Applied and Computational Harmonic Analysis **7** (1999), 292–304.

Density of Gabor Frames

Ole Christensen

Department of Mathematics, Technical University of Denmark, Building 303, 2800 Lyngby, Denmark
E-mail: olechr@mat.dtu.dk

Baiqiao Deng

Department of Mathematics, Columbus State University, Columbus, Georgia 31907
E-mail: Deng_Baiqiao@colstate.edu

and

Christopher Heil

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332
E-mail: heil@math.gatech.edu

Communicated by Guy Battle

Received February 3, 1998; revised August 3, 1998

A Gabor system is a set of time-frequency shifts $S(g,\Lambda) = \{e^{2\pi ibx} g(x-a)\}_{(a,b)\in\Lambda}$ of a function $g \in L^2(\mathbf{R}^d)$. We prove that if a finite union of Gabor systems $\bigcup_{k=1}^r S(g_k, \Lambda_k)$ forms a frame for $L^2(\mathbf{R}^d)$ then the lower and upper Beurling densities of $\Lambda = \bigcup_{k=1}^r \Lambda_k$ satisfy $D^-(\Lambda) \geq 1$ and $D^+(\Lambda) < \infty$. This extends recent work of Ramanathan and Steger. Additionally, we prove the conjecture that no collection $\bigcup_{k=1}^r \{g_k(x-a)\}_{a\in\Gamma_k}$ of pure translates can form a frame for $L^2(\mathbf{R}^d)$.

Key Words: Beurling density; frame; frame of translates; Gabor frame; Riesz basis.

1. INTRODUCTION

For each a, $b \in \mathbf{R}^d$, let T_a and M_b denote the translation and modulation operators on $L^2(\mathbf{R}^d)$ defined by

$$T_a g(x) \;=\; g(x-a) \qquad \text{and} \qquad M_b g(x) \;=\; e^{2\pi ibx} g(x),$$

where $bx = b_1x_1 + \cdots + b_dx_d$. A *time-frequency shift* is a composition of modulation and translation, i.e., it has the form

$$M_b T_a g(x) \;=\; e^{2\pi ibx} g(x-a).$$

If $\Gamma \subset \mathbf{R}^d$, then the collection of translates of g along Γ is defined to be

$$T(g, \Gamma) = \{T_a g\}_{a \in \Gamma}.$$

If $\Lambda \subset \mathbf{R}^{2d}$, then the collection of time-frequency shifts of g along Λ is defined to be

$$S(g, \Lambda) = \{M_b T_a g\}_{(a,b) \in \Lambda}.$$

We refer to $S(g, \Lambda)$ as the *Gabor system* generated by g and Λ.

Gabor systems which form frames for $L^2(\mathbf{R}^d)$ have a wide variety of applications. One important problem is therefore to determine sufficient conditions on g and Λ which imply that $S(g, \Lambda)$ is a frame. In the case that $d = 1$ and Λ is a regular lattice of the form $a\mathbf{Z} \times b\mathbf{Z}$, sufficient conditions for $S(g, \Lambda)$ to form a frame for $L^2(\mathbf{R})$ were found by Daubechies [4]. A generalization of this result requiring weaker assumptions, which also applies when $S(g, \Lambda)$ only forms a frame for its closed span instead of all of $L^2(\mathbf{R})$, was obtained recently in [2].

Our point of view in this paper is somewhat different, in that we are concerned with the connection between density properties of Λ and frame properties of $S(g, \Lambda)$, and the analogous problem for systems $T(g, \Gamma)$ of pure translates. For the case of Gabor systems, there is a rich literature on this subject, especially when Λ is the rectangular lattice $\Lambda = a\mathbf{Z}^d \times b\mathbf{Z}^d$. We briefly review here some of the main results connecting the density of Λ to properties of $S(g, \Lambda)$, and refer to the research-survey [1] for a more thorough historical discussion and a review of properties of Gabor systems. In addition to the papers that we discuss explicitly below, some relevant related articles include [9-11, 15].

For simplicity, consider the one-dimensional setting $d = 1$ and a rectangular lattice $\Lambda = a\mathbf{Z} \times b\mathbf{Z}$. In this case, Rieffel proved (as a corollary of results on C^* algebras) that $S(g, \Lambda)$ is incomplete in $L^2(\mathbf{R})$ if $ab > 1$ [14]. The algebraic structure of the lattice is crucial to this result, as the proof follows from computing the coupling constant of the von Neumann algebra generated by the operators $\{M_{mb} T_{na}\}_{m,n \in \mathbf{Z}}$. For the case that $ab > 1$ is rational, Daubechies provided a constructive proof of the incompleteness of $S(g, \Lambda)$ through the use of the Zak transform, which is again an algebraic tool highly dependent on the lattice structure of $\Lambda = a\mathbf{Z} \times b\mathbf{Z}$ [4]. Ramanathan and Steger introduced a technique that applies to countable, non-lattice sets Λ that are uniformly separated, i.e., there is a minimum distance δ between elements of Λ [13]. It is possible to define an upper Beurling density $D^+(\Lambda)$ and lower Beurling density $D^-(\Lambda)$ for such sets (the precise definition of density, along with other fundamental concepts used in this paper, is given in Section 2). For example, for the lattice $\Lambda = a\mathbf{Z} \times b\mathbf{Z}$ these two densities coincide and equal $1/(ab)$, hence this lattice is said to have uniform Beurling density $D(\Lambda) = 1/(ab)$. Ramanathan and Steger proved for arbitrary uniformly separated sets Λ that if $D^-(\Lambda) < 1$ then $S(g, \Lambda)$ is not a frame. Thus, in the case that $\Lambda = a\mathbf{Z} \times b\mathbf{Z}$, this can be viewed as a weak version of the Rieffel incompleteness result. On the other hand, the Ramanathan/Steger result applies to a far broader class of time-frequency translates than does the Rieffel result. Moreover, Ramanathan and Steger were able to recapture by their techniques the full Rieffel incompleteness result in the case that $\Lambda = a\mathbf{Z} \times b\mathbf{Z}$. In light of the above discussion, Ramanathan and Steger therefore conjectured that $S(g, \Lambda)$ must be incomplete whenever Λ is a uniformly separated set satisfying $D^-(\Lambda) < 1$. Walnut and Heil showed that this conjecture is false by constructing for each $\varepsilon > 0$ a function $g \in L^2(\mathbf{R})$ and a non-lattice $\Lambda \subset \mathbf{R}^2$ such that $S(g, \Lambda)$ is complete in $L^2(\mathbf{R})$ yet Λ has uniform Beurling density $D(\Lambda) < \varepsilon$ [1]. Hence the algebraic structure of Λ is in fact critical for the Rieffel incompleteness result.

In this paper, we extend and apply the Ramanathan/Steger density results. The extension is to higher dimensions, to multiple generating functions, and to completely arbitrary sets of time-frequency shifts. To state our result, for each $k = 1, \dots, r$ let g_k be an element of $L^2(\mathbf{R}^d)$ and let $\Lambda_k = \{(a_{k,i}, b_{k,i})\}_{i \in I_k}$ be a sequence of points in $\mathbf{R}^{2d}$. Unless specified otherwise, we place no restrictions on the sequences Λ_k. For example, the index set I_k may be countable or uncountable, and repetitions of points in Λ_k are allowed. For simplicity, we will write $\Lambda_k \subset \mathbf{R}^{2d}$, although we always mean that Λ_k is a sequence of points from $\mathbf{R}^{2d}$ and not merely a subset of $\mathbf{R}^{2d}$. Define an index set $I = \{(i,k) : i \in I_k,\, k = 1, \dots, r\}$ and sequence $\Lambda = \{(a_{k,i}, b_{k,i})\}_{(i,k)\in I} = \{(a_{k,i}, b_{k,i})\}_{i \in I_k, k=1,\dots,r}$, i.e., Λ is the sequence obtained by amalgamating $\Lambda_1, \dots, \Lambda_r$. For simplicity, we write $\Lambda = \bigcup_{k=1}^r \Lambda_k$, and say that Λ is the disjoint union of $\Lambda_1, \dots, \Lambda_r$. The Gabor system generated by $g_1, \dots, g_r$ and $\Lambda_1, \dots, \Lambda_r$ is then $\bigcup_{k=1}^r S(g_k, \Lambda_k)$, the disjoint union of the Gabor systems $S(g_k, \Lambda_k)$. With this notation, our first main result is the following.

Theorem 1.1. *For each $k = 1, \dots, r$, choose a nonzero function $g_k \in L^2(\mathbf{R}^d)$ and an arbitrary sequence $\Lambda_k \subset \mathbf{R}^{2d}$. Let Λ be the disjoint union of $\Lambda_1, \dots, \Lambda_r$.*

(a) *If $\bigcup_{k=1}^r S(g_k, \Lambda_k)$ possesses an upper frame bound for $L^2(\mathbf{R}^d)$, then $D^+(\Lambda) < \infty$.*

(b) *If $\bigcup_{k=1}^r S(g_k, \Lambda_k)$ is a frame for $L^2(\mathbf{R}^d)$, then $D^-(\Lambda) \geq 1$.*

We remark that the conclusion in part (a) of Theorem 1.1 that Λ has finite upper Beurling density is equivalent to the statement that Λ, and hence each Λ_k, is relatively uniformly separated, i.e., is a finite union of uniformly separated sequences (and hence must be countable). The proof of Theorem 1.1 is given in Section 3. The result of Ramanathan and Steger in [13] corresponds to the special case of Theorem 1.1(b) with $d = 1$ and $k = 1$ and with the added assumption that Λ is uniformly separated and satisfies $D^+(\Lambda) < \infty$.

One useful feature of Gabor frames $\bigcup_k S(g_k, \Lambda_k)$ generated by functions g_k that are well localized in both time and frequency is that if a function f is expanded in this frame, then a perturbation of f that is well localized in both time and frequency will have a local effect on the frame coefficients. By comparison, a frame of the form $\bigcup_k T(g_k, \Gamma_k)$ consisting solely of translates of finitely many functions g_k would have the desirable property that perturbations localized solely in time have localized effects on the frame coefficients. In this regard, Olson and Zalik proved that there do not exist any Riesz bases for $L^2(\mathbf{R})$ generated by translates of a single function [12], and Christensen conjectured that there are no frames for $L^2(\mathbf{R})$ of this form [3]. Since $T(g, \Gamma) = S(g, \Gamma \times \{0\})$, systems of translates can be considered to be special cases of Gabor systems. We show in Section 4 that Theorem 1.1 implies that there are no frames for $L^2(\mathbf{R}^d)$ of the form $\bigcup_k T(g_k, \Gamma_k)$. Moreover, we give a direct proof of the following refinement of this statement.

Theorem 1.2. *For each $k = 1, \dots, r$, choose a nonzero function $g_k \in L^2(\mathbf{R}^d)$ and an arbitrary sequence $\Gamma_k \subset \mathbf{R}^d$. Let Γ be the disjoint union of $\Gamma_1, \dots, \Gamma_r$.*

(a) *If $\bigcup_{k=1}^r T(g_k, \Gamma_k)$ possesses an upper frame bound for $L^2(\mathbf{R}^d)$, then $D^+(\Gamma) < \infty$.*

(b) *If $\bigcup_{k=1}^r T(g_k, \Gamma_k)$ possesses a lower frame bound for $L^2(\mathbf{R}^d)$, then $D^+(\Gamma) = \infty$.*

2. NOTATION AND PRELIMINARIES

2.1. General Notation

We use the Euclidean norm $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$ on $\mathbf{R}^d$. We denote the dot product on $\mathbf{R}^d$ by a simple juxtaposition, i.e., $xy = x_1 y_1 + \cdots + x_d y_d$.

For $x \in \mathbf{R}^d$ and $h > 0$ we let $Q_h(x)$ denote the cube centered at x with side lengths h:

$$Q_h(x) \;=\; \prod_{j=1}^{d} [x_j - h/2, x_j + h/2).$$

In particular, $\{Q_h(hn)\}_{n \in \mathbf{Z}^d}$ is a disjoint cover of $\mathbf{R}^d$. To distinguish between cubes in $\mathbf{R}^d$ and those in $\mathbf{R}^{2d}$, we write $\mathbf{Q}_h(x, y) = Q_h(x) \times Q_h(y)$ for a cube in $\mathbf{R}^{2d}$.

The Lebesgue measure of $E \subset \mathbf{R}^d$ is denoted by $|E|$. In particular, the volume of the cube $Q_h(x)$ is $|Q_h(x)| = h^d$. The number of points in $E \subset \mathbf{R}^d$ is denoted by $\#E$.

The L^2-inner product is $\langle f, g\rangle = \int f(x)\, \overline{g(x)}\, dx$. The short-time Fourier transform of $f \in L^2(\mathbf{R}^d)$ against $g \in L^2(\mathbf{R}^d)$ is

$$S_g f(a, b) \;=\; \langle f, M_b T_a g\rangle.$$

We have $S_g f \in L^2(\mathbf{R}^{2d}) \cap C_0(\mathbf{R}^{2d})$, with $\|S_g f\|_2 \;=\; \|f\|_2\, \|g\|_2$.

Given a closed subspace $V \subset L^2(\mathbf{R}^d)$, we let P_V denote the orthogonal projection onto V. Then for any $f \in L^2(\mathbf{R}^d)$,

$$\operatorname{dist}(f, V) \;=\; \|f - P_V f\|_2 \;=\; \inf_{u \in V} \|f - u\|_2.$$

2.2. Frames

A family of elements $\{f_i\}_{i \in I}$ is a *frame* for a Hilbert space H if there exist constants A, $B > 0$ such that

$$\forall\, f \in H, \quad A\, \|f\|^2 \;\le\; \sum_{i \in I} |\langle f, f_i\rangle|^2 \;\le\; B\, \|f\|^2. \tag{1}$$

The numbers A, B are called *frame bounds.* The *frame operator* $Sf \;=\; \sum_i \langle f, f_i\rangle\, f_i$ is a bounded, invertible, and positive mapping of H onto itself. This provides the *frame decomposition*

$$f \;=\; S^{-1} S f \;=\; \sum_{i \in I} \langle f, f_i\rangle\, \tilde{f}_i, \quad \forall\, f \in H, \tag{2}$$

where $\tilde{f}_i = S^{-1} f_i$. The family $\{\tilde{f}_i\}$ is also a frame for H, called the *dual frame* of $\{f_i\}$, and has frame bounds B^{-1}, A^{-1}. The utility of frames, as compared to sets of functions that are merely complete in $L^2(\mathbf{R}^d)$, often lies in the stable reconstruction formula (2).

Riesz bases are special cases of frames, and can be characterized as those frames which are biorthogonal to their dual frames, i.e., such that $\langle f_i, \tilde{f}_j\rangle = \delta_{ij}$.

An arbitrary family $\{f_i\}$ which satisfies the first inequality in (1) (and which may or may not satisfy the second inequality) is said to possess a lower frame bound. Likewise, a family $\{f_i\}$ which satisfies at least the second inequality in (1) is said to possess an upper frame bound. Such a family is also called a *Bessel sequence*.

Additional information on frames can be found in [4, 7].

2.3. Density

We now give several definitions related to the "density" of an arbitrary sequence $\Gamma = \{\gamma_i\}_{i\in I}$ of points of $\mathbf{R}^d$. The index set may be countable or uncountable, and since Γ is regarded as a sequence, repetitions of elements of Γ are allowed.

DEFINITION 2.1. Let $\Gamma = \{\gamma_i\}_{i\in I} \subset \mathbf{R}^d$.

(a) Γ is *δ-uniformly separated* if $\delta = \inf_{i\neq j} |\gamma_i - \gamma_j| > 0$. The number δ is the *separation constant*.

(b) Γ is *relatively uniformly separated* if it is a finite union of uniformly separated sequences Γ_k. More precisely, this means that I can be partitioned into disjoint sets $I_1, \dots, I_r$ such that each sequence $\Gamma_k = \{\gamma_i\}_{i\in I_k}$ is δ_k-uniformly separated for some $\delta_k > 0$.

DEFINITION 2.2. Let $\Gamma = \{\gamma_i\}_{i\in I} \subset \mathbf{R}^d$. For each $h > 0$, let $\nu^+(h)$ and $\nu^-(h)$ denote the largest and smallest numbers of points of Γ that lie in any $Q_h(x)$:

$$\nu^+(h) \;=\; \max_{x\in\mathbf{R}^d} \#\big(\Gamma \cap Q_h(x)\big) \qquad \text{and} \qquad \nu^-(h) \;=\; \min_{x\in\mathbf{R}^d} \#\big(\Gamma \cap Q_h(x)\big).$$

We have $0 \le \nu^-(h) \le \nu^+(h) \le \infty$ for each h. The *upper and lower Beurling densities* of Γ are then

$$D^+(\Gamma) \;=\; \limsup_{h\to\infty} \frac{\nu^+(h)}{h^d} \qquad \text{and} \qquad D^-(\Gamma) \;=\; \liminf_{h\to\infty} \frac{\nu^-(h)}{h^d}.$$

We have $0 \le D^-(\Gamma) \le D^+(\Gamma) \le \infty$. If $D^+(\Gamma) = D^-(\Gamma)$, then Γ is said to have *uniform Beurling density* $D(\Gamma) = D^+(\Gamma) = D^-(\Gamma)$.

Note that if Γ is the disjoint union of $\Gamma_1, \dots, \Gamma_r$, then we always have

$$\#\big(\Gamma \cap Q_h(x)\big) \;=\; \sum_{k=1}^{r} \#\big(\Gamma_k \cap Q_h(x)\big),$$

and therefore

$$\sum_{k=1}^{r} D^-(\Gamma_k) \;\le\; D^-(\Gamma) \;\le\; D^+(\Gamma) \;\le\; \sum_{k=1}^{r} D^+(\Gamma_k). \tag{3}$$

Some or all of the inequalities in (3) may be strict. For example, if Γ_1 is the set of negative integers, Γ_2 is the positive integers, and $\Gamma = \Gamma_1 \cup \Gamma_2$, then $D^-(\Gamma_1) = D^-(\Gamma_2) = 0$, $D^-(\Gamma) = D^+(\Gamma) = 1$, and $D^+(\Gamma_1) = D^+(\Gamma_2) = 1$.

The following lemma provides some equivalent ways to view the meaning of finite upper Beurling density.

LEMMA 2.3. *Let $\Gamma = \{\gamma_i\}_{i\in I}$ be any sequence of points in $\mathbf{R}^d$. Then the following statements are equivalent.*

(a) $D^+(\Gamma) < \infty$.

(b) Γ *is relatively uniformly separated.*

(c) *For some (and therefore every) $h > 0$, there is an integer $N_h > 0$ such that each cube $Q_h(hn)$ contains at most N_h points of Γ. That is,*

$$N_h \;=\; \sup_{n\in\mathbf{Z}^d} \#\big(\Gamma \cap Q_h(hn)\big) \;<\; \infty.$$

Proof. (a) $\Rightarrow$ (c). If $D^+(\Gamma) < \infty$ then $\nu^+(h)/h^d < \infty$ for some h.

(c) $\Rightarrow$ (b). Assume that there is an $h > 0$ such that each cube $Q_h(hn)$ contains at most N_h elements of Γ. Let $e_1, \dots, e_{2^d}$ denote the vertices of the unit cube $[0,1]^d$, and define

$$Z_k \;=\; (2\mathbf{Z})^d + e_k \qquad \text{and} \qquad B_k \;=\; \bigcup_{n\in Z_k} Q_h(hn).$$

Then $\mathbf{R}^d$ is the disjoint union of the 2^d sets B_k. Moreover, if $m,\, n \in Z_k$ with $m \neq n$, then

$$\operatorname{dist}(Q_h(hm), Q_h(hn)) \;=\; \inf\{|x-y| : x \in Q_h(hm),\, y \in Q_h(hn)\} \geq h.$$

Further, each cube $Q_h(hn)$ contains at most N_h elements of Γ, so the sequences $\{\gamma_i : \gamma_i \in B_k\}$ can be split into N_h uniformly separated sequences. Hence the entire sequence Γ can be split into $2^d N_h$ uniformly separated sequences.

(b) $\Rightarrow$ (a). Assume that Γ is relatively uniformly separated. Then we can partition I into sets $I_1, \dots, I_r$ in such a way that each sequence $\Gamma_k = \{\gamma_i\}_{i\in I_k}$ is δ_k-uniformly separated. Let $\delta = \min\{\delta_1/2, \dots, \delta_r/2\}$. Then any cube $Q_\delta(x)$ contains at most one element of Γ_k, and therefore contains at most r elements of Γ. Therefore, if h is any positive number then $Q_{h\delta}(x)$ can contain at most $r(h{+}1)^d$ elements of Γ. Hence $\nu^+(h\delta) \leq r(h+1)^d$ for each h, so

$$D^+(\Gamma) \;\leq\; \limsup_{h\to\infty} \frac{r(h+1)^d}{(h\delta)^d} \;=\; \frac{r}{\delta^d} \;<\; \infty. \quad \square$$

3. DENSITY OF GABOR FRAMES

We will prove Theorem 1.1 in this section. We consider part (a) and part (b) of the theorem separately. In particular, we begin by considering the special case of Theorem 1.1(a) when $r = 1$.

THEOREM 3.1. *Choose a nonzero $g \in L^2(\mathbf{R}^d)$ and a sequence $\Lambda \subset \mathbf{R}^{2d}$. If $S(g,\Lambda)$ possesses an upper frame bound, then Λ is relatively uniformly separated.*

Proof. Assume that Λ is not relatively uniformly separated. Choose any $f \in L^2(\mathbf{R}^d)$ with $\|f\|_2 = 1$, and note that

$$|\langle M_qT_pf, M_bT_ag\rangle| \;=\; |\langle f, M_{b-q}T_{a-p}g\rangle| \;=\; |S_gf(a-p, b-q)|.$$

Since S_gf is nonzero and continuous on $\mathbf{R}^{2d}$, it must be bounded away from zero on some cube, say,

$$\mu \;=\; \inf_{(x,y)\in\mathbf{Q}_h(c,d)} |S_gf(x,y)| \;>\; 0.$$

Now choose any $N > 0$. Then, by Lemma 2.3 applied to Λ, there exists some cube $\mathbf{Q}_h(p,q)$ which contains at least N elements of Λ. However, if $(a,b) \in \mathbf{Q}_h(p,q)$, then $(a-p+c, b-q+d) \in \mathbf{Q}_h(c,d)$, so

$$\sum_{(a,b)\in\Lambda\cap\mathbf{Q}_h(p,q)} |\langle M_{q-d}T_{p-c}f, M_bT_ag\rangle|^2 = \sum_{(a,b)\in\Lambda\cap\mathbf{Q}_h(p,q)} |S_gf(a-p+c, b-q+d)|^2 \;\geq\; N\mu^2.$$

Since $\|M_{q-d}T_{p-c}f\|_2 = 1$, it follows that $S(g,\Lambda)$ cannot possess an upper frame bound. □

Proof of Theorem 1.1(a). Suppose that $\bigcup_k S(g_k,\Lambda_k)$ possesses an upper frame bound. Then, by Theorem 3.1, each Λ_k is relatively uniformly separated. Hence Λ is a finite union of relatively uniformly separated sequences, therefore is itself relatively uniformly separated, and hence has finite upper Beurling density. □

We will now give the proof of Theorem 1.1(b). The insight provided by [13] is that Gabor frames possess a certain Homogeneous Approximation Property, or HAP. This is stated below in our general context as Theorem 3.4. The proof given by Ramanathan and Steger relied on weak convergence of translations of Λ. Gröchenig and Razafinjatovo proved an analogue of the HAP for frames of translates in the space of bandlimited functions [6]. Their proof was considerably shorter than the method of [13], but required a restriction on the frame generators. In our context of Gabor systems, this restriction stems from the fact that the local maximal function of the short-time Fourier transform S_gf is not necessarily square-integrable. We will provide a simple proof of the HAP which imposes no restriction on the generators.

Notation 3.2. Let $g_1,\dots,g_r \in L^2(\mathbf{R}^d)$ and $\Lambda_1,\dots,\Lambda_r \subset \mathbf{R}^{2d}$ be such that $\bigcup_k S(g_k,\Lambda_k)$ is a frame for $L^2(\mathbf{R}^d)$, with frame bounds A, B. Let

$$\{\tilde{g}_{k,a,b}\}_{(a,b)\in\Lambda_k, k=1,\dots,r}$$

denote the dual frame of $\bigcup_k S(g_k,\Lambda_k)$. In general, this dual frame need not consist of translates and modulates of some finite set of functions.

Given $h > 0$ and $(p,q) \in \mathbf{R}^{2d}$, let $W(h,p,q)$ denote the following subspace of $L^2(\mathbf{R}^d)$:

$$W(h,p,q) \;=\; \operatorname{span}\{\tilde{g}_{k,a,b} : (a,b) \in \mathbf{Q}_h(p,q)\cap\Lambda_k,\ k = 1,\dots,r\}. \tag{4}$$

This space is finite-dimensional because, by Theorem 1.1(a), each Λ_k is relatively uniformly separated.

Lemma 3.3. *Set $\varphi(x) = e^{-(\pi/2)x^2}$, and let $h > 0$ be fixed. Then there exists a constant K such that for each $f \in L^2(\mathbf{R}^d)$ and each $(p,q) \in \mathbf{R}^{2d}$,*

$$|\langle\varphi, M_qT_pf\rangle|^2 \;\leq\; K \iint_{\mathbf{Q}_h(p,q)} |\langle\varphi, M_yT_xf\rangle|^2\,dx\,dy.$$

Proof. The Bargmann transform

$$Bf(x+iy) \;=\; e^{(\pi/2)(x^2+y^2)}\, e^{\pi ixy}\langle M_yT_{-x}f, \varphi\rangle$$

maps $L^2(\mathbf{R}^d)$ into the space of entire functions on $\mathbf{C}^{2d}$ [5, p. 40]. Hence, by [8, Theorem 2.2.3], there exists a constant C, independent of f, such that

$$|Bf(0)| \;\le\; C \iint_{\mathbf{Q}_h(0,0)} |Bf(z)|\, dz. \tag{5}$$

Applying (5) to the function M_qT_pf therefore yields

$$\begin{aligned} |\langle \varphi, M_qT_pf\rangle|^2 \;&=\; |B(M_qT_pf)(0)|^2 \\ &\le\; C^2 \left(\iint_{\mathbf{Q}_h(0,0)} |e^{\frac{\pi}{2}(x^2+y^2)}\, \langle M_yT_{-x}(M_qT_pf), \varphi\rangle|\, dx\, dy \right)^2 \\ &\le\; C^2 \left(\iint_{\mathbf{Q}_h(0,0)} e^{\pi(x^2+y^2)}\, dx\, dy \right) \\ &\quad\; \times \left(\iint_{\mathbf{Q}_h(0,0)} |\langle M_{q+y}T_{p-x}f, \varphi\rangle|^2\, dx\, dy \right) \\ &=\; K \iint_{\mathbf{Q}_h(p,q)} |\langle M_yT_xf, \varphi\rangle|^2\, dx\, dy. \quad \square \end{aligned}$$

We can now state the HAP. Our simple proof follows by observing that the HAP for time-frequency shifts of a single function implies the HAP for all functions.

THEOREM 3.4 (Homogeneous Approximation Property). *Let $g_1, \dots, g_r \in L^2(\mathbf{R}^d)$ and $\Lambda_1, \dots, \Lambda_r \subset \mathbf{R}^{2d}$ be such that $\bigcup_{k=1}^r S(g_k, \Lambda_k)$ is a frame for $L^2(\mathbf{R}^d)$. Then for each $f \in L^2(\mathbf{R}^d)$,*

$$\forall\, \varepsilon > 0, \quad \exists\, R > 0, \quad \forall\, (p,q) \in \mathbf{R}^{2d}, \quad \operatorname{dist}\bigl(M_qT_pf,\, W(R,p,q)\bigr) \;<\; \varepsilon. \tag{6}$$

Proof. By Theorem 1.1(a), the assumption that $\bigcup_k S(g_k, \Lambda_k)$ is a frame implies that each Λ_k is relatively uniformly separated. By dividing each Λ_k into subsequences that are uniformly separated, we may assume without loss of generality that each Λ_k is δ_k-uniformly separated. Define $\delta = \min\{\delta_1/2, \dots, \delta_r/2\}$.

Let $\mathcal{H}$ be the subset of $L^2(\mathbf{R})$ consisting of all functions f for which (6) holds. It is easy to see that $\mathcal{H}$ is closed under finite linear combinations and L^2-limits. It therefore suffices to show that, for the Gaussian function $\varphi(x) = e^{-(\pi/2)x^2}$, all time-frequency shifts $M_tT_s\varphi$ belong to $\mathcal{H}$, for then $\mathcal{H} = L^2(\mathbf{R}^d)$ and the result follows.

Therefore, fix any $(s,t) \in \mathbf{R}^{2d}$, and consider any $(p,q) \in \mathbf{R}^{2d}$. The function $M_qT_p(M_tT_s\varphi)$ has the frame expansion

$$M_qT_p(M_tT_s\varphi) \;=\; \sum_{k=1}^r \sum_{(a,b)\in\Lambda_k} \langle M_qT_p(M_tT_s\varphi), M_bT_ag_k\rangle\, \tilde{g}_{k,a,b}.$$

By definition of distance and the fact that $\{\tilde{g}_{k,a,b}\}$ is itself a frame with upper frame bound A^{-1}, we have

$$\begin{aligned} &\operatorname{dist}\bigl(M_qT_p(M_tT_s\varphi),\, W(R,p,q)\bigr)^2 \\ &\quad\le\; \left\| M_qT_p(M_tT_s\varphi) \;-\; \sum_{k=1}^r \sum_{(a,b)\in\mathbf{Q}_R(p,q)\cap\Lambda_k} \langle M_qT_p(M_tT_s\varphi), M_bT_ag_k\rangle\, \tilde{g}_{k,a,b} \right\|_2^2 \end{aligned}$$

$$= \Bigl\| \sum_{k=1}^{r} \sum_{(a,b)\in\Lambda_k\setminus\mathbf{Q}_R(p,q)} \langle M_qT_p(M_tT_s\varphi), M_bT_ag_k\rangle\, \tilde{g}_{k,a,b} \Bigr\|_2^2$$

$$\leq\; A^{-1} \sum_{k=1}^{r} \sum_{(a,b)\in\Lambda_k\setminus\mathbf{Q}_R(p,q)} |\langle M_qT_p(M_tT_s\varphi), M_bT_ag_k\rangle|^2. \tag{7}$$

By Lemma 3.3, there exists a constant K such that

$$\begin{aligned} |\langle M_qT_p(M_tT_s\varphi), M_bT_ag_k\rangle|^2 \;&=\; |\langle \varphi, M_{b-q-t}T_{a-p-s}g_k\rangle|^2 \\ &\leq\; K \iint_{\mathbf{Q}_\delta(a-p-s,b-q-t)} |\langle\varphi, M_yT_xg_k\rangle|^2\,dx\,dy \\ &=\; K \iint_{\mathbf{Q}_\delta(p+s-a,q+t-b)} |S_\varphi g_k(x,y)|^2\,dx\,dy, \end{aligned} \tag{8}$$

where $S_\varphi g_k$ is the short-time Fourier transform of g_k against φ. Combining (7) and (8) with the fact that Λ_k is δ-separated, we conclude that

$$\begin{aligned} &\operatorname{dist}\bigl(M_qT_p(M_tT_s\varphi),\, W(R,p,q)\bigr)^2 \\ &\quad\leq\; A^{-1}K \sum_{k=1}^{r} \sum_{(a,b)\in\Lambda_k\setminus\mathbf{Q}_R(p,q)} \iint_{\mathbf{Q}_\delta(p+s-a,q+t-b)} |S_\varphi g_k(x,y)|^2\,dx\,dy \\ &\quad\leq\; A^{-1}K \sum_{k=1}^{r} \iint_{\mathbf{R}^{2d}\setminus\mathbf{Q}_{R-\delta}(s,t)} |S_\varphi g_k(x,y)|^2\,dx\,dy. \end{aligned} \tag{9}$$

Since each $S_\varphi g_k \in L^2(\mathbf{R}^{2d})$, the last quantity in (9) can be made arbitrarily small, independently of (p,q), by taking R large enough. ☐

Corollary 3.5 (Strong HAP). *Let $g_1,\dots,g_r \in L^2(\mathbf{R}^d)$ and $\Lambda_1,\dots,\Lambda_r \subset \mathbf{R}^{2d}$ be such that $\bigcup_{k=1}^r S(g_k,\Lambda_k)$ is a frame for $L^2(\mathbf{R}^d)$. Then for each $f \in L^2(\mathbf{R}^d)$ and each $\varepsilon > 0$, there exists a constant $R > 0$ such that*

$$\forall\,(p,q)\in\mathbf{R}^{2d}, \quad \forall\, h>0, \quad \forall\,(x,y)\in\mathbf{Q}_h(p,q), \quad \operatorname{dist}\bigl(M_yT_xf,\, W(h+R,p,q)\bigr) \;<\; \varepsilon.$$

Proof. Simply note that if $(x,y) \in \mathbf{Q}_h(p,q)$, then $W(R,x,y) \subset W(h+R,p,q)$, and therefore $\operatorname{dist}(M_yT_xf, W(h+R,p,q)) \leq \operatorname{dist}(M_yT_xf, W(R,x,y))$. ☐

We now use the Homogeneous Approximation Property to prove the following comparison between the density of a Gabor frame and the density of a Gabor Riesz basis. The double-projection idea of [13] is an important ingredient.

Theorem 3.6 (Comparison Theorem). *Let $g_1,\dots,g_r \in L^2(\mathbf{R}^d)$ and $\Lambda_1,\dots,\Lambda_r \subset \mathbf{R}^{2d}$ be such that $\bigcup_{k=1}^r S(g_k,\Lambda_k)$ is a frame for $L^2(\mathbf{R}^d)$. Let $\phi_1,\dots,\phi_s \in L^2(\mathbf{R}^d)$ and $\Delta_1,\dots,\Delta_s \subset \mathbf{R}^{2d}$ be such that $\bigcup_{k=1}^s S(\phi_k,\Delta_k)$ is a Riesz basis for $L^2(\mathbf{R}^d)$. Let Λ be the disjoint union of $\Lambda_1,\dots,\Lambda_r$ and let Δ be the disjoint union of $\Delta_1,\dots,\Delta_s$. Then*

$$D^-(\Delta) \;\leq\; D^-(\Lambda) \qquad \text{and} \qquad D^+(\Delta) \;\leq\; D^+(\Lambda).$$

Proof. We use the notation defined in Notation 3.2. Additionally, we denote the dual frame of $\bigcup_k S(\phi_k,\Delta_k)$ by $\{\tilde{\phi}_{k,a,b}\}_{(a,b)\in\Delta_k, k=1,\dots,s}$, and, in analogy to the subspaces $W(h,p,q)$ defined in (4), we set

$$V(h,p,q) \;=\; \operatorname{span}\{M_bT_a\phi_k : (a,b)\in\mathbf{Q}_h(p,q)\cap\Delta_k,\; k=1,\dots,s\}.$$

By Theorem 1.1(a), each Δ_k is relatively uniformly separated, and hence $V(h,p,q)$ is finite-dimensional. Since the elements of any frame are uniformly bounded in norm, we can find a constant C such that $\|\tilde{\phi}_{k,a,b}\| \le C$ for all k, a, and b.

Choose now any $\varepsilon > 0$. Then, by Corollary 3.5 applied to the frame $\bigcup_k S(g_k, \Lambda_k)$ and the function $f = \phi_k$, there exists $R_k > 0$ such that

$$\begin{gathered} \forall\, h > 0, \quad \forall\, (p,q) \in \mathbf{R}^{2d}, \quad \forall\, (x,y) \in \mathbf{Q}_h(p,q), \\ \operatorname{dist}\bigl(M_y T_x f,\, W(h + R_k, p, q)\bigr) \;<\; \frac{\varepsilon}{C}. \end{gathered} \tag{10}$$

Let $R = \max\{R_1, \dots, R_s\}$. Then (10) holds for each k when R_k is replaced by R.

Now let $h > 0$ and $(p,q) \in \mathbf{R}^{2d}$ be fixed. For simplicity, let us denote the orthogonal projections onto $V(h,p,q)$ and $W(h+R,p,q)$ by $P_V = P_{V(h,p,q)}$ and $P_W = P_{W(h+R,p,q)}$. Define $T \colon V(h,p,q) \to V(h,p,q)$ by

$$T \;=\; P_{V(h,p,q)}\, P_{W(h+R,p,q)} \;=\; P_V P_W.$$

Then, by the biorthogonality of $\bigcup_k S(\phi_k, \Delta_k)$ and $\{\tilde{\phi}_{k,a,b}\}$, the trace of T can be computed as

$$\operatorname{tr}(T) \;=\; \sum_{k=1}^{s} \sum_{(a,b) \in \mathbf{Q}_h(p,q) \cap \Delta_k} \langle T(M_b T_a \phi_k), \tilde{\phi}_{k,a,b} \rangle.$$

However, for $(a,b) \in \mathbf{Q}_h(p,q) \cap \Delta_k$, we have

$$\begin{aligned} \langle T(M_b T_a \phi_k), \tilde{\phi}_{k,a,b} \rangle \;&=\; \langle P_W M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle \\ &=\; \langle M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle + \langle (P_W - I) M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle. \end{aligned}$$

Now, $P_V M_b T_a \phi_k = M_b T_a \phi_k$ since $M_b T_a \phi_k \in V(h,p,q)$. Therefore,

$$\langle M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle \;=\; \langle P_V M_b T_a \phi_k, \tilde{\phi}_{k,a,b} \rangle \;=\; \langle M_b T_a \phi_k, \tilde{\phi}_{k,a,b} \rangle \;=\; 1,$$

the last equality following from biorthogonality. Further, by (10) we have

$$|\langle (P_W - I) M_b T_a \phi_k, P_V \tilde{\phi}_{k,a,b} \rangle| \;\le\; \|(P_W - I) M_b T_a \phi_k\|_2 \, \|P_V \tilde{\phi}_{k,a,b}\|_2 \;\le\; \frac{\varepsilon}{C} \cdot C \;=\; \varepsilon.$$

Hence, since Δ is the disjoint union of $\Delta_1, \dots, \Delta_s$,

$$\begin{aligned} \operatorname{tr}(T) \;&\ge\; \sum_{k=1}^{s} \sum_{(a,b) \in \mathbf{Q}_h(p,q) \cap \Delta_k} (1 - \varepsilon) \\ &=\; (1-\varepsilon) \sum_{k=1}^{s} \#\bigl(\mathbf{Q}_h(p,q) \cap \Delta_k\bigr) \\ &=\; (1-\varepsilon)\, \#\bigl(\mathbf{Q}_h(p,q) \cap \Delta\bigr). \end{aligned} \tag{11}$$

On the other hand, all eigenvalues of T satisfy $|\lambda| \le \|T\| \le 1$. Hence,

$$\begin{aligned} \operatorname{tr}(T) \;&\le\; \operatorname{rank}(T) \\ &\le\; \dim\bigl(W(h+R,p,q)\bigr) \end{aligned}$$

$$\leq \sum_{k=1}^{s} \#\big(\mathbf{Q}_{h+R}(p,q) \cap \Lambda_k\big)$$

$$= \#\big(\mathbf{Q}_{h+R}(p,q) \cap \Lambda\big). \tag{12}$$

Therefore, by combining (11) and (12), we see that for each $h > 0$ and each $(p,q) \in \mathbf{R}^{2d}$,

$$(1-\varepsilon)\,\#\big(\mathbf{Q}_h(p,q) \cap \Delta\big) \;\leq\; \#\big(\mathbf{Q}_{h+R}(p,q) \cap \Lambda\big).$$

As a consequence,

$$(1-\varepsilon)\,\frac{\#\big(\mathbf{Q}_h(p,q) \cap \Delta\big)}{h^{2d}} \;\leq\; \frac{\#\big(\mathbf{Q}_{h+R}(p,q) \cap \Lambda\big)}{(h+R)^{2d}}\,\frac{(h+R)^{2d}}{h^{2d}}.$$

It follows that

$$(1-\varepsilon)\,D^-(\Delta) \;\leq\; D^-(\Lambda) \qquad \text{and} \qquad (1-\varepsilon)\,D^+(\Delta) \;\leq\; D^+(\Lambda),$$

and since ε is arbitrary, the theorem is proved. □

The proof of part (b) of Theorem 1.1 is now immediate.

Proof of Theorem 1.1(b). Define $\phi = \chi_{Q_1(0)}$ and $\Delta = \mathbf{Z}^d$. Then $S(\phi,\Delta)$ is an orthonormal basis for $L^2(\mathbf{R}^d)$. Therefore, Theorem 3.6 implies that $D^-(\Lambda) \geq D^-(\Delta) = 1$. □

We remark that we cannot replace the conclusion $D^-(\Lambda) \geq 1$ of Theorem 1.1(b) by the stronger statement that $\sum_{k=1}^{r} D^-(\Lambda_k) \geq 1$. For example, consider again the orthonormal basis $S(\phi,\Delta)$ defined by $\phi = \chi_{Q_1(0)}$ and $\Delta = \mathbf{Z}^d$. We do have $D^-(\Delta) \geq 1$. However, if we define $\Delta_1 = \{n = (n_1,\dots,n_d) \in \mathbf{Z}^d : n_1 \geq 0\}$ and $\Delta_2 = \{n = (n_1,\dots,n_d) \in \mathbf{Z}^d : n_1 < 0\}$, then $S(\phi,\Delta_1) \cup S(\phi,\Delta_2)$ is an orthonormal basis for $L^2(\mathbf{R}^d)$, yet $D^-(\Delta_1) = D^-(\Delta_2) = 0$.

We conclude this section with the following consequence of the Comparison Theorem for Gabor Riesz bases.

Corollary 3.7. *Assume that $\phi_1,\dots,\phi_s \in L^2(\mathbf{R}^d)$ and $\Delta_1,\dots,\Delta_s \subset \mathbf{R}^{2d}$ are such that $\bigcup_{k=1}^{s} S(\phi_k,\Delta_k)$ is a Riesz basis for $L^2(\mathbf{R}^d)$. Let Δ be the disjoint union of $\Delta_1,\dots,\Delta_s$. Then $D^+(\Delta) = D^-(\Delta) = 1$; i.e., Δ has uniform Beurling density $D(\Delta) = 1$.*

Proof. Let $g = \chi_{Q_1(0)}$ and $\Lambda = \mathbf{Z}^d$. Then $S(g,\Lambda)$ is an orthonormal basis, and hence a frame, for $L^2(\mathbf{R}^d)$. Therefore, Theorem 3.6 applied to this frame and to the Riesz basis $\bigcup_k S(\phi_k,\Delta_k)$ implies that $D^-(\Delta) \leq D^-(\Lambda) = 1$ and $D^+(\Delta) \leq D^+(\Lambda) = 1$. By symmetry, we also have $1 = D^-(\Lambda) \leq D^-(\Delta)$ and $1 = D^+(\Lambda) \leq D^+(\Delta)$. □

4. FRAMES OF TRANSLATES

We will prove Theorem 1.2 in this section.

First, however, we observe that Theorem 1.1 already implies that there are no frames consisting of translates of finitely many functions. To see this, assume that $g_1,\dots,g_r \in L^2(\mathbf{R}^d)$ and $\Gamma_1,\dots,\Gamma_r \subset \mathbf{R}^d$ were such that $\bigcup_k T(g_k,\Gamma_k)$ was a frame for $L^2(\mathbf{R}^d)$. Considering that $S(g_k,\Gamma_k \times \{0\}) = T(g_k,\Gamma_k)$, we see that Theorem 1.1(b) implies that

$D^-(\Gamma \times \{0\}) \geq 1$, where Γ is the disjoint union of $\Gamma_1, \ldots, \Gamma_r$. However, this is certainly a contradiction, since $D^-(\Gamma \times \{0\}) = 0$.

Indeed, Theorem 1.1(b) implies that whenever $\bigcup_k S(g_k, \Lambda_k)$ is a frame, the disjoint union Λ cannot contain arbitrarily large gaps, since if for each radius h there existed a point $(x, y) \in \mathbf{R}^{2d}$ such that $\mathbf{Q}_h(x, y)$ contained no points of Λ, then we would have $D^-(\Lambda) = 0$ and therefore could not have a frame. Thus, the collection Λ of time-frequency translates must be "spread" throughout the entire time-frequency plane $\mathbf{R}^{2d}$. For example, Λ could not be restricted to a banded set like $\mathbf{R}^d \times Q_1(0)$, or to a single "quadrant" in $\mathbf{R}^{2d}$.

We now give the proof of Theorem 1.2.

Proof of Theorem 1.2. (a) Assume that $\bigcup_k T(g_k, \Gamma_k)$ possessed an upper frame bound. Then by Theorem 1.1(a), we have $D^+(\Gamma \times \{0\}) < \infty$. This implies that $\Gamma \times \{0\}$ is relatively uniformly separated as a subset of $\mathbf{R}^{2d}$. Hence Γ is relatively uniformly separated as a subset of $\mathbf{R}^d$, and therefore $D^+(\Gamma) < \infty$.

(b) We will prove the contrapositive statement. Assume that $D^+(\Gamma) < \infty$. Then Γ, and therefore each Γ_k, is relatively uniformly separated. Hence for each k we can write Γ_k as the union of subsequences Δ_{kj} for $j = 1, \ldots, s_k$, each of which is δ_{kj}-separated. Define $\delta = \min\{\delta_{kj}/2\}$. Then fix any $h < \delta$, and define $Q = Q_h(0)$. Note that the cubes $\{Q + a\}_{a \in \Delta_{kj}}$ are disjoint, and define

$$B_{kj} = \bigcup_{a \in \Delta_{kj}} (Q + a).$$

Then,

$$\begin{aligned} \sum_{k=1}^{r} \sum_{a \in \Gamma_k} |\langle \chi_Q, T_a g_k \rangle|^2 &= \sum_{k=1}^{r} \sum_{j=1}^{s_k} \sum_{a \in \Delta_{kj}} |\langle \chi_Q, \chi_Q T_a g_k \rangle|^2 \\ &\leq \sum_{k=1}^{r} \sum_{j=1}^{s_k} \sum_{a \in \Delta_{kj}} \|\chi_Q\|_2^2 \, \|\chi_Q T_a g_k\|_2^2 \\ &= \|\chi_Q\|_2^2 \sum_{k=1}^{r} \sum_{j=1}^{s_k} \int_{B_{kj}} |g_k(x)|^2 \, dx. \end{aligned}$$

However, for each fixed k and j, the function $\chi_{B_{kj}}(x) \, |g_k(x)|^2$ converges to zero pointwise a.e. as $h \to 0$, and is dominated by the integrable function $|g_k(x)|^2$. It therefore follows from the Lebesgue Dominated Convergence Theorem that $\lim_{h \to 0} \int_{B_{kj}} |g_k(x)|^2 \, dx = 0$. Hence $\bigcup_k T(g_k, \Gamma_k)$ cannot possess a lower frame bound. □

ACKNOWLEDGMENTS

The authors thank Christopher Lennard for helpful discussions and insights. The third author also thanks Jay Ramanathan for several discussions.

The first author was supported by a grant from the Danish Research Council. The second author was supported by a Faculty Development Grant from the Georgia Institute of Technology. The third author was partially supported by the National Science Foundation under grant DMS-9401340.

The research for this paper was performed while the first and second authors were visiting the School of Mathematics at the Georgia Institute of Technology. These authors thank the School for its hospitality and support.

REFERENCES

[1] J. J. Benedetto, C. Heil, and D. F. Walnut, Differentiation and the Balian–Low theorem, *J. Fourier Anal. Appl.* **1** (1995), 355–402.

[2] P. G. Casazza and O. Christensen, Weyl–Heisenberg frames for subspaces of $L^2(\mathbf{R})$, *preprint* (1998).

[3] O. Christensen, Frames containing a Riesz basis and approximation of the frame coefficients, *J. Math. Anal. Appl.* **199** (1995), 256–270.

[4] I. Daubechies, The wavelet transform, time-frequency localization and signal analysis, *IEEE Trans. Inform. Theory* **39** (1990), 961–1005.

[5] G. B. Folland, "Harmonic Analysis in Phase Space", Princeton University Press, Princeton, NJ, 1989.

[6] K. Gröchenig and H. Razafinjatovo, On Landau's necessary density conditions for sampling and interpolation of band-limited functions, *J. London Math. Soc. (2)* **54** (1996), 557–565.

[7] C. Heil and D. Walnut, Continuous and discrete wavelet transforms, *SIAM Rev.* **31** (1989), 628–666.

[8] L. Hörmander, "An Introduction to Complex Analysis in Several Variables", North-Holland, New York, 1973.

[9] A. J. E. M. Janssen, Signal analytic proofs of two basic results on lattice expansions, *Appl. Comp. Harm. Anal.* **1** (1994), 350–354.

[10] A. J. E. M. Janssen, A density theorem for time-continuous filter banks, *in* "Signal and Image Representation in Combined Spaces" (Y. Zeevi and R. Coifman, eds.), Academic Press, San Diego, 1998, pp. 513–523.

[11] H. Landau, On the density of phase-space expansions, *IEEE Trans. Inform. Th.* **39** (1993), 1152–1156.

[12] T. E. Olson and R. A. Zalik, Nonexistence of a Riesz basis of translates, *in* "Approximation Theory," Lecture Notes in Pure and Applied Math., vol. 138, Dekker, New York, 1992, pp. 401–408.

[13] J. Ramanathan and T. Steger, Incompleteness of sparse coherent states, *Appl. Comp. Harm. Anal.* **2** (1995), 148–153.

[14] M. Rieffel, Von Neumann algebras associated with pairs of lattices in Lie groups, *Math. Ann.* **257** (1981), 403–418.

[15] K. Seip, Sampling and interpolation in the Bargmann–Fock space, I, *J. Reine Angew. Math.* **429** (1992), 91–106.