A DISCRETE ZAK TRANSFORM

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Abstract. A discrete version of the Zak transform is defined and used to analyze discrete Weyl–Heisenberg frames, which are nonorthogonal systems in the space of square-summable sequences that, although not necessarily bases, provide representations of square-summable sequences as sums of the frame elements. While the general theory is essentially similar to the continuous case, major differences occur when specific Weyl–Heisenberg frames are evaluated. In particular, it is shown that while Weyl–Heisenberg frames in the continuous case are bases only if they are generated by functions that are not smooth or have poor decay, it is possible in the discrete case to construct Weyl–Heisenberg frames that are bases and are generated by sequences with good decay. The sampled Gaussian provides an example of such a signal.

1. Introduction

The recent paper [HW1] surveyed the current literature on frames in $L^2(\mathbf{R})$. Frames are generalizations of orthonormal bases and $L^2(\mathbf{R})$ is the space of finite-energy signals on the real line. The intent of the frame constructions described in [HW1] is to provide a mathematical foundation for simultaneous time and frequency analysis in signal processing. However, before these frames can be implemented as algorithms on a digital computer, the theory must be extended to the type of signals actually used in digital signal processing, namely, discrete or sampled signals. D. Walnut, in [W2], began this work by describing frame constructions for $\ell^2(\mathbf{Z})$, the space of square-summable sequences indexed by $\mathbf{Z}$, the set of integers. Each such sequence can be thought of as a sampled finite-energy signal. This paper continues in the same spirit by extending another aspect of the continuous-signal theory to discrete signals, namely, we define a discrete version of the so-called Zak transform. We show that the definition of this transform and the general theorems obtained by applying it to Weyl–Heisenberg frames are essentially the same as in the continuous case, but that very different results can occur when specific Weyl–Heisenberg frames are evaluated. In particular, discrete Weyl–Heisenberg frames appear to be more tractable than their continuous counterparts, indicating that they have good potential for digital signal processing analysis.

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To give a more precise meaning to the above remarks, let us first recall the definition of orthonormal basis. If H is a Hilbert space (e.g., $L^2(\mathbf{R})$ or $\ell^2(\mathbf{Z})$) with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, then a set of vectors $\{e_n\}$ in H is an *orthonormal basis* if

1. $\langle e_m, e_n \rangle = \begin{cases} 1, & \text{if } m = n; \\ 0, & \text{if } m \neq n; \end{cases}$

2. $\sum |\langle x, e_n \rangle|^2 = \|x\|^2$ for all $x \in H$.

A fundamental property of orthonormal bases is that $x = \sum \langle x, e_n \rangle e_n$ for every $x \in H$, so that information about x is stored in the set of scalars $\{\langle x, e_n \rangle\}$. Because the orthonormality condition (1) is a stringent and sensitive condition, we define frames, which have no orthonormality condition and are therefore often easier to generate. A set of vectors $\{x_n\}$ in H is a *frame* if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \ \leq \ \sum_n |\langle x, x_n \rangle|^2 \ \leq \ B\|x\|^2$$

for all $x \in H$. As exposited in [HW1], the *frame operator* $Sx = \sum \langle x, x_n \rangle x_n$ is then well-defined, though not necessarily the identity, as is the case for orthonormal bases. However, S is an isomorphism (i.e., is bijective, continuous, and has a continuous inverse), and every $x \in H$ can be written $x = \sum \langle x, S^{-1}x_n \rangle x_n$, so x is again characterized by its "frame transform" $\{\langle x, S^{-1}x_n \rangle\}$.

The paper [HW1] discussed the problem of finding certain frames for $L^2(\mathbf{R})$. In particular, the frames of interest are those in which the frame elements are easily generated from a single fixed function through a combination of the elementary operations of translation, modulation, and dilation. For example, a *Weyl–Heisenberg system* for $L^2(\mathbf{R})$ is a set of the form $\{G_{mn}\}_{m,n \in \mathbf{Z}}$, where $G_{mn}(t) = e^{2\pi i m(t-na)/b} G(t - na)$ for some fixed function $G \in L^2(\mathbf{R})$ and fixed $a, b > 0$. An *affine system* is a set of the form $\{H_{mn}\}_{m,n \in \mathbf{Z}}$, where $H_{mn}(t) = a^{n/2} H(a^n t - mb)$ for some fixed $H \in L^2(\mathbf{R})$ and fixed $a > 1$, $b > 0$. Because of the difficulty involved in time-compressing sampled signals, discrete versions of affine frames are fairly complex (see [W2]). This makes discrete Weyl–Heisenberg frames correspondingly more attractive, and so we will not mention affine frames again in this paper. The discrete analogue of Weyl–Heisenberg systems is a set of the form $\{g_{mn}\}_{n \in \mathbf{Z}, m=0,\dots,a-1}$, where $g_{mn}(k) = e^{2\pi i m(k-na)/b} g(k - na)$ for some fixed $g \in \ell^2(\mathbf{Z})$ and fixed integers $a, b > 0$.

For quantum mechanical reasons, J. Zak in the 1960s proved results about the properties of the Weyl–Heisenberg system $\{G_{mn}\}$ when G was the Gaussian signal $G(t) = e^{-\pi t^2}$ and when $a = b = 1$. In particular, he showed that the set $\{G_{mn}\}$ was complete (in fact, overcomplete by one element) in the sense that its linear span was dense in $L^2(\mathbf{R})$ (see [Z]). However, I. Daubechies, A. Grossmann, and Y. Meyer later showed that the Gaussian does not generate a Weyl–Heisenberg frame (see [DGM]). This is unfortunate since, by the uncertainty principle, the Gaussian is the function in $L^2(\mathbf{R})$ which is best localized both in time and frequency, in the sense of minimizing the value

$$\frac{\|tG(t)\|_2 \, \|\gamma\hat{G}(\gamma)\|_2}{\|G\|_2}. \tag{UP}$$

Thus it is the signal which should provide best results when attempting simultaneous time and frequency signal analysis. Recently, it has been shown that no function $G \in L^2(\mathbf{R})$ for which (UP) is even finite can generate a Weyl–Heisenberg frame with $a = b$ (see [BHW] for history and review of proofs, also [Bal; Bat; D; DJ; L]). It can be shown that Weyl–Heisenberg frames are bases (i.e., the frame representations are unique) only when $a = b$. Thus the only Weyl–Heisenberg frames that will have this desirable feature are those for which (UP) is infinite, which implies that G is either not smooth or does not decay quickly. This makes Weyl–Heisenberg frames in $L^2(\mathbf{R})$ somewhat unattractive. However, we show in this paper that the situation for discrete Weyl–Heisenberg frames is quite different. For example, we show that the discrete Weyl–Heisenberg system generated by the sampled Gaussian signal, $g(k) = e^{-\pi k^2}$ for $k \in \mathbf{Z}$, does generate a frame when $a = b$, and that this frame is a basis for $\ell^2(\mathbf{Z})$.

2. Notation

$\mathbf{R}$ is the real line thought of as the time axis, and $\hat{\mathbf{R}}$ is its dual group, the real line thought of as the frequency axis. $\mathbf{Z}$ is the set of integers, and $\mathbf{C}$ the complex numbers. $\mathbf{Z}_+$ is the set of positive integers, and $\mathbf{R}_+ = (0, \infty)$ the set of positive real numbers. Given $a \in \mathbf{Z}_+$, we set

$$\mathbf{Z}_a \;=\; \{0, 1, \dots, a-1\}.$$

We will attempt to distinguish between sequences and functions by using lowercase letters for the former and uppercase for the latter. A *sequence* is a mapping f defined on the integers, while a *function* is a mapping F defined on the real line.

$\ell^2(\mathbf{Z})$ is the Hilbert space of all square-summable sequences, with norm and inner product

$$\|f\|_2 \;=\; \biggl(\sum_{k\in\mathbf{Z}} |f(k)|^2\biggr)^{1/2} \qquad \text{and} \qquad \langle f, g\rangle \;=\; \sum_{k\in\mathbf{Z}} f(k)\,\overline{g(k)}.$$

The Banach spaces $\ell^p(\mathbf{Z})$ for $1 \le p < \infty$ are defined analogously, and $\ell^\infty(\mathbf{Z})$ is the space of all bounded sequences.

$L^2(\mathbf{R})$ is the Hilbert space of all square-integrable functions, with norm and inner product

$$\|F\|_2 \;=\; \biggl(\int_{\mathbf{R}} |F(t)|^2\,dt\biggr)^{1/2} \qquad \text{and} \qquad \langle F, G\rangle \;=\; \int_{\mathbf{R}} F(t)\,\overline{G(t)}\,dt.$$

The Banach spaces $L^p(\mathbf{R})$ for $1 \le p < \infty$ are defined analogously, and $L^\infty(\mathbf{R})$ is the space of all essentially bounded functions.

Given a function F we define the *translation* of F by $a \in \mathbf{R}$ as

$$T_a F(t) \;=\; F(t-a).$$

Modulation is defined as

$$E_b F(t) \;=\; e^{2\pi i b t}\, F(t)$$

for $b \in \mathbf{R}$. Translation and modulation are also defined on sequences, as long as $a \in \mathbf{Z}$ and $b \in \mathbf{R}$. We also use the symbol E_b by itself to refer to the *exponential function* $E_b(t) = e^{2\pi i b t}$. The *two-dimensional exponentials* are $E_{(a,b)}(t,\omega) \;=\; e^{2\pi i a t} e^{2\pi i b \omega}$.

Given a Hilbert space H and sets of vectors $\{x_n\}$ and $\{y_n\}$, we define the following terms.

1. $\{x_n\}$ and $\{y_n\}$ are *biorthonormal* if

$$\langle x_m, y_n\rangle \;=\; \delta_{mn} \;=\; \begin{cases} 1, & \text{if } m = n; \\ 0, & \text{if } m \ne n. \end{cases}$$

2. $\{x_n\}$ is *complete* if its linear span is dense in H, or equivalently, if the only $x \in H$ with $\langle x, x_n\rangle = 0$ for all n is $x = 0$.

3. $\{x_n\}$ is *minimal* if $x_m \notin \overline{\text{span}}\{x_n\}_{n\ne m}$ for any m (where $\overline{\text{span}}$ stands for the closed linear span).

4. $\{x_n\}$ is a *basis* if for each $x \in H$ there exist unique scalars $\{c_n\}$ such that $x = \sum c_n x_n$. The basis is *unconditional* if every rearrangement of these series converge. It is *bounded* if $0 < \inf \|x_n\| \le \sup \|x_n\| < \infty$.

3. Frames

We review some of the properties of frames. More details can be found in [H2; HW1; D; DS; Y].

Definition 3.1 [DS]**.** A set of vectors $\{x_n\}$ in a Hilbert space H is a *frame* if there exist numbers $A, B > 0$ such that for every $x \in H$ we have

$$A\|x\|^2 \;\le\; \sum_n |\langle x, x_n\rangle|^2 \;\le\; B\|x\|^2.$$

The numbers A, B are the *frame bounds*. The frame is *tight* if $A = B$. The frame is *exact* if it ceases to be a frame whenever any single element is deleted from the sequence.

Given two operators $S, T\colon H \to H$ we write $S \ge T$ if $\langle Sx, x\rangle \ge \langle Tx, x\rangle$ for all $x \in H$. We denote by I the identity map on H, i.e., $Ix = x$ for all $x \in H$.

Theorem 3.2 [DS]. *Given a sequence $\{x_n\}$ in a Hilbert space H, the following two statements are equivalent:*

1. *$\{x_n\}$ is a frame with bounds A, B.*
2. *$Sx = \sum \langle x, x_n\rangle x_n$ is a bounded linear operator with $AI \le S \le BI$, called the frame operator for $\{x_n\}$.*

Corollary 3.3 [DS].

1. *S is invertible and $B^{-1}I \le S^{-1} \le A^{-1}I$.*
2. *$\{S^{-1}x_n\}$ is a frame with bounds B^{-1}, A^{-1}, called the dual frame.*
3. *Every $x \in H$ can be written $x = \sum \langle x, S^{-1}x_n\rangle x_n = \sum \langle x, x_n\rangle S^{-1}x_n$.*

Theorem 3.4 [DS]. *The removal of a vector from a frame leaves either a frame or an incomplete set. In particular, given m fixed,*

$$\langle x_m, S^{-1}x_m\rangle \ne 1 \;\Rightarrow\; \{x_n\}_{n\ne m} \text{ is a frame;}$$
$$\langle x_m, S^{-1}x_m\rangle = 1 \;\Rightarrow\; \{x_n\}_{n\ne m} \text{ is incomplete.}$$

Theorem 3.5 [Y].

1. *Inexact frames are not bases, i.e., the representations in Corollary 2.3 part 3 are not unique.*
2. *A sequence $\{x_n\}$ in a Hilbert space H is an exact frame for H if and only if it is a bounded unconditional basis. This occurs if and only if there exists an orthonormal basis $\{e_n\}$ and an isomorphism $U\colon H \to H$ such that $x_n = Ue_n$ for all n.*

4. The Zak Transform

Two good articles on the properties of the continuous version of the Zak transform are [J1; J2].

Definition 4.1.

1. The (continuous) *Zak transform* of a function F is (formally)

$$ZF(t, \omega) \;=\; \sum_{j\in\mathbf{Z}} F(t + ja)\, e^{2\pi i j\omega}$$

for $(t, \omega) \in \mathbf{R} \times \hat{\mathbf{R}}$, and where $a \in \mathbf{R}_+$ is a fixed parameter.

2. The (discrete) *Zak transform* of a sequence f is (formally)

$$zf(k,\omega) \;=\; \sum_{j\in\mathbf{Z}} f(k+ja)\, e^{2\pi i j\omega}$$

for $(k,\omega) \in \mathbf{Z} \times \hat{\mathbf{R}}$, and where $a \in \mathbf{Z}_+$ is a fixed parameter.

The series defining ZF or zf can converge in various senses. For example, if F is a continuous function with compact support, then ZF clearly converges pointwise since it reduces everywhere to a finite sum.

First note that ZF and zf are *quasiperiodic* in the following sense:

1. For all $(t,\omega) \in \mathbf{R} \times \hat{\mathbf{R}}$ we have

$$\begin{aligned} ZF(t+a,\omega) &= e^{-2\pi i\omega}\, ZF(t,\omega); \\ ZF(t,\omega+1) &= ZF(t,\omega). \end{aligned}$$

2. For all $(k,\omega) \in \mathbf{Z} \times \hat{\mathbf{R}}$ we have

$$\begin{aligned} zf(k+a,\omega) &= e^{-2\pi i\omega}\, zf(k,\omega); \\ zf(k,\omega+1) &= zf(k,\omega). \end{aligned}$$

Therefore, ZF is completely determined by its values on the fundamental rectangle

$$\mathbf{Q} \;=\; [0,a) \times [0,1),$$

while zf is completely determined by its values on the set

$$\mathbf{q} \;=\; \mathbf{Z}_a \times [0,1) \;=\; \{0,\dots,a-1\} \times [0,1).$$

Note that $\mathbf{q}$ is the subset of $\mathbf{Q}$ consisting of vertical lines with integer abscissa coordinates. Moreover, if F is a continuous function, and f is the integer sampled values of F, i.e., $f(k) = F(k)$ for $k \in \mathbf{Z}$, then

$$ZF(k,\omega) \;=\; zf(k,\omega)$$

for all $(k,\omega) \in \mathbf{q}$ (in fact, for all $(k,\omega) \in \mathbf{Z} \times \hat{\mathbf{R}}$).

Definition 4.2. We let $L^2(\mathbf{Q})$ be the Hilbert space of all mappings $\Phi\colon \mathbf{Q} \to \mathbf{C}$ for which

$$\|\Phi\|_2 \;=\; \left(\int_0^a \int_0^1 |\Phi(t,\omega)|^2\, dt\, d\omega\right)^{1/2} \;<\; \infty,$$

with the usual inner product. Similarly, $\ell^2(\mathbf{q})$ is the Hilbert space of mappings $\varphi\colon \mathbf{q} \to \mathbf{C}$ for which

$$\|\varphi\|_2 \;=\; \left(\sum_{k \in \mathbf{Z}_a} \int_0^1 |\varphi(k,\omega)|^2 \, d\omega \right)^{1/2} \;<\; \infty,$$

again with the expected inner product.

The proof of the following theorem for Z can be found in [HW1]. The proof for the discrete case is virtually identical, so will be omitted.

Theorem 4.3.

1. *Z is a unitary map of $L^2(\mathbf{R})$ onto $L^2(\mathbf{Q})$, with the series defining Zf converging in the norm of $L^2(\mathbf{Q})$.*

2. *z is a unitary map of $\ell^2(\mathbf{Z})$ onto $\ell^2(\mathbf{q})$, with the series defining zf converging in the norm of $\ell^2(\mathbf{q})$.*

5. Analysis of Weyl–Heisenberg Frames with the Zak Transform

Definition 5.1.

1. Given a fixed function $G \in L^2(\mathbf{R})$ and given $a, b \in \mathbf{R}_+$, the (continuous) *Weyl–Heisenberg system* generated by (G, a, b) is $\{G_{mn}\}_{m,n \in \mathbf{Z}}$, where

   $$G_{mn}(t) \;=\; T_{na} E_{\frac{m}{b}} G(t) \;=\; e^{2\pi i m(t-na)/b}\, G(t-na).$$

   We call G the *mother wavelet* and a, b the *system parameters*.

2. Given a fixed sequence $g \in \ell^2(\mathbf{Z})$ and given $a, b \in \mathbf{Z}_+$, the (discrete) *Weyl–Heisenberg system* generated by (g, a, b) is $\{g_{mn}\}_{m \in \mathbf{Z}_b, n \in \mathbf{Z}}$, where

   $$g_{mn}(k) \;=\; T_{na} E_{\frac{m}{b}} g(k) \;=\; e^{2\pi i m(k-na)/b}\, g(k-na).$$

   We call g the *mother sequence* and a, b the *system parameters*.

We are most interested, of course, in Weyl–Heisenberg *frames*. We refer to [HW1] for examples and conditions for the existence of continuous Weyl–Heisenberg frames, and to [W2] for discrete W–H frames.

Since Z and z are unitary maps, we can use them to transform problems from the time domain into a "Zak transform domain". More precisely, the unitarity implies that a discrete W–H system $\{g_{mn}\}$ in $\ell^2(\mathbf{Z})$ is complete, a frame, an exact frame, an orthonormal basis, etc., if and only if the same is true of the system $\{zg_{mn}\}$ in $\ell^2(\mathbf{q})$, with analogous remarks holding for continuous systems. This transformation turns out to be particularly useful in the case $a = b$, for zg_{mn} then has the following simple form.

Theorem 5.2. *Given $g \in \ell^2(\mathbf{Z})$ and $a \in \mathbf{Z}_+$. If $b = a$ then*

$$zg_{mn} \;=\; E_{(\frac{m}{a},n)} \cdot zg.$$

Proof. We compute:

$$\begin{aligned} zg_{mn}(k,\omega) &= \sum_j g_{mn}(k+ja)\, e^{2\pi i j\omega} \\ &= \sum_j e^{2\pi i m(k+ja-na)/a}\, g(k+ja-na)\, e^{2\pi i j\omega} \\ &= e^{2\pi i mk/a} \sum_j g(k+ja-na)\, e^{2\pi i j\omega} \\ &= e^{2\pi i mk/a} \sum_j g(k+ja)\, e^{2\pi i (j+n)\omega} \\ &= e^{2\pi i mk/a} e^{2\pi i n\omega} \sum_j g(k+ja)\, e^{2\pi i j\omega} \\ &= E_{(\frac{m}{a},n)}(k,\omega) \cdot zg(k,\omega). \quad \square \end{aligned}$$

An analogous result holds for the continuous case. The importance of Theorem 5.2 is that the set of functions $\{a^{-1/2} E_{(\frac{m}{a},n)}\}_{m \in \mathbf{Z}_a, n \in \mathbf{Z}}$ forms an orthonormal basis for $\ell^2(\mathbf{q})$. Thus the question of when $\{g_{mn}\}$ can form a frame for $\ell^2(\mathbf{Z})$ when $a = b$ is transformed by z into the question of when is possible to take an orthonormal basis for $\ell^2(\mathbf{q})$, multiply each element of the basis by the single function zg, and still obtain a frame for $\ell^2(\mathbf{q})$. This imposes severe restrictions on the function zg, which are given precisely in the following theorem.

Theorem 5.3. *Given a fixed $g \in L^2(\mathbf{R})$ and $a \in \mathbf{Z}_+$. If we define $b = a$ then the following statements hold.*

1. *$\{g_{mn}\}$ is complete in $\ell^2(\mathbf{Z})$ if and only if $zg \neq 0$ a.e.*
2. *$\{g_{mn}\}$ is minimal and complete in $\ell^2(\mathbf{Z})$ if and only if $\frac{1}{zg} \in \ell^2(\mathbf{q})$.*
3. *$\{g_{mn}\}$ is a frame for $\ell^2(\mathbf{Z})$ with frame bounds A, B if and only if*

   $$0 < a^{-1}A \leq |zg|^2 \leq a^{-1}B < \infty \text{ a.e.}$$

   In this case the frame is exact.
4. *$\{g_{mn}\}$ is an orthonormal basis for $\ell^2(\mathbf{Z})$ if and only if $|zg|^2 = a^{-1}$ a.e.*

Proof. 1. Assume $\{g_{mn}\}$ was complete in $\ell^2(\mathbf{Z})$. Then $\{zg_{mn}\}$ is complete in $\ell^2(\mathbf{q})$ by the unitarity of z. Define φ on $\mathbf{q}$ by

$$\varphi(k,\omega) \;=\; \begin{cases} 1, & \text{if } zg(k,\omega) = 0; \\ 0, & \text{if } zg(k,\omega) \neq 0. \end{cases}$$

Then

$$\begin{aligned} \langle zg_{mn}, \varphi \rangle &= \langle E_{(\frac{m}{a},n)} \cdot zg, \varphi \rangle \\ &= \langle E_{(\frac{m}{a},n)}, \overline{zg} \cdot \varphi \rangle \\ &= \langle E_{(\frac{m}{a},n)}, 0 \rangle \\ &= 0 \end{aligned}$$

for all m, n. But $\{zg_{mn}\}$ is complete, so this implies $\varphi = 0$ a.e., which implies $zg = 0$ a.e.

Now assume $zg \neq 0$ a.e., and suppose $\varphi \in \ell^2(\mathbf{q})$ is such that $\langle zg_{mn}, \varphi \rangle = 0$ for all m, n. Then

$$0 \;=\; \langle zg_{mn}, \varphi \rangle \;=\; \langle E_{(\frac{m}{a},n)} \cdot zg, \varphi \rangle \;=\; \langle E_{(\frac{m}{a},n)}, \overline{zg} \cdot \varphi \rangle.$$

Since $\overline{zg} \cdot \varphi \in \ell^1(\mathbf{q})$ and $\{E_{(\frac{m}{a},n)}\}$ is complete in $\ell^1(\mathbf{q})$, it follows that $\varphi = 0$ a.e., so $\{zg_{mn}\}$ is complete is $\ell^2(\mathbf{q})$. Since z is unitary, this implies that $\{g_{mn}\}$ is complete in $\ell^2(\mathbf{Z})$.

2. Assume $\{g_{mn}\}$ is minimal and complete in $\ell^2(\mathbf{Z})$, so the same is true of $\{zg_{mn}\}$ in $\ell^2(\mathbf{q})$. By part 1, we also know that $zg \neq 0$ a.e. Now, the minimality implies (see [S]) that there is a set of functions $\{\varphi_{mn}\}$ in $\ell^2(\mathbf{q})$ which are biorthonormal to $\{zg_{mn}\}$, i.e.,

$$\langle zg_{mn}, \varphi_{m'n'} \rangle \;=\; \delta_{mm'}\,\delta_{nn'} \;=\; \begin{cases} 1, & \text{if } m = m' \text{ and } n = n'; \\ 0, & \text{otherwise.} \end{cases}$$

Now $\langle zg_{mn}, \varphi_{m'n'} \rangle = \langle E_{(\frac{m}{a},n)}, \overline{zg} \cdot \varphi_{m'n'} \rangle$, and $\overline{zg} \cdot \varphi_{mn} \in \ell^1(\mathbf{q})$ while $E_{(\frac{m}{a},n)} \in \ell^\infty(\mathbf{q})$. Thus the set $\{\overline{zg} \cdot \varphi_{mn}\} \subset \ell^1(\mathbf{q})$ is biorthonormal to the set $\{E_{(\frac{m}{a},n)}\} \subset \ell^\infty(\mathbf{q})$. But there is a unique sequence in $\ell^1(\mathbf{q})$ which is biorthonormal to $\{E_{(\frac{m}{a},n)}\} \subset \ell^\infty(\mathbf{q})$, namely $\{a^{-1} E_{(\frac{m}{a},n)}\}$. Therefore, $\overline{zg} \cdot \varphi_{mn} = a^{-1} E_{(\frac{m}{a},n)}$, so $E_{(\frac{m}{a},n)} / \overline{zg} = a^{-1} \varphi_{mn}$ for all m, n. In particular, $1/\overline{zg} \in \ell^2(\mathbf{q})$.

Conversely, assume $1/zg \in \ell^2(\mathbf{q})$. Then $zg \neq 0$ a.e., so $\{g_{mn}\}$ is complete by part 1. Let $\tilde{g} = a^{-1} Z^{-1}(1/\overline{zg}) \in \ell^2(\mathbf{Z})$. Then

$$\begin{aligned} \langle g_{mn}, \tilde{g}_{m'n'} \rangle &= \langle zg_{mn}, z\tilde{g}_{m'n'} \rangle \\ &= \langle E_{(\frac{m}{a},n)} \cdot zg, E_{(\frac{m'}{a},n')} \cdot z\tilde{g} \rangle \\ &= a^{-1} \langle E_{(\frac{m}{a},n)}, E_{(\frac{m'}{a},n')} \rangle \\ &= \delta_{mm'}\,\delta_{nn'}. \end{aligned}$$

Thus $\{g_{mn}\}$ and $\{\tilde{g}_{mn}\}$ are biorthonormal in $\ell^2(\mathbf{Z})$. The existence of a biorthonormal set implies immediately that $\{g_{mn}\}$ is minimal (see [S]).

3. Similar to the continuous Zak argument presented in [HW1].

4. Follows easily from 3. □

An analogous continuous version of Theorem 5.3 can be found in [H2] and [HW1].

We are especially interested in parts 3 and 4 of Theorem 5.3. Part 3 implies that, in the $a = b$ case, any W–H frame will be exact, hence a *basis* for $\ell^2(\mathbf{Z})$. This is desirable, since in this case we have unique decompositions of $\ell^2(\mathbf{Z})$, i.e., the frame representations in Corollary 2.3 part 3 are unique. In both the continuous and discrete cases, it can be shown that if the ratio a/b is greater than 1 then it is impossible to form W–H frames, while if it is less than 1, any W–H frame will be inexact, i.e., the frame representations will not be unique. This is why we have concentrated our attention on the $a = b$ case. Our goal now is to determine what sorts of functions will or will not satisfy the conditions of Theorem 5.3. To do this, we must determine when the Zak transform of a sequence or function will have zeroes. In the continuous case, it can be shown that if $G \in L^2(\mathbf{R})$ is such that ZG is continuous on the plane $\mathbf{R} \times \hat{\mathbf{R}}$, then ZG has a zero, and so G cannot generate a W–H frame for the case $a = b$. It is shown in [H2] that if G is continuous and satisfies the mild decay condition

$$\sum_{k \in \mathbf{Z}} \sup_{t \in [k,k+1]} |g(t)| \; < \; \infty,$$

then ZG will be continuous. Thus only "unpleasant" functions will generate W–H frames for $L^2(\mathbf{R})$. In particular, the Gaussian function $G(t) = e^{-\pi t^2}$ will not generate a W–H frame when $a = b$.

The situation is more complicated in the discrete case, for here zg is defined only on disconnected vertical lines. Therefore, there is no concept of continuity in the k variable, a fact which will allow us to construct discrete W–H frames from sequences which do have good decay.

In the sequel, when we speak of the continuity of zg, we will mean only that zg is continuous in the ω variable on each vertical line.

Theorem 5.4. *If $g \in \ell^1(\mathbf{Z})$ then zg is well-defined; in fact, the series defining $zg(k, \cdot)$ converges uniformly for each $k \in \mathbf{Z}$ to a continuous function.*

Proof. Given any fixed $k \in \mathbf{Z}$, we have the following for each $\omega \in \hat{\mathbf{R}}$:

$$|zg(k,\omega)| \; = \; \left| \sum_{k \in \mathbf{Z}} g(k + ja)\, e^{2\pi i j \omega} \right|$$

$$\begin{aligned} &\leq \sum_{k\in\mathbf{Z}} |g(k+ja)| \\ &\leq \sum_{k\in\mathbf{Z}} |g(k)| \\ &< \infty. \end{aligned}$$

Therefore $zg(k,\cdot)$ converges uniformly, and so the limit must be continuous since each term $g(k+ja)\,e^{2\pi ij\omega}$ is continuous. $\square$

Although not needed, we mention the following additional facts about the Zak transform of ℓ^1-sequences. The proof is similar to the continuous version in [H2].

Theorem 5.5. *Given $1 \leq p < 2$, the Zak transform is a linear, continuous, injective mapping of $\ell^p(\mathbf{Z})$ into $\ell^p(\mathbf{q})$ with $\|z\| = 1$. Moreover,* $\mathrm{Range}(z)$ *is dense in $\ell^p(\mathbf{q})$, but z is not surjective, and $z^{-1}\colon \mathrm{Range}(z) \to \ell^p(\mathbf{Z})$ is not continuous.*

Theorem 5.6. *The Zak transform cannot be defined as a map of $\ell^q(\mathbf{Z})$ into any $\ell^r(\mathbf{q})$ when $q > 2$.*

We now examine the properties of sequences whose Zak transform is continuous.

Theorem 5.7. *Assume $g \in \ell^2(\mathbf{Z})$ is such that zg is continuous.*

1. *If g is odd (i.e., $g(-k) = -g(k)$ for $k \in \mathbf{Z}$), then $zg(0,0) = zg(0,1/2) = 0$. If in addition a is even, then $zg(a/2,0) = 0$.*

2. *If g is even (i.e., $g(-k) = g(k)$ for $k \in \mathbf{Z}$), then $zg(a/2,1/2) = 0$.*

Proof. We prove only 2 since 1 is similar. Since g is even we can compute:

$$\begin{aligned} zg(-k,-\omega) &= \sum_{k\in\mathbf{Z}} g(-k+ja)\,e^{-2\pi ij\omega} \\ &= \sum_{k\in\mathbf{Z}} g(-k-ja)\,e^{2\pi ij\omega} \\ &= \sum_{k\in\mathbf{Z}} g(k+ja)\,e^{2\pi ij\omega} \\ &= zg(k,\omega). \end{aligned}$$

Since a is even, $a/2$ is an integer. The quasiperiodicity of zg therefore implies that

$$\begin{aligned} zg(a/2,1/2) &= zg(-a/2,-1/2) \\ &= e^{-2\pi i(-1/2)} zg(a/2,-1/2) \\ &= e^{\pi i} zg(a/2,1/2) \end{aligned}$$

$$= -zg(a/2, 1/2).$$

Therefore $zg(a/2, 1/2) = 0$. □

As a result of this theorem, no sequence $g \in \ell^1(\mathbf{Z})$ which is odd can generate a W–H frame for any integer $a \in \mathbf{Z}_+$ if we take $b = a$. Also, no even sequence $g \in \ell^1(\mathbf{Z})$ can generate a W–H frame for any *even* values of $a \in \mathbf{Z}_+$ (with $b = a$). In particular, the sampled Gaussian sequence $g(k) = e^{-\pi k^2}$ will not generate a frame for any even $a \in \mathbf{Z}_+$. This still leaves the question of odd values of a, which we address in the next section.

6. The Gaussian

The *Guassian function and sequence* are:

$$\begin{aligned} G(t) &= e^{-rt^2}, \qquad t \in \mathbf{R}; \\ g(k) &= e^{-rk^2}, \qquad k \in \mathbf{Z}; \end{aligned}$$

where $r > 0$ is fixed. Since G is well-behaved, we know that ZG will be continuous and must therefore have at least one zero in the fundamental rectangle $\mathbf{Q}$. Direct calculation gives

$$\begin{aligned} ZG(t, \omega) &= \sum_j G(t + ja)\, e^{2\pi i j\omega} \\ &= \sum_j e^{-rt^2}\, e^{-2rtja}\, e^{-rj^2a^2}\, e^{2\pi i j\omega} \\ &= e^{-rt^2} \sum_j \bigl(e^{-ra^2}\bigr)^{j^2} e^{4\pi i j\left(\frac{\omega}{2} - \frac{rta}{2\pi i}\right)} \\ &= e^{-rt^2}\, \theta_3\bigl(\tfrac{\omega}{2} + \tfrac{irta}{2\pi}, e^{-ra^2}\bigr), \end{aligned}$$

where θ_3 is the third Jacobi theta function

$$\theta_3(w, q) = 1 + 2\sum_{j=1}^{\infty} q^{j^2} \cos(4\pi j w) = \sum_{j=-\infty}^{\infty} q^{j^2} e^{4\pi i j w}.$$

(see [R] for an exposition of the properties of the four theta functions). Now, the zeroes of $\theta_3(\cdot, e^{-ra^2})$ occur precisely at the points

$$\frac{1}{4} + \frac{ra^2 i}{4\pi} + \frac{m}{2} + \frac{nra^2 i}{2\pi},$$

for $m, n \in \mathbf{Z}$. These correspond to the values $t = \frac{a}{2} + na$ and $\omega = \frac{1}{2} + m$. Hence ZG has a single zero in $\mathbf{Q}$.

Now, zg is given by $zg(k,\omega) = ZG(k,\omega)$ for $(k,\omega) \in \mathbf{q}$. If a is even then the point $(a/2, 1/2)$ lies in $\mathbf{q}$, so zg has a zero. Since ZG is continuous and zg is its restriction, this implies that zg is not bounded below, hence cannot generate a frame for the case $a = b$. However, if a is *odd* then the point $(a/2, 1/2)$ does not lie in $\mathbf{q}$, so zg has *no* zeroes in $\mathbf{q}$. As zg is continuous on $\mathbf{q}$, this implies that it is bounded above and below on $\mathbf{q}$, and therefore does generate a W–H frame for $\ell^2(\mathbf{Z})$ if $b = a$. Moreover, by Theorem 5.3, this frame is exact, so is a bounded unconditional basis for $\ell^2(\mathbf{Z})$ by Theorem 2.5.

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