the Strang–Fix conditions of order* n.

ii) f *has accuracy* n.

Proof: We will apply the Poisson Summation Formula in the form

$$\sum_k g_s(k) \ = \ \sum_k \hat{g}_s(2k\pi) \tag{6.10}$$

to the functions $g_s(x) = x^s f(t+x)$ for $0 \le s \le n-1$. Note that

$$g_s(k) \ = \ k^s f(t+k),$$

and that

$$\begin{aligned} \hat{g}_s(\omega) &= i^s\, D^s(e^{i\omega t}\hat{f}(\omega)) \\ &= i^s \sum_{r=0}^{s} \binom{s}{r} i^r\, t^r \hat{f}^{(s-r)}(\omega)\, e^{i\omega t}. \end{aligned} \tag{6.11}$$

i) $\Rightarrow$ ii). Assume that f satisfies the Strang–Fix conditions of order n. Then we see from (6.11) that $\hat{g}_s(2k\pi) = 0$ for $k \neq 0$. On the other hand, for $k = 0$ we have

$$\hat{g}_s(0) \;=\; \sum_{r=0}^{s} \left[\binom{s}{r} \, i^{s+r} \hat{f}^{(s-r)}(0) \right] t^r .$$

This is a polynomial in t whose leading coefficient is $(-1)^s \hat{f}(0) \neq 0$. Hence, by (6.10),

$$\sum_k k^s f(t+k) \;=\; \sum_k g_s(k) \;=\; \sum_k \hat{g}_s(2k\pi) \;=\; \hat{g}_s(0)$$

is a polynomial in t of degree s. Hence $\mathcal{S}(f)$ contains at least one polynomial of degree $s = n-1$, and therefore by Proposition 1 contains all polynomials of degree less than n. Hence f has accuracy n.

ii) $\Rightarrow$ i). Assume that f has accuracy $n > 0$. We will use induction on s to show that (6.9) holds.

Consider first the case $s = 0$. By Proposition 2, we know that the constant polynomial can be reproduced using coefficients which are themselves constant. Hence we must have $\sum f(x+k) = c$ for some constant c, and this constant must be nonzero since translates of f are independent. Therefore, by (6.10) for the case $s = 0$,

$$\sum_k e^{i2k\pi t} \hat{f}(2k\pi) \;=\; \sum_k \hat{g}_0(2k\pi) \;=\; \sum_k g_0(k) \;=\; \sum_k f(t+k) \;=\; c.$$

This is only possible if $\hat{f}(0) = c$ and $\hat{f}(2k\pi) = 0$ for $k \neq 0$.

For the inductive step, assume that for some $0 \leq s \leq n-1$ we have

$$\hat{f}(0) \neq 0 \quad \text{and} \quad \hat{f}^{(r)}(2k\pi) = 0, \ \forall\, k \in \mathbf{Z} - \{0\}, \ 0 \leq r \leq s-1.$$

Then by (6.11),

$$\begin{aligned} \sum_k k^s f(t+k) \;&=\; \sum_k g_s(k) \\ &=\; \sum_k \hat{g}_s(2k\pi) \\ &=\; i^s \sum_{k\neq 0} \hat{f}^{(s)}(2k\pi)\, e^{i2k\pi t} \;+\; \sum_{r=1}^{s} \left[\binom{s}{r} \, i^{s+r} \hat{f}^{(s-r)}(0) \right] t^r . \end{aligned}$$

However, by Corollary 1 and our assumption of accuracy, we know that $\sum_k k^s f(t+k)$ is a polynomial in t of degree s. This is only possible if $\sum_{k\neq 0} \hat{f}^{(s)}(2k\pi)\, e^{i2k\pi t} \equiv 0$, which implies that $\hat{f}^{(s)}(2k\pi) = 0$ for $k \neq 0$. ∎

2.3 Refinable Functions

Definition 4 A compactly supported function f is *refinable* if there exists a sequence of complex numbers $\{c_k\}_{k\in\mathbf{Z}}$ such that f satisfies the *refinement equation*

$$f(x) \;=\; \sum_k c_k\, f(2x-k).$$

We will restrict our attention to the case where only finitely many c_k are nonzero. By translating f if necessary, we may without loss of generality assume that there exists a positive integer N such that $c_k = 0$ when $k \neq 0, ..., N$. That is, we can assume that the refinement equation has the form

$$f(x) \;=\; \sum_{k=0}^{N} c_k\, f(2x-k). \tag{6.12}$$

The *symbol* of this refinement equation is the 2π-periodic function

$$m_0(\omega) \;=\; \sum_{k=0}^{N} c_k\, e^{-i\omega k}.$$

If $f \in \mathcal{L}^2(\mathbf{R})$ is refinable, then $\hat{f}$ must satisfy

$$\hat{f}(2\omega) \;=\; \frac{1}{2} m_0(\omega)\, \hat{f}(\omega). \tag{6.13}$$

The motivation from wavelet theory for studying the accuracy of refinable functions is that wavelets associated with a multiresolution analysis are constructed from a refinable function called the *scaling function*. The wavelet inherits most of its properties from the scaling function. For example, if the scaling function has high accuracy then the multiresolution analysis will have good approximation properties.

The accuracy or other properties of refinable functions can be characterized in terms of the properties of the mask $\{c_k\}$ or in terms of properties of the symbol m_0. We will first present in Section 2.4 a Fourier characterization of accuracy for refinable functions, and then in Section 2.5 present a time-domain approach.

2.4 Strang–Fix Conditions for Refinable Functions

The next theorem shows that the Strang–Fix conditions for a refinable function are equivalent to the requirement that the symbol have zeros at π, i.e., that $m_0^{(s)}(\pi) = 0$ for $s = 0, \ldots, n-1$. Equivalently, this states that $\bigl(\frac{1+e^{i\omega}}{2}\bigr)^{n-1}$ is a factor of m_0.

Theorem 2 *Assume that $f \in \mathcal{L}^2(\mathbf{R})$ satisfies* (6.3)*. Then the following statements are equivalent:*

i) f satisfies the Strang–Fix conditions of order n.

ii) π is a zero of order n of m_0, i.e., $m_0^{(s)}(\pi) = 0$ for $0 \le s \le n-1$.

Proof:
i) $\Rightarrow$ ii). Assume that f satisfies the Strang–Fix conditions of order n. We will show that ii) holds by using induction on s.

Consider first the case $s = 0$. Since $\hat{f}(0) \ne 0$, it follows from (6.13) that $m_0(0) = 2$. Therefore, using (6.13) again and the fact that m_0 is 2π-periodic, we have that

$$\begin{aligned} \sum_k |\hat{f}(2k\pi)|^2 &= \sum_k \frac{1}{4}|m_0(k\pi)|^2\, |\hat{f}(k\pi)|^2 \\ &= \frac{1}{4}|m_0(0)|^2 \sum_k |\hat{f}(2k\pi)|^2 \;+\; \frac{1}{4}|m_0(\pi)|^2 \sum_k |\hat{f}(\pi+2k\pi)|^2 \\ &= \sum_k |\hat{f}(2k\pi)|^2 \;+\; \frac{1}{4}|m_0(\pi)|^2 \sum_k |\hat{f}(\pi+2k\pi)|^2. \qquad (6.14) \end{aligned}$$

Now, since f has independent translates, we have from equation (6.4) that $\sum |\hat{f}(\pi+2k\pi)|^2 > 0$. It therefore follows from (6.14) that $m_0(\pi) = 0$.

For the inductive step, assume that for some $0 \le s \le n-1$ we have

$$m_0^{(r)}(\pi) = 0, \qquad 0 \le r \le s-1.$$

Taking the sth derivative of (6.13), evaluating at $\omega = \pi + 2k\pi$ and using the facts that $\hat{f}^{(r)}(2k\pi) = 0$ and $m_0(\pi) = 0$, we have that

$$\forall\, k \in \mathbf{Z}, \quad m_0^{(s)}(\pi)\, \hat{f}(\pi+2k\pi) \;=\; 0.$$

Since $\sum |\hat{f}(\pi+2k\pi)|^2 > 0$, we must have $\hat{f}(\pi+2k\pi) \ne 0$ for some k, and therefore $m_0^{(s)}(\pi) = 0$.

ii) $\Rightarrow$ i). Assume that $m_0^{(s)}(\pi) = 0$ for $0 \le s \le n-1$. We will use induction on s to prove that the Strang–Fix conditions in (6.9) hold.

Consider first the case $s = 0$. Choose any $k \ne 0$, and write $k = 2^j \ell$ with $j \ge 0$ and ℓ odd. Iterating (6.13), we have

$$\hat{f}(2k\pi) \;=\; \hat{f}(2^{j+1}\pi\ell) \;=\; \frac{1}{2^{j+1}} m_0(2^j\pi\ell) \cdots m_0(2\pi\ell) m_0(\pi\ell) \hat{f}(\pi\ell).$$

However, m_0 is 2π-periodic, so $m_0(\pi\ell) = m_0(\pi) = 0$. Hence $\hat{f}(2k\pi) = 0$ for $k \ne 0$. On the other hand, we know that $\sum |\hat{f}(2k\pi)|^2 > 0$, so we must have $\hat{f}(0) \ne 0$.

For the inductive step, assume that for some $0 \le s \le n-1$ we have $\hat{f}^{(r)}(2k\pi) = 0$ for all $k \ne 0$ and $0 \le r \le s-1$. Taking the derivative of

(6.13), we find that

$$2^s \hat{f}^{(s)}(2k\pi) \;=\; \frac{1}{2}\sum_{r=0}^{s} \binom{s}{r} m_0^{(r)}(k\pi)\, \hat{f}^{(s-r)}(k\pi). \tag{6.15}$$

Now, if k is odd, then we have by hypothesis ii) that $m_0^{(r)}(k\pi) = m_0^{(r)}(\pi) = 0$ for $0 \le r \le s$, and therefore $\hat{f}^{(s)}(2k\pi) = 0$ by (6.15). On the other hand, if $k \neq 0$ is even then we have by the inductive hypothesis that $\hat{f}^{(s-r)}(k\pi) = 0$ for $0 < r \le s$. Hence (6.15) reduces for this case to

$$2^s \hat{f}^{(s)}(2k\pi) \;=\; \frac{1}{2} m_0(k\pi)\, \hat{f}^{(s)}(k\pi). \tag{6.16}$$

Therefore, if we write $k = 2^j \ell$ with $j > 0$ and ℓ odd, we can iterate (6.16) to obtain

$$(2^s)^{j+1} \hat{f}^{(s)}(2k\pi) \;=\; \frac{1}{2} m_0(2^j \ell\pi) \cdots m_0(\ell\pi) \hat{f}^{(s)}(\ell\pi) \;=\; 0,$$

since $m_0(\ell\pi) = m_0(\pi) = 0$. ∎

2.5 A Time-Domain Approach for Refinable Functions

In this section we again consider the accuracy of refinable functions, but we approach the question in the time domain rather than the frequency domain. We introduce a convenient matrix notation, which leads to the definition of a fundamental operator associated with the refinement equation. This operator is a bi-infinite matrix L whose entries are coefficients of the refinement equation, specifically,

$$L \;=\; [c_{2i-j}]_{i,j\in\mathbf{Z}} \;=\; \begin{bmatrix} \ddots & & & & & & & & & \\ \cdots & c_3 & c_2 & c_1 & c_0 & & & & & \\ & & \cdots & c_3 & c_2 & c_1 & c_0 & & & \\ & & & & \cdots & c_3 & c_2 & c_1 & c_0 & \\ & & & & & & & & & \ddots \end{bmatrix}.$$

Note that only finitely many entries of any given row or column of L are nonzero. Moreover, there is a double shift between the rows of L; thus L is a "downsampled Toeplitz operator" or a "two-slanted matrix."

For each $x \in \mathbf{R}$, let $F(x)$ be the infinite column vector with components $f(x+k)$, i.e.,

$$F(x) \;=\; [f(x+k)]_{k\in\mathbf{Z}} \;=\; \begin{bmatrix} \vdots \\ f(x-1) \\ f(x) \\ f(x+1) \\ \vdots \end{bmatrix}.$$

Note that for each given x, only finitely many components $f(x+k)$ can be nonzero, since f has compact support.

If f satisfies the refinement equation (6.12), then

$$\begin{aligned} L\,F(2x) &= [c_{2i-j}]_{i,j\in\mathbf{Z}}\;[f(2x+j)]_{j\in\mathbf{Z}} \\ &= \left[\sum_j c_{2i-j}\,f(2x+j)\right]_{i\in\mathbf{Z}} \\ &= \left[\sum_k c_k\,f(2x+2i-k)\right]_{i\in\mathbf{Z}} \;=\; [f(x+i)]_{i\in\mathbf{Z}} \;=\; F(x). \end{aligned}$$

The converse is also true, so the refinement equation (6.12) can be rewritten in the compact matrix form

$$L\,F(2x) \;=\; F(x). \tag{6.17}$$

In order to state a result giving necessary and/or sufficient conditions for a refinable function to have accuracy n, we need to introduce some notation. Given a finite list of scalars $\{v_0,\dots,v_{n-1}\}$, we will associate the polynomials

$$y_{[s]}(x) \;=\; \sum_{r=0}^{s}\binom{s}{r}(-x)^{s-r}\,v_r, \qquad 0\le s\le n-1. \tag{6.18}$$

Note that $y_{[s]}$ has degree s if and only if $v_0\ne 0$, and that $y_{[0]}(x)\equiv v_0$. By applying the binomial theorem, we obtain the following useful property of these polynomials:

$$y_{[s]}(x+y) \;=\; \sum_{t=0}^{s}\binom{s}{t}(-y)^{s-t}\,y_{[t]}(x). \tag{6.19}$$

Next, we place the evaluations of the polynomial $y_{[s]}$ at integers into an infinite row vector that we call $Y_{[s]}$, i.e.,

$$Y_{[s]} \;=\; (y_{[s]}(k))_{k\in\mathbf{Z}} \;=\; (\dots,\; y_{[s]}(-1),\; y_{[s]}(0),\; y_{[s]}(1),\; \dots). \tag{6.20}$$

With this notation, our time-domain characterization of accuracy is as follows.

Theorem 3 *Assume that $f\in\mathcal{L}^1(\mathbf{R})$ satisfies the standard hypotheses* (6.3). *If f is refinable and if $\hat f(0)\ne 0$, then the following statements are equivalent:*

a) f has accuracy n.

b) *There exist scalars* $\{v_0, \dots, v_{n-1}\}$ *such that* $v_0 \neq 0$ *and such that the infinite row vector* $Y_{[n-1]}$ *defined by* (6.18) *and* (6.20) *is a left eigenvector for* L *for the eigenvalue* $2^{-(n-1)}$, *i.e.,*

$$Y_{[n-1]} \;=\; 2^{n-1}\, Y_{[n-1]}\, L, \tag{6.21}$$

or equivalently,

$$y_{[n-1]}(\ell) \;=\; 2^{n-1} \sum_k y_{[n-1]}(k)\, c_{2k-\ell}, \qquad \ell \in \mathbf{Z}. \tag{6.22}$$

c) *There exist scalars* $\{v_0, \dots, v_{n-1}\}$ *such that* $v_0 \neq 0$ *and such that the infinite row vectors* $Y_{[s]}$ *defined by* (6.18) *and* (6.20) *are left eigenvectors for* L *for the eigenvalues* 2^{-s}, *i.e.,*

$$Y_{[s]} \;=\; 2^s\, Y_{[s]}\, L, \qquad 0 \leq s \leq n-1, \tag{6.23}$$

or equivalently,

$$y_{[s]}(\ell) \;=\; 2^s \sum_k y_{[s]}(k)\, c_{2k-\ell}, \quad 0 \leq s \leq n-1 \text{ and } \ell \in \mathbf{Z}. \tag{6.24}$$

d) *There exist scalars* $\{v_0, \dots, v_{n-1}\}$ *such that* $v_0 \neq 0$ *and such that*

$$\begin{aligned} v_s \;&=\; \sum_k \sum_{t=0}^{s} (-1)^{s-t} \binom{s}{t} (2k)^{s-t}\, 2^t\, v_t\, c_{2k} \\ &=\; \sum_k \sum_{t=0}^{s} (-1)^{s-t} \binom{s}{t} (2k+1)^{s-t}\, 2^t\, v_t\, c_{2k+1}. \end{aligned} \tag{6.25}$$

e) $\sum_k c_k = 2$ *and* $\sum_k (-1)^k\, k^s\, c_k = 0$ *for* $0 \leq s \leq n-1$.

Remark 1 The equations in statement e) of Theorem 3 are the *sum rules.*

Remark 2 Note that condition c) of Theorem 3 implies that in order for f to provide accuracy n, it is necessary that $1, \frac{1}{2}, \dots, \frac{1}{2^{n-1}}$ be eigenvalues of L. However, this eigenvalue condition alone is not sufficient for accuracy. The extra requirement needed is that the left eigenvectors $Y_{[s]}$ have the special polynomial structure described in (6.18) and (6.20). Moreover, it will be clear from the proof that if the numbers $\{v_0, \dots, v_{n-1}\}$ are scaled by the nonzero factor $v_0 \hat{f}(0)$, then $x^s = \sum y_{[s]}(k)\, f(x+k)$ for each $0 \leq s \leq n-1$. Thus, the components of the left eigenvectors $Y_{[s]}$ are precisely the scalars that are used to reproduce x^s from f.

Remark 3 Note that condition d) of Theorem 3 implies that the scalars $\{v_0, \ldots, v_{n-1}\}$ are entirely determined from the coefficients c_k by a finite set of finite linear equations. Further, these equations have a triangular form, i.e., once $v_0, \ldots, v_{s-1}$ have been found, the equations can be solved for v_s.

Proof of Theorem 3:

a) $\Rightarrow$ b). Assume that f has accuracy n. Then, by Proposition 2, there exist polynomials u_s of degree s such that

$$x^s \;=\; \sum_k u_s(k)\, f(x+k), \qquad 0 \le s \le n-1. \tag{6.26}$$

Set $v_s = u_s(0)$. Then the proof of Proposition 2, and equation (6.6) in particular, implies that

$$u_{n-1}(x) \;=\; \sum_{r=0}^{n-1} \binom{n-1}{r} (-x)^{n-1-r}\, v_r \;=\; y_{[n-1]}(x).$$

Since $y_{[n-1]} = u_{n-1}$ has degree $n-1$, we must therefore have $v_0 \neq 0$. Moreover, we can rewrite (6.26) for $s = n-1$ as

$$x^{n-1} \;=\; \sum_k y_{[n-1]}(k)\, f(x+k) \;=\; Y_{[n-1]}\, F(x). \tag{6.27}$$

Therefore, by using the refinement equation in the form $F(x) = L\,F(2x)$, we see that

$$\begin{aligned}
Y_{[n-1]}\, F(2x) \;&=\; (2x)^{n-1} && \text{by (6.27)}\\
&=\; 2^{n-1}\, x^{n-1} && \\
&=\; 2^{n-1}\, Y_{[n-1]}\, F(x) && \text{by (6.27)}\\
&=\; 2^{n-1}\, Y_{[n-1]}\, L\, F(2x) && \text{by (6.17).}
\end{aligned}$$

Now, both $Y_{[n-1]}\, F(2x)$ and $2^{n-1}\, Y_{[n-1]}\, L\, F(2x)$ are linear combinations of the translates $\{f(2x+k)\}_{k\in\mathbf{Z}}$. For example,

$$Y_{[n-1]}\, F(2x) \;=\; \sum y_{[n-1]}(k)\, f(2x+k).$$

Replacing x by $x/2$ and considering our assumption that the integer translates of f are independent, this implies that the coefficients of the linear combinations $Y_{[n-1]}\, F(x)$ and $2^{n-1}\, Y_{[n-1]}\, L\, F(x)$ must be equal, i.e., that $Y_{[n-1]} = 2^{n-1}\, Y_{[n-1]}\, L$.

b) $\Rightarrow$ c). Assume that there exist scalars $\{v_0, \ldots, v_{n-1}\}$ such that $v_0 \neq 0$ and $Y_{[n-1]} = 2^{n-1}\, Y_{[n-1]}\, L$. We must show that the equations in (6.24) are also satisfied.

Choose any $j, \ell \in \mathbf{Z}$. Then:

$$\begin{aligned}
&\sum_{s=0}^{n-1} \binom{n-1}{s} (-2j)^{n-1-s} \Big(2^s \sum_k y_{[s]}(k)\, c_{2k-\ell}\Big) \\
&\quad= \sum_{s=0}^{n-1} 2^{n-1} \binom{n-1}{s} (-j)^{n-1-s} \sum_k y_{[s]}(k)\, c_{2k-\ell} \\
&\quad= 2^{n-1} \sum_k \Big(\sum_{s=0}^{n-1} \binom{n-1}{s} (-j)^{n-1-s}\, y_{[s]}(k) \Big)\, c_{2k-\ell} \\
&\quad= 2^{n-1} \sum_k y_{[n-1]}(j+k)\, c_{2k-\ell} && \text{by (6.19)} \\
&\quad= 2^{n-1} \sum_k y_{[n-1]}(k)\, c_{2k-(2j+\ell)} \\
&\quad= y_{[n-1]}(2j+\ell) && \text{by hypothesis b)} \\
&\quad= \sum_{s=0}^{n-1} \binom{n-1}{s} (-2j)^{n-1-s}\, y_{[s]}(\ell) && \text{by (6.19).}
\end{aligned}$$

Since both the first and last lines of the formula above are polynomials in j, we must therefore have that $y_{[s]}(\ell) = 2^s \sum_k y_{[s]}(k)\, c_{2k-\ell}$ for $0 \le s \le n-1$ and $\ell \in \mathbf{Z}$. Thus c) holds.

c) $\Rightarrow$ a). Assume that there exist scalars $\{v_0, \dots, v_{n-1}\}$ such that $v_0 \ne 0$ and $Y_{[s]} = 2^s\, Y_{[s]}\, L$ for $0 \le s \le n-1$. We must show that f has accuracy n.

For each $0 \le s \le n-1$, define a function $G_{[s]}(x)$ by

$$G_{[s]}(x) \;=\; \sum_k y_{[s]}(k)\, f(x+k) \;=\; Y_{[s]}\, F(x). \tag{6.28}$$

Note that for each fixed x, only finitely many terms of the series in (6.28) are nonzero. Using the hypothesis $Y_{[s]} = 2^s\, Y_{[s]}\, L$ and the refinement equation in the form $L\, F(2x) = F(x)$, we have that

$$\begin{aligned}
G_{[s]}(2x) \;=\; Y_{[s]}\, F(2x) &= 2^s\, Y_{[s]}\, L\, F(2x) \\
&= 2^s\, Y_{[s]}\, F(x) \\
&= 2^s\, G_{[s]}(x).
\end{aligned} \tag{6.29}$$

Therefore $G_{[s]}(x)$ behaves similarly to x^s when dilated by 2. We will show that, in fact, there is a constant C independent of s such that $G_{[s]}(x) = Cx^s$ for $0 \le s \le n-1$.

We proceed by induction on s. Consider first the case $s = 0$. Since $y_{[0]}(k) = v_0$ for every k, the function $G_{[0]}$ is defined by the formula

$$G_{[0]}(x) \;=\; v_0 \sum_k f(x+k).$$

Hence, $G_{[0]}(x)$ is 1-periodic. Further, by equation (6.29) we have $G_{[0]}(2x) = G_{[0]}(x)$. Thus $G_{[0]}(x)$ satisfies

$$G_{[0]}(2x) \;=\; G_{[0]}(x) \qquad \text{and} \qquad G_{[0]}(x-\ell) \;=\; G_{[0]}(x), \quad \ell \in \mathbf{Z}. \tag{6.30}$$

Therefore, if we define a map $\tau\colon [0,1) \to [0,1)$ by

$$\tau(x) \;=\; 2x \;(\mathrm{mod}\ 1) \;=\; \begin{cases} 2x, & 0 \le x < 1/2, \\ 2x-1, & 1/2 \le x < 1, \end{cases}$$

then (6.30) implies that $G_{[0]}(\tau(x)) = G_{[0]}(x)$ for each $x \in [0,1)$. However, τ is ergodic mapping of $[0,1)$ into itself. It therefore follows from the Ergodic Theorem [Wal82, Theorem 1.6] that $G_{[0]}$ is constant a.e. on $[0,1)$. By periodicity, we therefore have $G_{[0]}(x) = C$ a.e. on $\mathbf{R}$. Moreover, we can evaluate this constant explicitly:

$$\begin{aligned} C \;=\; \int_0^1 G_{[0]}(x)\,dx \;&=\; v_0 \sum_k \int_0^1 f(x+k)\,dx \\ &=\; v_0 \int_{\mathbf{R}} f(x)\,dx \;=\; v_0 \hat{f}(0) \;\neq\; 0. \end{aligned}$$

For the inductive step, assume that for some $0 \le s \le n-1$ we have $G_{[t]}(x) = Cx^t$ a.e. for $0 \le t \le s-1$. Then:

$$\begin{aligned} G_{[s]}(x-\ell) \;&=\; \sum_k y_{[s]}(k)\, f(x-\ell+k) \\ &=\; \sum_k y_{[s]}(k+\ell)\, f(x+k) \\ &=\; \sum_k \sum_{t=0}^{s} \binom{s}{t} (-\ell)^{s-t}\, y_{[t]}(k)\, f(x+k) \qquad \text{by (6.19)} \\ &=\; \sum_{t=0}^{s} \binom{s}{t} (-\ell)^{s-t}\, G_{[t]}(x) \\ &=\; G_{[s]}(x) \;+\; \sum_{t=0}^{s-1} \binom{s}{t} (-\ell)^{s-t}\, Cx^t \qquad \text{by induction} \\ &=\; G_{[s]}(x) \;+\; C \sum_{t=0}^{s} \binom{s}{t} (-\ell)^{s-t}\, x^t \;-\; Cx^s \\ &=\; G_{[s]}(x) \;+\; C(x-\ell)^s \;-\; Cx^s \qquad \text{binomial theorem.} \end{aligned}$$

Therefore, if we define

$$H_{[s]}(x) \;=\; G_{[s]}(x) - Cx^s,$$

then we have shown that

$$H_{[s]}(x-\ell) \;=\; H_{[s]}(x), \quad \ell \in \mathbf{Z},$$

i.e., $H_{[s]}(x)$ is 1-periodic. Further, it follows from (6.29) and the statement $(2x)^s = 2^s x^s$ that

$$H_{[s]}(2x) \;=\; 2^s\, H_{[s]}(x).$$

The combination of the two preceeding equations implies that

$$H_{[s]}(\tau(x)) \;=\; 2^s\, H_{[s]}(x), \qquad x \in [0,1). \tag{6.31}$$

We will show that this implies that $H_{[s]}(x) \equiv 0$, which consequently implies the desired fact that $G_{[s]}(x) = Cx^s$ a.e.

Let $E \subset [0,1)$ be a set of positive measure on which $H_{[s]}$ is bounded, say $|H_{[s]}(x)| \le M$ for $x \in E$. Since τ is ergodic, we know from the Ergodic Theorem [Wal82, p. 35], that for almost every $x \in [0,1)$,

$$\lim_{n\to\infty} \frac{\#\{0 < k \le n : \tau^k(x) \in E\}}{n} \;=\; |E| \;>\; 0. \tag{6.32}$$

Fix any $x \in [0,1)$ such that (6.32) holds. Then there exists an increasing sequence $\{n_j\}_{j=1}^\infty$ of positive integers such that $\tau^{n_j}(x) \in E$ for each j. Hence for each j we have by (6.31) and the boundedness of $H_{[s]}$ that

$$M \;\ge\; |H_{[s]}(\tau^{n_j}(x))| \;=\; |(2^s)^{n_j}\, H_{[s]}(x)|.$$

Therefore we must have $H_{[s]}(x) = 0$ a.e. on $[0,1)$, and since $H_{[s]}$ is 1-periodic, it must therefore vanish a.e. on $\mathbf{R}$. Hence $G_{[s]}(x) = Cx^s$ a.e., which completes the proof.

a), b), c) $\Leftrightarrow$ d). The proof of these equivalences involves techniques that are essentially similar to those used in the arguments above, and therefore will be omitted.

c) $\Rightarrow$ e). Assume that there exist scalars $\{v_0, \dots, v_{n-1}\}$ such that $v_0 \ne 0$ and $Y_{[s]} = 2^s\, Y_{[s]}\, L$ for $0 \le s \le n-1$. To show that e) holds, we will proceed by induction on s.

For the case $s = 0$, note first that $y_{[0]}(x) = v_0$ for every x. Therefore, equation (6.24) for the case $s = 0$ implies that for each $\ell \in \mathbf{Z}$ we have

$$v_0 \;=\; y_{[0]}(\ell) \;=\; \sum_k y_{[0]}(k)\, c_{2k-\ell} \;=\; v_0 \sum_k c_{2k-\ell}. \tag{6.33}$$

Since $v_0 \ne 0$, it follows by setting $\ell = 0$ and $\ell = 1$ in (6.33) that $\sum c_{2k} = 1 = \sum c_{2k-1}$. Hence we have that $\sum c_k = 2$ and that $\sum (-1)^k\, c_k = 0$.

Assume now, inductively, that for some $0 \le s \le n-1$ we have

$$\sum_k (2k)^r\, c_{2k} \;=\; \sum_k (2k-1)^r\, c_{2k-1}, \qquad 0 \le r \le s-1.$$

We will compute $v_s = y_{[s]}(0)$ in two different ways. First,

$$\begin{aligned}
v_s &= y_{[s]}(0) \\
&= 2^s \sum_k y_{[s]}(k)\, c_{2k} && \text{by (6.24)} \\
&= 2^s \sum_k \sum_{t=0}^{s} \binom{s}{t} (-k)^{s-t}\, v_t\, c_{2k} && \text{by (6.18)} \\
&= \sum_k \sum_{t=0}^{s} \binom{s}{t} (-1)^{s-t}\, (2k)^{s-t}\, 2^t\, v_t\, c_{2k} \\
&= (-1)^s\, v_0 \sum_k (2k)^s\, c_{2k} \;+\; \sum_{t=1}^{s} \binom{s}{t} (-1)^{s-t}\, 2^t\, v_t \sum_k (2k)^{s-t}\, c_{2k}.
\end{aligned}$$

Second, making use of the identity $\binom{s}{t}\binom{t}{r} = \binom{s}{r}\binom{s-r}{t-r}$, we have

$$\begin{aligned}
v_s &= y_{[s]}(1-1) \\
&= \sum_{t=0}^{s} \binom{s}{t} y_{[t]}(1) && \text{by (6.19)} \\
&= \sum_{t=0}^{s} \binom{s}{t} 2^t \sum_k y_{[t]}(k)\, c_{2k-1} && \text{by (6.24)} \\
&= \sum_k \sum_{t=0}^{s} \binom{s}{t} 2^t \sum_{r=0}^{t} \binom{t}{r} (-k)^{t-r}\, v_r\, c_{2k-1} && \text{by (6.18)} \\
&= \sum_k \sum_{t=0}^{s} \sum_{r=0}^{t} \binom{s}{r} \binom{s-r}{t-r} (-2k)^{t-r}\, 2^r\, v_r\, c_{2k-1} \\
&= \sum_k \sum_{r=0}^{s} \binom{s}{r} \sum_{t=r}^{s} \binom{s-r}{t-r} (-2k)^{t-r}\, 2^r\, v_r\, c_{2k-1} && \text{interchange sums} \\
&= \sum_k \sum_{r=0}^{s} \binom{s}{r} (1-2k)^{s-r}\, 2^r\, v_r\, c_{2k-1} && \text{binomial theorem} \\
&= (-1)^s\, v_0 \sum_k (2k-1)^s\, c_{2k-1} \;+ \\
&\qquad \sum_{r=1}^{s} \binom{s}{r} (-1)^{s-r}\, 2^r\, v_r \sum_k (2k-1)^{s-r}\, c_{2k-1}.
\end{aligned}$$

The second terms in the last line of both of these calculations are equal by the inductive hypothesis. Further, v_0 is a nonzero scalar, so this implies that

$$\sum_k (2k)^s\, c_{2k} \;=\; \sum_k (2k-1)^s\, c_{2k-1},$$

which completes the induction.

e) $\Rightarrow$ c). Assume that e) holds. We will inductively define scalars v_s for $s = 0, \dots, n-1$ so that the polynomials $y_{[s]}$ satisfy (6.24).

Begin by setting $v_0 = 1$. Note that by hypothesis e), the numbers

$$m_{s,t} \;=\; (-1)^{s-t} \binom{s}{t} \sum_k (2k-\ell)^{s-t}\, c_{2k-\ell}$$

are independent of $\ell \in \mathbf{Z}$ when $0 \le s-t \le n-1$. Therefore, once $v_0, \dots, v_{s-1}$ have been defined, we can define v_s by the equation

$$v_s \;=\; 2^s\, v_s \;+\; \sum_{t=0}^{s-1} m_{s,t}\, 2^t\, v_t.$$

With this definition, we have for each $0 \le s \le n-1$ that

$$\begin{aligned} v_s \;&=\; 2^s\, v_s \sum_k c_{2k-\ell} \;+\; \sum_{t=0}^{s-1} (-1)^{s-t} \binom{s}{t} \sum_k (2k-\ell)^{s-t}\, c_{2k-\ell} \\ &=\; \sum_k \sum_{t=0}^{s} (-1)^{s-t} \binom{s}{t} (2k-\ell)^{s-t}\, 2^t\, v_t\, c_{2k-\ell}. \end{aligned}$$

Therefore,

$$\begin{aligned} & 2^s \sum_k y_{[s]}(k)\, c_{2k-\ell} \\ &\quad=\; 2^s \sum_k \sum_{r=0}^{s} \binom{s}{r} (-k)^{s-r}\, v_r\, c_{2k-\ell} \qquad\qquad \text{by (6.18)} \\ &\quad=\; \sum_k \sum_{r=0}^{s} \binom{s}{r} (-1)^{s-r}\, (2k-\ell+\ell)^{s-r}\, 2^r\, v_r\, c_{2k-\ell} \\ &\quad=\; \sum_k \sum_{r=0}^{s} \binom{s}{r} (-1)^{s-r} \sum_{t=r}^{s} \binom{s-r}{t-r} (2k-\ell)^{t-r}\, \ell^{s-t}\, 2^r\, v_r\, c_{2k-\ell} \\ &\quad=\; \sum_k \sum_{t=0}^{s} \sum_{r=0}^{t} (-1)^{s-r} \binom{s}{t} \binom{t}{r} (2k-\ell)^{t-r}\, \ell^{s-t}\, 2^r\, v_r\, c_{2k-\ell} \\ &\quad=\; \sum_{t=0}^{s} (-1)^{s-t} \binom{s}{t} \ell^{s-t} \sum_k \sum_{r=0}^{t} (-1)^{t-r} \binom{t}{r} (2k-\ell)^{t-r}\, 2^r\, v_r\, c_{2k-\ell} \\ &\quad=\; \sum_{t=0}^{s} \binom{s}{t} (-\ell)^{s-t}\, v_t \\ &\quad=\; y_{[s]}(\ell), \end{aligned}$$

so (6.24) holds for $0 \le s \le n-1$. ■

2.6 Accuracy and Orthogonal Wavelets

In this section we will study the relationship between accuracy and properties of orthogonal wavelets. We briefly recall the construction of such wavelets, referring to [Dau92] for a detailed description.

The construction of an orthogonal wavelet basis begins by choosing a refinable function $\varphi \in \mathcal{L}^2(\mathbf{R})$ satisfying the refinement equation

$$\varphi(x) \;=\; \sum_{k=0}^{N} c_k \, \varphi(2x-k), \tag{6.34}$$

which has the further property that its integer translates $\{\varphi(x+k)\}_{k\in\mathbf{Z}}$ are orthonormal. A necessary condition, which is only "rarely" insufficient, for φ to have orthonormal translates is that $\sum_k c_k \, \bar{c}_{k-2j} \;=\; 2\delta_{0j}$. Once a scaling function with orthonormal translates is chosen, the corresponding wavelet is

$$\psi(x) \;=\; \sum_k (-1)^{k-1} \, \bar{c}_{-k-1} \, \varphi(2x-k). \tag{6.35}$$

Equivalently, ψ is defined by the equation

$$\hat{\psi}(2\omega) \;=\; \frac{1}{2} m_1(\omega) \, \hat{\varphi}(\omega), \tag{6.36}$$

where

$$m_1(\omega) \;=\; \frac{1}{2} \sum_k (-1)^{k-1} \, \bar{c}_{-k-1} \, e^{-i\omega k} \;=\; e^{i\omega} \, \overline{m_0(\omega+\pi)}. \tag{6.37}$$

This wavelet has the remarkable property that $\{2^{n/2}\psi(2^n x+k)\}_{n,k\in\mathbf{Z}}$ forms an orthonormal basis for $\mathcal{L}^2(\mathbf{R})$. Further, $V_0 = \operatorname{span}\{\varphi(x+k)\}$ and $W_0 = \operatorname{span}\{\psi(x+k)\}$ are orthogonal complements in $\mathcal{L}^2(\mathbf{R})$.

High accuracy of the scaling function is a desirable feature for an orthogonal wavelet basis. For example, it leads to good approximation properties for the subspaces $V_j = \operatorname{span}\{2^{j/2}\varphi(2^j x + k)\}$ which define the associated multiresolution analysis. Further, the smoothness of the scaling function and wavelet is limited by the accuracy of the scaling function (see Section 5). The following result lists some implications of accuracy for scaling functions. We will make use of the fact that if a scaling function φ has orthonormal translates, then we must have $\hat{\varphi}(0) = 1$.

Theorem 4 *Assume that $\varphi \in \mathcal{L}^2(\mathbf{R})$ satisfies the refinement equation* (6.34) *and has orthonormal translates. Let ψ be the associated wavelet, defined by* (6.35). *Then the following are equivalent.*

i) $\displaystyle\int x^s \psi(x)\, dx = 0$ *for* $s = 0, \ldots, n-1$.

ii) $m_0^{(s)}(\pi) = 0$ *for* $s = 0, \ldots, n-1$.

iii) $\sum_k (2k)^s\, c_{2k} = \sum_k (2k+1)^s\, c_{2k+1}$ *for* $s = 0, \ldots, n-1$.

iv) φ *has accuracy* n.

Proof:

The equivalence of ii), iii), and iv) follows upon combining Theorems 2, 3, and 4.

i) $\Rightarrow$ ii) Assume that $\int x^s \psi(x)\, dx = 0$ for $s = 0, \ldots, n-1$. Note that both $\hat{\varphi}$ and $\hat{\psi}$ are continuous functions since φ and ψ have compact support. In light of (6.37), it therefore suffices to show that $m_1^{(s)}(0) = 0$ for $s = 0, \ldots, n-1$. We proceed by induction on s. For the case $s = 0$, we have $0 = \hat{\psi}(0) = \frac{1}{2} m_1(0) \hat{\varphi}(0) = \frac{1}{2} m_1(0)$.

Assume now that for some $0 \le s \le n-1$ we have $m_1^{(r)}(0) = 0$ for $r = 0, \ldots, s-1$. Differentiating both sides of (6.36), we have that

$$2^s \hat{\psi}^{(s)}(2\omega) \;=\; \frac{1}{2} \sum_{r=0}^{s} \binom{s}{r}\, m_1^{(r)}(\omega)\, \hat{\varphi}^{(s-r)}(\omega).$$

Therefore, by the inductive hypothesis and the fact that $\hat{\varphi}(0) = 1$, we have

$$0 \;=\; 2^s \hat{\psi}^{(s)}(0) \;=\; \frac{1}{2} \sum_{r=0}^{s} \binom{s}{r}\, m_1^{(r)}(0)\, \hat{\varphi}^{(s-r)}(0) \;=\; \frac{1}{2} m_1^{(s)}(0).$$

iv) $\Rightarrow$ i) Assume that φ has accuracy n. Fix any s with $0 \le s \le n-1$. Then there exist scalars α_k such that $x^s = \sum \alpha_k\, \varphi(x+k)$. Since φ and ψ are both compactly supported, we can interchange the integral and the sum in the following calculation:

$$\begin{aligned} \int x^s \psi(x)\, dx \;&=\; \int \overline{\sum_k \alpha_k\, \varphi(x+k)}\, \psi(x)\, dx \\ &=\; \sum_k \bar{\alpha}_k \int \psi(x)\, \overline{\varphi(x+k)}\, dx \;=\; 0, \end{aligned}$$

since φ and ψ have orthogonal translates. ■

3 Higher Dimensions, One Function - Sum Rules

In this section we present the statements of results which generalize some of the theorems of earlier sections to refinement equations in higher dimensions. We try to indicate how the techniques used previously can be extended to this more general settings. Refinable functions with domain $\mathbf{R}^2$ in particular play important roles in applications such as image processing.

We will omit the details and proofs of most results; these can be found in [CHM96] and [CHM97]. In Section 4 we will further extend these results to the case of multiple refinable functions.

We will use the standard multi-index notation $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, where $x = (x_1, \dots, x_d)^{\mathrm{t}} \in \mathbf{R}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d)$ is a vector of nonnegative integers. The degree of x^α is $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The number of multi-indices α of degree s is $d_s = \binom{s+d-1}{d-1}$. We write $\beta \le \alpha$ if $\beta_i \le \alpha_i$ for $i = 1, \dots, d$.

One way to generalize the refinement equation (6.1) to higher dimensions is simply to retain the equation as written, but to allow x to be an element of $\mathbf{R}^d$ instead of $\mathbf{R}$. In this case many results carry over with little change. Such refinement equations have been studied in detail, for example in [CDM91]. However, the uniform dilation $2x$ appearing in (6.1) is often too limiting, and therefore we would like to allow dilations which involve rotations, shears, etc. To do this, we replace the factor 2 by a *dilation matrix* A. Such a matrix must satisfy:

a) $A(\mathbf{Z}^d) \subset \mathbf{Z}^d$, and

b) A is *expansive*, i.e., $|\lambda| > 1$ for all eigenvalues λ of A.

In this case $m = |\det(A)|$ is necessarily an integer, and therefore the quotient group $\mathbf{Z}^d/A(\mathbf{Z}^d)$ has order m. We will say that a *full set of digits* $d_1, \dots, d_m \in \mathbf{Z}^d$ is a complete set of representatives of $\mathbf{Z}^d/A(\mathbf{Z}^d)$. In this case, $\mathbf{Z}^d$ is partitioned into the disjoint cosets

$$\Gamma_i \;=\; A(\mathbf{Z}^d) - d_i \;=\; \{Ak - d_i : k \in \mathbf{Z}^d\}, \qquad i = 1, \dots, m.$$

For example, in the one-dimensional case $d = 1$ with $A = m$, the numbers $0, \dots, m-1$ are a full set of digits. We remark that the lattice $\mathbf{Z}^d$ is chosen for convenience only; any full-rank lattice $\Gamma \subset \mathbf{R}^d$ could be used instead with appropriate modifications of the results. However, such a general lattice can always be reduced to the lattice $\mathbf{Z}^d$ by a change of basis.

Using the above notation, the refinement equation that we will study in this section has the form

$$f(x) \;=\; \sum_{k \in \Lambda} c_k\, f(Ax - k), \qquad x \in \mathbf{R}^d, \tag{6.38}$$

where Λ is a finite subset of $\mathbf{Z}^d$, the c_k are complex scalars, and f maps $\mathbf{R}^d$ into $\mathbf{C}$. We say that f has accuracy n if every multivariate polynomial $q(x) = q(x_1, \dots, x_d)$ of degree strictly less than n can be written exactly as an infinite linear combination of the translates $\{f(x+k)\}_{k \in \mathbf{Z}^d}$. We remark that although there remain close connections between accuracy and order of approximation, some of the implications valid in one dimension do not carry over to higher dimensions.

There are two simple but key properties that recur in the one-dimensional proofs in Section 2, and especially in the proof of Theorem 3. These are the homogeneity of dilation and the binomial theorem, i.e., the facts that for x, $y \in \mathbf{R}$ we have

$$(2x)^s \;=\; 2^s x^s \qquad \text{and} \qquad (x-y)^s \;=\; \sum_{t=0}^{s} \binom{s}{t} (-y)^{s-t}\, x^t. \tag{6.39}$$

To illustrate the difficulties that occur in higher dimensions with a general dilation matrix, let us consider a specific two-dimensional example. With $d = 2$, take A to be the *quincunx matrix* $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The construction of wavelets using this matrix has received special attention, e.g., [KV92], [GM92], [CD93], [Vil94]. Note that $m = \det(A) = 2$ for this matrix.

Consider now the dilation by A of a given monomial, say of $x^\alpha = x_1^2 x_2$ where $\alpha = (2,1)$. Unfortunately, $(Ax)^\alpha$ is no longer itself a monomial; indeed, since

$$Ax \;=\; \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;=\; \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix},$$

we have

$$\begin{aligned} (Ax)^\alpha \;=\; \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}^\alpha \;&=\; (x_1 - x_2)^2\,(x_1 + x_2) \\ &=\; x_1^3 - x_1^2 x_2 - x_1 x_2^2 + x_2^3 \\ &=\; x^{(3,0)} - x^{(2,1)} - x^{(1,2)} + x^{(0,3)}. \end{aligned}$$

Hence to express $(Ax)^\alpha$ we require terms x^β with all $|\beta| \le |\alpha|$.

To overcome this difficulty, let us consider *all* the monomials x^α of a given degree together, by collecting them into a vector. For example, for $|\alpha| = 3$, define the vector of all monomials of degree 3 to be

$$X_{[3]}(x) \;=\; \begin{bmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{bmatrix}.$$

Then,

$$X_{[3]}(Ax) \;=\; \begin{bmatrix} (x_1 - x_2)^3 \\ (x_1 - x_2)^2 (x_1 + x_2) \\ (x_1 - x_2)(x_1 + x_2)^2 \\ (x_1 + x_2)^3 \end{bmatrix} \;=\; \begin{bmatrix} 1 & -3 & 3 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{bmatrix}.$$

Hence, if we define $A_{[3]}$ to be the 4×4 matrix which appears in the line above, then we have that

$$X_{[3]}(Ax) \;=\; A_{[3]}\, X_{[3]}(x).$$

Except that we must keep in mind that $A_{[3]}$ is a matrix instead of a scalar, this equation is formally analogous to the one-dimensional equation $(2x)^3 = 2^3x^3$, and allows us to proceed with proofs that are similar in structure to the one-dimensional versions.

It is now clear how to define an analogous matrix $A_{[s]}$ corresponding to an arbitrary dilation matrix A and a given degree s. We first define $X_{[s]}(x) = [x^\alpha]_{|\alpha|=s}$ to be the vector of all monomials of degree s. The ordering of these monomials is unimportant, as long as the same ordering is used consistently. Recalling that there are $d_s = \binom{s+d-1}{d-1}$ multi-indices α of degree s, we then let $A_{[s]}$ be the $d_s \times d_s$ matrix which has the property that $X_{[s]}(Ax) = A_{[s]}\, X_{[s]}(x)$. It is easy to see that such a matrix will always exist. The following result lists some of the remarkable properties of these matrices $A_{[s]}$.

Lemma 1 *Let A, B be arbitrary $d \times d$ matrices.*

a) If $d = 1$ (so A is a scalar), then $A_{[s]} = A^s$.

b) $A_{[0]}$ is the scalar 1, and $A_{[1]} = A$.

c) $(AB)_{[s]} = A_{[s]}\, B_{[s]}$. Hence, if A is invertible then so is $A_{[s]}$, and $(A_{[s]})^{-1} = (A^{-1})_{[s]}$.

d) Let $\lambda = (\lambda_1, \dots, \lambda_d)^t$ be the vector whose entries are the eigenvalues of A. Then the eigenvalues of $A_{[s]}$ are $[\lambda^\alpha]_{|\alpha|=s}$. Hence, if A is expansive and $s > 0$, then $A_{[s]}$ is expansive.

The above remarks provide one way to generalize the first half of (6.39) to higher dimensions. The next problem is to generalize the second half, i.e., to generalize the binomial theorem. Using our vectors $X_{[s]}(x)$ containing all monomials of degree s, it is now clear how to proceed. We simply must understand the behavior of $X_{[s]}(x)$ under translation by an element $y \in \mathbf{R}^d$. First, define the following extension of binomial coefficients to higher dimensions:

$$\binom{\alpha}{\beta} = \begin{cases} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}, & \text{if } \beta_i \le \alpha_i \text{ for every } i, \\ 0, & \text{if } \beta_i > \alpha_i \text{ for some } i. \end{cases}$$

Then for any multi-index α of degree $|\alpha| = s$ we have:

$$\begin{aligned} &(x-y)^\alpha \\ &\quad = \quad (x_1 - y_1)^{\alpha_1} \cdots (x_d - y_d)^{\alpha_d} \\ &\quad = \quad \prod_{i=1}^{d} \sum_{\beta_i=0}^{\alpha_i} \binom{\alpha_i}{\beta_i} (-y_i)^{\alpha_i - \beta_i}\, x_i^{\beta_i} \end{aligned}$$

$$= \sum_{\beta_1=0}^{\alpha_1} \cdots \sum_{\beta_d=0}^{\alpha_d} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} (-y_1)^{\alpha_1-\beta_1} \cdots (-y_d)^{\alpha_d-\beta_d}\, x_1^{\beta_1} \cdots x_d^{\beta_d}$$

$$= \sum_{|\beta|\le s} (-1)^{|\alpha|-|\beta|} \binom{\alpha}{\beta} y^{\alpha-\beta}\, x^{\beta}. \tag{6.40}$$

Now, for each integer $0 \le t \le s$ and each $y \in \mathbf{R}^d$, define a $d_s \times d_t$ matrix

$$Q_{[s,t]}(y) = (-1)^{s-t} \left[\binom{\alpha}{\beta} y^{\alpha-\beta} \right]_{|\alpha|=s,|\beta|=t}.$$

Note that we can view $Q_{[s,t]}$ as a matrix of polynomials, each entry of which is either 0 or is a monomial of degree $s-t$ in y. The utility of this matrix of polynomials is that, by (6.40),

$$\begin{aligned} X_{[s]}(x-y) &= [(x-y)^{\alpha}]_{|\alpha|=s} \\ &= \left[\sum_{t=0}^{s} \sum_{|\beta|=t} (-1)^{s-t} \binom{\alpha}{\beta} y^{\alpha-\beta}\, x^{\beta} \right]_{|\alpha|=s} \\ &= \sum_{t=0}^{s} Q_{[s,t]}(y)\, X_{[t]}(x). \end{aligned}$$

This equation plays the same role for higher dimensions that the binomial theorem plays for one dimension.

The following lemma lists basic properties of the matrix of polynomials $Q_{[s,t]}$, and its interaction with the matrices $A_{[s]}$ defined above.

Lemma 2 *a)* $Q_{[s,t]}(x+y) = \sum_{u=t}^{s} Q_{[s,u]}(y)\, Q_{[u,t]}(x)$.

b) $Q_{[s,t]}(Ay) = A_{[s]}\, Q_{[s,t]}(y)\, A_{[t]}^{-1}$.

Using this machinery, many of the results for one dimension can be extended to higher dimensions with arbitrary dilation matrices. Since in the next section we will consider the additional generalization to multiple functions, we include here the statement of only one result on the characterization of accuracy for a single multivariate refinable function. This result generalizes the one-dimensional sum rules of Theorem 3, part e).

Theorem 5 (Sum Rules) *Assume that* $f\colon \mathbf{R}^d \to \mathbf{C}$ *satisfies the refinement equation* (6.38)*, that* f *is integrable and compactly supported, and that translates of* f *along* $\mathbf{Z}^d$ *are independent. Let* $m = |\det(A)|$*, and let* $d_1, \dots, d_m \in \mathbf{Z}^d$ *be a full set of digits. Set* $\Gamma_i = A(\mathbf{Z}^d) - d_i$*. Then the following statements are equivalent.*

i) f *has accuracy* n.

ii) $\displaystyle\sum_{k\in\mathbf{Z}^d} c_k = m$ *and* $\displaystyle\sum_{k\in\Gamma_1} k^{\alpha} c_k = \cdots = \sum_{k\in\Gamma_m} k^{\alpha} c_k$ *for* $0 \le |\alpha| \le n-1$.

4 Higher Dimensions with Multiple Functions

In this section we state the generalizations of previous results to the case of multiple refinable functions $f_1, \dots, f_r$. Since we have already discussed the transition to higher dimensions, we will assume that each f_i has domain $\mathbf{R}^d$. We will omit the proofs of these results, referring mainly to [CHM96] for details. In order to transmit the main ideas, we have not presented the results in their weakest form.

A finite collection of functions $f_1, \dots, f_r$ are refinable if each function f_i can be rewritten as a finite linear combination of the rescaled and translated functions $f_j(Ax - k)$. That is, there exist scalars $c_{i,j,k}$ such that

$$f_i(x) \;=\; \sum_{j=1}^{r} \sum_{k \in \Lambda} c_{i,j,k}\, f_j(Ax - k), \qquad i = 1, \dots, r, \tag{6.41}$$

where Λ is some finite subset of $\mathbf{Z}^d$. We can write this more compactly if we define a vector-valued function $f \colon \mathbf{R}^d \to \mathbf{C}^r$ by $f(x) = (f_1(x), \dots, f_r(x))^{\mathrm{t}}$. Then we can rewrite (6.41) as

$$f(x) \;=\; \sum_{k \in \Lambda} c_k\, f(Ax - k), \tag{6.42}$$

where the c_k are now some appropriate $r \times r$ *matrices*. Note that this equation is the same as (6.38) except that f is now vector-valued and the c_k are matrices.

The *accuracy* of the collection of functions $f_1, \dots, f_r$, or simply the accuracy of f, is the the largest integer n such that all multivariate polynomials $p(x) = p(x_1, \dots, x_d)$ with $\deg(p) < n$ lie in the shift-invariant space

$$\mathcal{S}(f) \;=\; \left\{ \sum_{k \in \mathbf{Z}^d} \sum_{i=1}^{r} \alpha_{k,i}\, f_i(x + k) : \alpha_{k,i} \in \mathbf{C} \right\}.$$

We can write this more compactly by using row vectors and column vectors. We let $\mathbf{C}^r$ be the space of all column vectors of length r; for example, $f(x) = (f_1(x), \dots, f_r(x))^{\mathrm{t}} \in \mathbf{C}^r$ for each $x \in \mathbf{R}^d$. We let $\mathbf{C}^{1\times r}$ be the space of all row vectors of length r; for example, we can set $\alpha_k = (\alpha_{k,1}, \dots, \alpha_{k,r}) \in \mathbf{C}^{1\times r}$. With this notation, $\mathcal{S}(f)$ can be rewritten as

$$\mathcal{S}(f) \;=\; \left\{ \sum_{k \in \mathbf{Z}^d} \alpha_k\, f(x + k) : \alpha_k \in \mathbf{C}^{1\times r} \right\}. \tag{6.43}$$

We will use this type of vector notation throughout this section, and we will speak interchangeably of the function $f \colon \mathbf{R}^d \to \mathbf{C}^r$ and the collection of functions $f_1, \dots, f_r$. For example, we say that translates of $f_1, \dots, f_r$ are independent, or simply that translates of f are independent, if $\sum b_k\, f(x + k) = 0$ with $b_k \in \mathbf{C}^{1\times r}$ implies $b_k = 0$ for every k.

A further notational convenience is to allow "vectors" that are indexed by $\mathbf{Z}^d$, and to regard these as behaving like row vectors or column vectors. For example, given the column vector $f(x) = (f_1(x), \dots, f_r(x))^{\mathrm{t}} \in \mathbf{C}^r$, we define an "infinite column vector" $F(x)$ by the formula

$$F(x) \;=\; [f(x+k)]_{k\in\mathbf{Z}^d}.$$

Next, we need to generalize the notation that we introduced immediately preceeding the statement of Theorem 3. We shall often be given a finite list of row vectors $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| \le n-1\}$, indexed by multi-indices α of degree less than n. We group these vectors by degree to form matrices of size $d_s \times r$ that we call $v_{[s]}$, i.e.,

$$v_{[s]} \;=\; [v_\alpha]_{|\alpha|=s} \;=\; \begin{bmatrix} v_{\alpha_1,1} & \cdots & v_{\alpha_1,r} \\ \vdots & \ddots & \vdots \\ v_{\alpha_{d_s},1} & \cdots & v_{\alpha_{d_s},r} \end{bmatrix}.$$

To these we associate the matrix of polynomials

$$y_{[s]}(x) \;=\; \left[\sum_{t=0}^{s} \sum_{|\beta|=t} (-1)^{s-t} \binom{\alpha}{\beta} x^{\alpha-\beta} v_\beta \right]_{|\alpha|=s} \;=\; \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}.$$

Note that

$$v_{[t]}\, f(x+k) \;=\; [v_\alpha]_{|\alpha|=s}\, f(x+k) \;=\; [v_\alpha f(x+k)]_{|\alpha|=s}.$$

Since $v_\alpha f(x+k) = v_{\alpha,1} f_1(x+k) + \cdots v_{\alpha,r} f_r(x+k)$, we see that $v_{[t]}\, f(x+k)$ is a vector whose entries are finite linear combinations of translates of $f_1, \dots, f_r$. For simplicity, we shall simply say that it is a vector of finite linear combinations of translates of f.

Next, we place the evaluations of the matrix of polynomials $y_{[s]}$ at lattice points into an "infinite row vector" that we call $Y_{[s]}$, i.e.,

$$Y_{[s]} \;=\; (y_{[s]}(k))_{k\in\mathbf{Z}^d}.$$

Although $Y_{[s]}$ is indexed by $\mathbf{Z}^d$, we treat it as a row vector in calculations. For example, the product of the row vector $Y_{[s]}$ with the column vector $F(x)$ is computed like the usual dot product of a row vector with a column vector:

$$\begin{aligned} Y_{[s]}\, F(x) &= (y_{[s]}(k))_{k\in\mathbf{Z}^d}\, [f(x+k)]_{k\in\mathbf{Z}} \\ &= \sum_{k\in\mathbf{Z}^d} y_{[s]}(k)\, f(x+k) \\ &= \sum_{k\in\mathbf{Z}^d} \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}\, f(x+k). \end{aligned}$$

This is a vector whose entries are infinite linear combinations of translates of f. Therefore, if we knew, say, that $Y_{[s]}\,F(x) = X_{[s]}(x)$, then we would know that every monomial x^α of degree $|\alpha| = s$ could be exactly reproduced from lattice translates of f. Hence, if we could achieve this statement for $s = 0, \dots, n-1$, then we could conclude that f has accuracy n.

4.1 Results for arbitrary functions

In this section we state some results that are valid for all functions $f\colon \mathbf{R}^d \to \mathbf{C}^r$, without the need to assume refinability.

The following result generalizes Proposition 2.

Theorem 6 ([CHM96]) *Assume that $f\colon \mathbf{R}^d \to \mathbf{C}^r$ has compact support and that translates of f along $\mathbf{Z}^d$ are independent. If f has accuracy n, then there exist row vectors $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| \le n-1\}$ such that $v_0 \ne 0$ and*

$$X_{[s]}(x) \;=\; \sum_{k\in\mathbf{Z}^d} y_{[s]}(k)\, f(x+k) \;=\; Y_{[s]}\,F(x), \qquad 0 \le s \le n-1.$$

In particular, if p is any polynomial with $\deg(p) < n$, *then there exists a unique row vector of polynomials $u_p\colon \mathbf{R}^d \to \mathbf{C}^{1\times r}$, with* $\deg(u_p) = \deg(p)$, *such that*

$$p(x) \;=\; \sum_{k\in\mathbf{Z}^d} u_p(k)\, f(x+k).$$

Note that since translates of f are assumed to be independent, the coefficients $y_{[s]}(k)$ in statement ii) of Theorem 6 which reproduce the monomials of degree s are unique.

Remark 4 Suppose that $f\colon \mathbf{R}^d \to \mathbf{C}^r$ is compactly supported with independent translates, and that f has accuracy p. Let $\Pi_{n,r}$ be the space of all row vectors of polynomials $p\colon \mathbf{R}^d \to \mathbf{C}^r$ with $\deg(p) < n$. Then Theorem 6 implies that the linear mapping $\mathbf{u}\colon \Pi_{n,1} \to \Pi_{n,r}$ defined by $\mathbf{u}(p) = u_p$ is injective and preserves degree. The dimensions of $\Pi_{n,1}$ and $\Pi_{n,r} = \Pi_{n,1} \times \cdots \times \Pi_{n,1}$ are equal only when $r = 1$. Therefore $\mathbf{u}$ is surjective if and only if $r = 1$. As a consequence, if $r = 1$ then for each polynomial $u \in \Pi_{n,1}$ we have that the function $q(x) = \sum_{k\in\mathbf{Z}^d} u(k)\, f(x+k)$ is itself a multivariate polynomial with $\deg(u) = \deg(q)$. However, $\mathbf{u}$ cannot be surjective when $r > 1$. As a consequence, if $r > 1$ then there must exist polynomials $u \in \Pi_{n,r}$ such that $q(x) = \sum_{k\in\mathbf{Z}^d} u(k)\, f(x+k)$ is not a polynomial (one example is given in [CHM96]).

The next result extends Proposition 3.

Theorem 7 ([CHM96]) *Assume that $f: \mathbf{R}^d \to \mathbf{C}^r$ has compact support, and let α be any multi-index. If $u: \mathbf{R}^d \to \mathbf{C}^{1\times r}$ is a row vector of polynomials such that*

$$x^\alpha \;=\; \sum_{k\in\mathbf{Z}^d} u(k)\, f(x+k),$$

then for each $0 \le \beta \le \alpha$ we have

$$x^\beta \;=\; C_\beta \sum_{k\in\mathbf{Z}^d} (D^{\alpha-\beta}u)(k)\, f(x+k),$$

where

$$D^\gamma u \;=\; \left(\frac{\partial^{|\gamma|}}{\partial x^\gamma}u_1, \dots, \frac{\partial^{|\gamma|}}{\partial x^\gamma}u_r\right)$$

and

$$C_\gamma \;=\; (-1)^{|\alpha-\gamma|}\,\frac{\gamma!}{\alpha!} \;=\; (-1)^{|\alpha-\gamma|}\,\frac{\gamma_1!}{\alpha_1!}\cdots\frac{\gamma_d!}{\alpha_d!}.$$

4.2 Strang–Fix Conditions

We briefly discuss the Strang–Fix conditions in higher dimensions. For a single function, i.e., when $r = 1$, if $f: \mathbf{R}^d \to \mathbf{C}$ is a compactly supported function in $\mathcal{L}^2(\mathbf{R}^d)$, then f satisfies the Strang–Fix conditions of order n if

$$\hat{f}(0) \neq 0 \quad \text{and} \quad \hat{f}^{(s)}(2k\pi) = 0,\ \forall\, k \in \mathbf{Z}^d - \{0\},\ \ 0 \le s \le n-1. \tag{6.44}$$

When f is continuously differentiable, these conditions are equivalent to accuracy [SF73].

In the general case, if $f = (f_1, \dots, f_r)^{\mathrm{t}}: \mathbf{R}^d \to \mathbf{C}^r$ is a vector of compactly supported functions, we say that f satisfies the Strang–Fix conditions of order n if there exists a function g which is a finite linear combination of the translates of $f_1, \dots, f_r$, i.e.,

$$g(x) \;=\; \sum_{i=1}^{r} \sum_{k=N_1}^{N_2} a_{k,i}\, f_i(x+k),$$

and which satisfies the Strang–Fix conditions (6.44). The following result states that the Strang–Fix conditions are still equivalent to accuracy.

Theorem 8 (see [Jia95]) *Assume that $f_1, \dots, f_r$ are compactly supported functions in $\mathcal{L}^2(\mathbf{R}^d)$, and that translates of $f = (f_1, \dots, f_r)^t$ are independent. Then the following statements are equivalent:*

i) f has accuracy n.

ii) f satisfies the Strang–Fix conditions of order n.

4.3 Strang–Fix Conditions and Refinable Functions

The relationship between the Strang–Fix conditions and accuracy for *refinable* functions depends on whether one function or multiple functions are being considered.

If $f: \mathbf{R}^d \to \mathbf{C}$ is refinable, i.e., we are considering only *one* function, then the Strang–Fix conditions relate to accuracy as in the one-dimensional case. The proof of this result can be found in [Jia97] and [CGV97]

Theorem 9 *Let* $m = |\det(A)|$ *and let* $d_1 = 0, d_2, \dots, d_m \in \mathbf{Z}^d$ *be a full set of digits. Define* $B = (A^{-1})^t$*. If* $f: \mathbf{R}^d \to \mathbf{C}$ *satisfies the refinement equation* (6.38)*, then the following statements are equivalent:*

a) f *satisfies the Strang–Fix conditions of order* n*.*

b) $m_0^{(\alpha)}(2\pi B d_i) = 0$ *for* $0 \le |\alpha| \le n-1$ *and* $i = 1, \dots, m$*.*

For the case of multiple functions in *one* dimension, Plonka [Plo97] proved the following characterization of accuracy. Note that the symbol $m_0(\omega) = \sum c_k e^{-i\omega k}$ is now matrix-valued, since the c_k are $r \times r$ matrices.

Theorem 10 *Assume that* $f = (f_1, \dots, f_r)^t: \mathbf{R} \to \mathbf{C}^r$ *is continuous and compactly supported, and that translates of* f *are independent. Then the following statements are equivalent:*

a) f *has accuracy* n*.*

b) There exist row vectors $Y^0, \dots, Y^{n-1} \in \mathbf{C}^{1\times r}$ *such that for* $s = 0, \dots, n-1$*,*

$$\begin{aligned} \sum_{t=0}^{s} \binom{s}{t} Y^t\,(2i)^{t-s}\,m_0^{(s-t)}(0) &= \frac{1}{2^s} Y^t, \\ \sum_{t=0}^{s} \binom{s}{t} Y^t\,(2i)^{t-s}\,m_0^{(s-t)}(\pi) &= 0. \end{aligned}$$

Additionally, Plonka found that accuracy implies a fundamental factorization of the matrix-valued symbol m_0. This generalizes the one-dimensional, single-function case, and has led to important advances in the construction of multiwavelets [PS95], [MS97].

4.4 A Time-Domain Characterization of Accuracy

An equivalent time-domain version of Theorem 10 of the previous section appears in [HSS96] and was the starting point for the following result, which generalizes Theorem 3 both to higher dimensions and to multiple functions. We use the notation introduced at the beginning of Section 4.

Additionally, we must generalize the bi-infinite matrix L defined in Section 2.5. This follows the same type of generalizations as used before: the index set becomes $\mathbf{Z}^d$ instead of $\mathbf{Z}$, and the dilation 2 is replaced by the dilation matrix A. Hence L becomes the "$\mathbf{Z}^d$ by $\mathbf{Z}^d$ matrix"

$$L \;=\; [c_{Ai-j}]_{i,j\in\mathbf{Z}^d}.$$

We compute the product of this "matrix" L with an "infinite column vector" such as $F(x)$ by adapting the rules of ordinary matrix-vector multiplication. For example,

$$L\,F(x) \;=\; [c_{Ai-j}]_{i,j\in\mathbf{Z}^d}\,[f(x+j)]_{j\in\mathbf{Z}^d} \;=\; \left[\sum_{j\in\mathbf{Z}^d} c_{Ai-j}\,f(x+j)\right]_{i\in\mathbf{Z}^d}.$$

With this notation, the refinement equation can be recast, analogously to the one-dimensional case, in the form

$$L\,F(Ax) \;=\; F(x), \qquad x\in\mathbf{R}^d.$$

Together with our earlier machinery, this allows the techniques used in the proof of Theorem 3 to be extended to the general setting, and yields the following result.

Theorem 11 ([CHM96]) *Let $m = |\det(A)|$, and let $d_1,\dots,d_m \in \mathbf{Z}^d$ be a full set of digits. Assume that $f = (f_1,\dots,f_r)^t\colon \mathbf{R}^d \to \mathbf{C}^r$ satisfies the refinement equation* (6.42)*, that f is integrable and compactly supported, and that translates of $f_1,\dots,f_r$ along $\mathbf{Z}^d$ are independent. Define $\Gamma_i = A(\mathbf{Z}^d) - d_i$. Then the following statements are equivalent.*

a) f has accuracy n.

b) There exist row vectors $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| \le n-1\}$ such that $v_0 \neq 0$ and

$$Y_{[n-1]} \;=\; A_{[n-1]}\,Y_{[n-1]}\,L.$$

c) There exist row vectors $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| \le n-1\}$ such that $v_0 \neq 0$ and

$$Y_{[s]} = A_{[s]}\,Y_{[s]}\,L, \qquad 0 \le s \le n-1.$$

d) There exist row vectors $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| \le n-1\}$ such that $v_0 \neq 0$ and

$$v_{[s]} \;=\; \sum_{k\in\Gamma_i}\sum_{t=0}^{s} Q_{[s,t]}(k)\,A_{[t]}\,v_{[t]}\,c_k, \qquad 0 \le s \le n-1,\ i = 1,\dots,m.$$

If $r = 1$, then the statements above are further equivalent to the following statement:

e) $\displaystyle\sum_{k\in\mathbf{Z}^d} c_k = m \quad$ *and* $\quad \displaystyle\sum_{k\in\Gamma_1} k^\alpha c_k = \cdots = \sum_{k\in\Gamma_m} k^\alpha c_k, \;\; 0 \le |\alpha| \le n-1.$

Since only finitely many matrices c_k are nonzero, the summations in statement d) of Theorem 11 are all finite.

For the case $s = 0$, statement d) in Theorem 11 reduces to the requirement that

$$v_0 \;=\; v_0 \sum_{k\in\Gamma_i} c_k, \qquad i = 1, \ldots, m.$$

Since $\mathbf{Z}^d$ is the disjoint union of the cosets Γ_i, this implies that $v_0 = v_0\Delta$, where $\Delta = \sum_{k\in\Lambda} c_k = m_0(0)$. Hence v_0 is a left 1-eigenvector of this matrix Δ.

An important implication of statement d) in Theorem 11 is that *the vectors v_α are determined directly by the matrices c_k and can be computed without explicit knowledge of f.* These vectors determine the coefficients $y_{[s]}(k)$ needed to reproduce the vector of monomials $X_{[s]}(x)$ from translates of f. Hence these coefficients can be derived directly from the matrices c_k. Further, the system of equations in statement d) has a block triangular structure, i.e., the equation for $v_{[s]}$ involves only $v_{[0]}, \ldots, v_{[s]}$. Hence the system can be checked recursively: $v_{[s+1]}$ is solved for after $v_{[0]}, \ldots, v_{[s]}$ have been found. Each step that can be solved implies one more degree of accuracy.

Finally, note that the condition $Y_{[s]} = A_{[s]}\, Y_{[s]}\, L$ in statement c) of Theorem 11 is no longer an eigenvector equation, as it is in one dimension, because $A_{[s]}$ is now a matrix and not a scalar. However, by changing to a basis in which $A_{[s]}$ is in Jordan form, it is possible to derive necessary conditions on the eigenvalues of the matrix L similar to those that hold in one dimension.

Proposition 5 ([CHM97]) *Let $\lambda = (\lambda_1, \ldots, \lambda_d)^t$ be the vector containing all eigenvalues of A. If there exist row vectors $Y_{[s]} \in ((\mathbf{C}^{1\times r})^{d_s\times 1})^{1\times\mathbf{Z}^d}$ such that $Y_{[s]} = A_{[s]}\, Y_{[s]}\, L$ for $0 \le s \le n-1$, then $\lambda^{-\alpha}$ is a left eigenvalue for L for each multi-index α with $0 \le |\alpha| \le n-1$.*

Considering Theorem 11 and Proposition 5 together, we see that if f has accuracy n, then $\lambda^{-\alpha}$ must be a left eigenvalue for L for each $0 \le |\alpha| \le n-1$. An example from [JRZ96] shows that even in the case $d = 1$, $r = 1$, the existence of such eigenvalues alone is not sufficient to imply accuracy for f; the corresponding left eigenvectors must have the polynomial structure specified in Theorem 11.

Since L is an infinite matrix, it is conceivable that the determination of its eigenvalues could be a difficult task. In fact, the eigenvalues and eigenvectors of L are completely determined by a particular finite submatrix L_0 of L [JRZ96], [CHM97]. Therefore, we have the following alternative test for accuracy: Once an upper bound for n has been computed by checking the eigenvalues of L_0, the left eigenvectors for L_0 lead to the vectors $Y_{[n-1]}$ such that $Y_{[n-1]} = A_{[n-1]}\, Y_{[n-1]}\, L$. If these vectors have a polynomial structure, then the accuracy is n. If they do not have a polynomial structure, then the test must be repeated replacing n by $n-1$. This test does require the computation of the eigenvalues of a finite matrix, which cannot be done using only systems of linear equations.

5 Implications of Accuracy

5.1 Accuracy and Order of Approximation

The concept of accuracy has been studied in the context of approximation theory and is closely related to properties of approximation of shift-invariant spaces. In this section we will discuss the connection between accuracy and *order of approximation*. Excellent reviews on this topic and on other related concepts are the papers [Jia95] and [dB90]. These also contain extensive and useful bibliographies.

Let $f_1, \dots, f_r \in \mathcal{L}^q(\mathbf{R}^d)$ be a fixed set of functions, and define $f = (f_1, \dots, f_r)^{\mathrm{t}}$ as usual. Let $\mathcal{S}(f)$ be the shift-invariant space defined by (6.43). Define $\mathcal{S} = \mathcal{S}(f) \cap \mathcal{L}^q(\mathbf{R}^d)$, and set $\mathcal{S}^h = \{g(x/h) : g \in \mathcal{S}\}$. Let $W_n^q(\mathbf{R}^d)$ denote the Sobolev space consisting of all functions whose weak derivatives up to order n all lie in $\mathcal{L}^q(\mathbf{R}^d)$. Then we say that $\mathcal{S}(f)$ provides *$\mathcal{L}^q$-approximation order n* if for each $g \in W_n^q(\mathbf{R}^d)$ there exists a constant c_g such that

$$\forall\, h > 0, \qquad \inf_{k \in \mathcal{S}^h} \|g - k\|_q \;\le\; c_g h^n.$$

The following result states that if the functions $f_1, \dots, f_r$ are compactly supported and have linearly independent translates, then order of approximation and accuracy are equivalent concepts.

Theorem 12 *Assume that $f_1, \dots, f_r \in \mathcal{L}^q(\mathbf{R}^d)$ are compactly supported, and that translates of $f = (f_1, \dots, f_r)^t$ are independent. Then the following statements are equivalent:*

i) f has accuracy n.

ii) $\mathcal{S}(f)$ provides $\mathcal{L}^q$-approximation order n for each $1 \le q \le +\infty$.

de Boor and Höllig [dBH83] showed that the assumption of linear independence is necessary. In particular, they gave an example of two functions defined on $\mathbf{R}^2$ which together have accuracy 4 but order of approximation 3. If we drop the assumption on linear independence, the result remains true (with appropriate minor hypothesis) when either $d = 1$ or $r = 1$ [Jia95].

5.2 Accuracy and Smoothness

One motivation from wavelet theory for studying accuracy is that the accuracy of the scaling function is related to the smoothness of the corresponding wavelet. In particular, the scaling function and wavelet have the same amount of smoothness, and accuracy is a necessary condition for the scaling function to be smooth. Hence, in order to construct smooth wavelets, we need scaling functions which have sufficiently high accuracy.

The following result which for simplicity, we present here only for the one-dimensional, single-function case, is due to Meyer [Mey92].

Theorem 13 *Let ϕ be a compactly supported scaling function with orthonormal integer translates, and let ψ be the corresponding wavelet. If ψ is k-times continuously differentiable, then $\int x^s \psi_i(x)\,dx = 0$ for $s = 0, \ldots, k$.*

In light of Theorem 4, the zero-moment condition of this theorem implies that the associated scaling function φ has accuracy $k+1$.

We conclude by presenting some results which show that smoothness of a refinable function implies accuracy, for the particular case of the uniform dilation $A = 2I$.

Theorem 14 ([CDM91]) *Let $f \in \mathbf{C}^k(\mathbf{R}^d)$ be a compactly supported refinable function such that $\hat{f}(0) \neq 0$. Then f has accuracy $k+1$.*

Theorem 15 ([Jia96a]) *If $f \in W_1^k(\mathbf{R}^d)$ is a compactly supported refinable function such that $\hat{f}(0) \neq 0$, then f has accuracy $k+1$.*

This result is extended to higher dimensions with isotropic dilation matrices A in [Jia97]. The case of multiple functions in higher dimensions, again with the uniform dilation $A = 2I$, is discussed in [Ron97].

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