Chapter 2

Multiwavelets in $\mathbf{R}^n$ with an arbitrary dilation matrix

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ABSTRACT We present an outline of how the ideas of self-similarity can be applied to wavelet theory, especially in connection to wavelets associated with a multiresolution analysis of $\mathbb{R}^n$ allowing arbitrary dilation matrices and no restrictions on the number of scaling functions.

2.1 Introduction

Wavelet bases have proved highly useful in many areas of mathematics, science, and engineering. One of the most successful approaches for the construction of such a basis begins with a special functional equation, the *refinement equation*. The solution to this refinement equation, called the scaling function, then determines a multiresolution analysis, which in turn determines the wavelet and the wavelet basis. In order to construct wavelet bases with prescribed properties, we must characterize those particular refinement equations which yield scaling functions that possess some specific desirable property. Much literature has been written on this topic for the classical one-dimensional, single-function, two-scale refinement equation, but when we move from the one-dimensional to the higher-dimensional setting or from the single wavelet to the multiwavelet setting it becomes increasingly difficult to find and apply such characterizations.

Our goal in this paper is to outline some recent developments in the construction of higher-dimensional wavelet bases that exploit the fact that the refinement equation is a statement that the scaling function satisfies a certain kind of self-similarity. In the classical one-dimensional case with dilation factor two, there are a variety of tools in addition to self-similarity which can be used to analyze the refinement equation. However, many of these tools become difficult or impossible to apply in the multidimensional setting with a general dilation matrix, whereas self-similarity becomes an even more natural and important tool in this setting. By viewing scaling functions as particular cases of "generalized self-similar functions," we showed in [5] that the tools of functional analysis can be applied to analyze refinement equations in the general higher-dimensional and multi-function setting. We derived conditions for the existence of continuous or L^p solutions to the refinement equation in this general setting, and showed how these conditions can be combined with the analysis of the accuracy of scaling functions from [4], [3] to construct new examples of nonseparable (non-tensor product) two-dimensional multiwavelets using a quincunx dilation matrix.

We will sketch some of the ideas and results from [5] in this paper, attempting to provide some insights into the techniques without dwelling on the mass of technical details that this generality necessitates. We emphasize that this work is intimately tied and connected to the vast literature on wavelets and refinement equations, and while we cannot trace those connections here, a full discussion with extensive references is presented in [5]. In particular, the important and fundamental contributions of Daubechies, Lagarias, Wang, Jia, Jiang, Shen, Plonka, Strela, and many others are discussed in [5].

2.2 Self-Similarity

The seed for this approach can be traced back to Bajraktarevic [1], who in 1957 studied solutions to equations of the form

$$\mathbf{u}(x) = \mathcal{O}(x,\, (\mathbf{u} \circ g_1)(x),\, \ldots,\, (\mathbf{u} \circ g_m)(x)) \tag{2.2.1}$$

where $g_i : X \to X$ and $\mathcal{O} : X \times E^m \to E$, and the solution $\mathbf{u} : X \to E$ lies in some function space $\boldsymbol{\mathcal{F}}$. Bajraktarevic proved that, under mild conditions on $\mathcal{O}$ and the g_i, there is a unique solution to (2.2.1). (See also [9].) A generalized version of this equation of the form

$$\mathbf{u}(x) = \mathcal{O}(x,\, \varphi_1(x, (\mathbf{u} \circ g_1)(x)),\, \ldots,\, \varphi_m(x, (\mathbf{u} \circ g_m)(x))), \tag{2.2.2}$$

where $\varphi_i : X \times E \to E$, was studied in [6]. We will state one uniqueness result below, and then in later sections demonstrate the fundamental connection between (2.2.2) and wavelets. If there exists a set B that is *self-similar* with respect to the functions g_i, i.e., if $B = \bigcup_{i=1}^m g_i^{-1}(B)$, then we refer to the solution $\mathbf{u}$ of (2.2.2) as a *generalized self-similar function*. This is because at a given point $x \in B$, the value of $\mathbf{u}(x)$ is obtained by combining the values of $\mathbf{u}(g_i(x))$ through the action of the operator $\mathcal{O}$, with each $g_i(x)$ lying in B.

In order to state the uniqueness result, we require the following notation. Let X be a closed subset of $\mathbb{R}^n$, and let $\|\cdot\|$ be any fixed norm on $\mathbb{C}^r$. Then we define $L^\infty(X, \mathbb{C}^r)$ to be the Banach space of all mappings $\mathbf{u} : X \to \mathbb{C}^r$ such that

$$\|\mathbf{u}\|_{L^\infty} = \sup_{x \in X} \|g(x)\| < \infty.$$

This definition is independent of the choice of norm on $\mathbb{C}^r$ in the sense that each choice of norm for $\mathbb{C}^r$ yields an equivalent norm for $L^\infty(X, \mathbb{C}^r)$. If E

is a nonempty closed subset of $\mathbb{C}^r$, then $L^\infty(X,E)$ will denote the closed subset of $L^\infty(X,\mathbb{C}^r)$ consisting of functions which take values in E. We say that a function $\mathbf{u}: X \to E$ is *stable* if $\mathbf{u}(B)$ is a bounded subset of E whenever B is a bounded subset of X.

The following result is a special case of more general results proved in [6]. In particular, we will consider here only uniform versions of this result; it is possible to formulate L^p and other versions as well.

Theorem 2.2.1. *Let X be a compact subset of $\mathbb{R}^n$, and let E be a closed subset of $\mathbb{C}^r$. Let $\|\cdot\|$ be any norm on $\mathbb{C}^r$. Let $m \geq 1$, and assume that functions w_i, φ_i, and $\mathcal{O}$ are chosen with the following properties.*

- *For each $i = 1, \dots, m$, let $w_i: X \to X$ be continuously differentiable, injective maps.*
- *Let $\varphi_i: X \times E \to E$ for $i = 1, \dots, m$ satisfy the Lipschitz-like condition*

$$\max_{1\leq i\leq m} \|\varphi_i(x,u) - \varphi_i(x,v)\| \leq C\,\|u-v\|. \tag{2.2.3}$$

- *Let $\mathcal{O}: X \times E^m \to E$ be non-expansive for each $x \in X$, i.e.,*

$$\|\mathcal{O}(x,u_1,\dots,u_m) - \mathcal{O}(x,v_1,\dots,v_m)\| \leq \max_{1\leq i\leq m} \|u_i - v_i\|. \tag{2.2.4}$$

Let t_0 be an arbitrary point in E. For $u \in L^\infty(X,E)$, define

$$T\mathbf{u}(x) = \mathcal{O}(x,\, \varphi_1(x,\mathbf{u}(w_1^{-1}(x))),\, \dots,\, \varphi_m(x,\mathbf{u}(w_m^{-1}(x)))),$$

where we interpret

$$\mathbf{u}(w_i^{-1}(x)) = t_0 \quad \text{if } x \notin w_i(X).$$

If $\mathcal{O}$ and the φ_i are stable, then T maps $L^\infty(X,E)$ into itself, and satisfies

$$\|T\mathbf{u} - T\mathbf{v}\|_{L^\infty} \leq C\,\|\mathbf{u}-\mathbf{v}\|_{L^\infty}.$$

In particular, if $C < 1$, then T is contractive, and there exists a unique function $\mathbf{v}^* \in L^\infty(X,E)$ such that $T\mathbf{v}^* = \mathbf{v}^*$, and, moreover, $\mathbf{v}^*$ is continuous. Further, if $C < 1$ and $\mathbf{v}^{(0)}$ is any function in $L^\infty(X,E)$, then the iteration $\mathbf{v}^{(i+1)} = T\mathbf{v}^{(i)}$ converges to $\mathbf{v}^*$ in $L^\infty(X,E)$.

2.3 Refinement Equations

The connection between Theorem 2.2.1 and wavelets is provided by the now-classical concept of multiresolution analysis (MRA). To construct a multiresolution analysis in $\mathbb{R}^n$, one begins with a *refinement equation* of the form

$$f(x) = \sum_{k\in\Lambda} c_k\, f(Ax-k), \qquad x \in \mathbb{R}^n, \tag{2.3.1}$$

where Λ is a subset of the lattice $\mathbb{Z}^n$ and A is a *dilation matrix*, i.e., $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$ and every eigenvalue λ of A satisfies $|\lambda| > 1$. We assume now that A, Λ, and c_k are fixed for the remainder of this paper.

A solution of the refinement equation is called a *scaling function* or a *refinable function*. If f is scalar-valued then the coefficients c_k are scalars, while if one allows vector-valued ($f : \mathbb{R}^n \to \mathbb{C}^r$) or matrix-valued ($f : \mathbb{R}^n \to \mathbb{C}^{r\times\ell}$) functions then the c_k are $r \times r$ matrices. We will consider the case $f : \mathbb{R}^n \to \mathbb{C}^r$ in this paper. We say that the number r is the *multiplicity* of the scaling function f.

The fact that A can be any dilation matrix (instead of just a "uniform" dilation such as $2I$) means that the geometry of $\mathbb{R}^n$ must be carefully considered with respect to the action of A. Note that since $A(\mathbb{Z}^n) \subset \mathbb{Z}^n$, the dilation matrix A necessarily has integer determinant. We define

$$m = |\det(A)|.$$

By [13], to each scaling function that generates a MRA there will be associated $(m-1)$ "mother wavelets," so it is desirable for some applications to consider "small" m.

The *refinement operator* associated with the refinement equation is the mapping S acting on vector functions $\mathbf{u} : \mathbb{R}^n \to \mathbb{C}^r$ defined by

$$S\mathbf{u}(x) = \sum_{k\in\Lambda} c_k\, \mathbf{u}(Ax-k), \qquad x \in \mathbb{R}^n. \tag{2.3.2}$$

A scaling function is thus a fixed point of S.

We will focus on compactly supported solutions of the refinement equation, and therefore will require that the subset Λ be finite. Let us consider the support of a solution to the refinement equation in this case. For each $k \in \mathbb{Z}^n$, let $w_k : \mathbb{R}^n \to \mathbb{R}^n$ denote the contractive map

$$w_k(x) = A^{-1}(x+k). \tag{2.3.3}$$

Now let $\mathcal{H}(\mathbb{R}^n)$ denote the set of all nonempty, compact subsets of $\mathbb{R}^n$ equipped with the Hausdorff metric. Then it can be shown that the mapping w on $\mathcal{H}(\mathbb{R}^n)$ defined by

$$w(B) = \bigcup_{k\in\Lambda} w_k(B) = A^{-1}(B+\Lambda)$$

is a contractive mapping of $\mathcal{H}(\mathbb{R}^n)$ into itself [11]. Hence there is a unique compact set K_Λ such that

$$K_\Lambda = w(K_\Lambda) = \bigcup_{k\in\Lambda} A^{-1}(K_\Lambda + k).$$

In the terminology of Iterated Function Systems, the set K_Λ is the attractor of the IFS generated by the collection $\{w_k\}_{k\in K}$. It can be shown that if f is a compactly supported solution of the refinement equation, then necessarily $\operatorname{supp}(f) \subset K_\Lambda$ [5].

Let

$$D = \{d_1, \ldots, d_m\}$$

be a *full set of digits* with respect to A and $\mathbb{Z}^n$, i.e., a complete set of representatives of the order-m group $\mathbb{Z}^n/A(\mathbb{Z}^n)$. Because D is a full set of digits, the lattice $\mathbb{Z}^n$ is partitioned into the m disjoint cosets

$$\Gamma_d = A(\mathbb{Z}^n) - d = \{Ak - d : k \in \mathbb{Z}^n\}, \qquad d \in D.$$

Let Q be the attractor of the of the IFS generated by $\{w_d\}_{d\in D}$, i.e., Q is the unique nonempty compact set satisfying

$$Q = K_D = \bigcup_{d\in D} A^{-1}(Q + d).$$

We will say that Q is a *tile* if its $\mathbb{Z}^n$ translates cover $\mathbb{R}^n$ with overlaps of measure 0. In that case, the Lebesgue measure of Q is 1 [2], and the characteristic function of Q generates a MRA in $\mathbb{R}^n$ [10]. This MRA is the n-dimensional analogue of the Haar MRA in $\mathbb{R}$, because if we consider dilation by 2 in $\mathbb{R}$ with digit set $D = \{0, 1\}$, and set

$$w_0(x) = \frac{1}{2}x \qquad \text{and} \qquad w_1(x) = \frac{1}{2}x + \frac{1}{2},$$

then the set $[0, 1]$ satisfies

$$[0, 1] = w_0([0, 1]) \bigcup w_1([0, 1])$$

and therefore is the attractor for the IFS $\{w_0, w_1\}$. Note that $[0, 1]$ is a tile, and that the Lebesgue measure of $[0, 1]$ is 1.

Example 2.3.1. Tiles may have a fractal boundaries. For example, if we consider the dilation matrix

$$A_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and digit set $D = \{(0, 0),\ (1, 0)\}$, then the tile Q is the celebrated "twin dragon" fractal shown on the left in Figure 2.1. On the other hand, if

$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and $D = \{(0, 0),\ (1, 0)\}$, then the tile Q is the parallelogram with vertices - $\{(0, 0),\ (1, 0),\ (2, 1),\ (1, 1)\}$ pictured on the right in Figure 2.1. For these two matrices A_1 and A_2, the sublattices $A_1(\mathbb{Z}^2)$ and $A_2(\mathbb{Z}^2)$ coincide. This sublattice is called the *quincunx sublattice* of $\mathbb{Z}^2$. As a consequence, these two matrices A_1, A_2 are often referred to as *quincunx dilation matrices.*

It is not always the case that, given an arbitrary dilation matrix A, there exists a set of digits such that the associated attractor of $\{w_d\}_{d\in D}$ is a tile [14], [12]. We will not address this question here, and will only consider dilation matrices for which a tile Q exists, and we assume that the digit set D has been chosen in such a way that Q is a tile. Without loss of generality, we can assume that $0 \in D$, and therefore the tile Q will contain the origin [5].

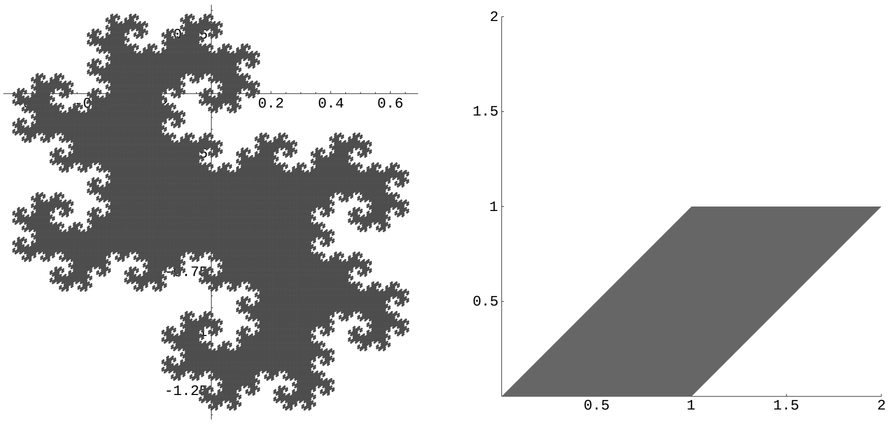

FIGURE 2.1. Twin Dragon and Parallelogram Attractors

Let us now return to setting up the notation required to connect the refinement equation (2.3.1) to Theorem 2.2.1.

Since $\operatorname{supp}(f) \subset K_\Lambda$, which is compact, and since Q is a tile and therefore covers $\mathbb{R}^n$ by translates, there exists a finite subset $\Omega \subset \mathbb{Z}^n$ such that

$$K_\Lambda \subset Q + \Omega = \bigcup_{\omega \in \Omega} (Q + \omega) = \{q + \omega : q \in Q,\, \omega \in \Omega\}.$$

Consider now any function $g \colon \mathbb{R}^n \to \mathbb{C}^r$ such that $\operatorname{supp}(g) \subset K_\Lambda$. Define the *folding* of g to be the function $\Phi g \colon Q \to (\mathbb{C}^r)^\Omega$ given by

$$\Phi g(x) = [g(x+k)]_{k \in \Omega}, \qquad x \in Q.$$

If for $k \in \Omega$ we write $(\Phi g)_k(x) = g(x+k)$ for the kth component of $\Phi g(x)$, then this folding has the property that $(\Phi g)_{k_1}(x_1) = (\Phi g)_{k_2}(x_2)$ whenever x_1, $x_2 \in Q$ and k_1, $k_2 \in \Omega$ are such that $x_1 + k_1 = x_2 + k_2$ (it can be shown that such points x_1, x_2 would necessarily have to lie on the boundary of Q [5]).

Here (and whenever we deal with vectors indexed by general sets) we consider that Ω has been ordered in some way; the choice of ordering is not important as long as the same ordering is used throughout. We use square brackets, e.g., $[u_k]_{k\in\Omega}$, to denote column vectors, and round brackets, e.g., $(u_k)_{k\in\Omega}$, to denote row vectors.

Since Q is the attractor of the IFS $\{w_d\}_{d\in D}$, it satisfies

$$Q = \bigcup_{d \in D} A^{-1}(Q + d).$$

Moreover, since Q is a tile, if $d_1 \neq d_2$ then $A^{-1}(Q+d_1) \cap A^{-1}(Q+d_2)$ has measure zero, and in fact it can be shown that these sets can intersect only along their boundaries. We will require subsets Q_d of $A^{-1}(Q+d)$ whose union is Q but which have *disjoint* intersections, i.e., such that

$$\bigcup_{d\in D} Q_d = Q \quad \text{and} \quad Q_{d_1} \cap Q_{d_2} = \emptyset \text{ if } d_1 \neq d_2.$$

A precise method for creating these sets Q_d is given in [5].

For each $d \in D$, define a matrix T_d by

$$T_d = [c_{Aj-k+d}]_{j,k\in\Omega}.$$

Note that T_d consists of an $\Omega \times \Omega$ collection of $r \times r$ blocks, i.e., $T_d \in (\mathbb{C}^{r\times r})^{\Omega\times\Omega}$. Assume that E is a subset (but not necessarily a subspace) of $(\mathbb{C}^r)^\Omega$ that is invariant under each matrix T_d (we will specify E precisely later). Then for each $d \in D$ we can define $\varphi_d : Q \times E \to E$ by

$$\varphi_d(x, e) = T_d e, \tag{2.3.4}$$

and define $\mathcal{O} : Q \times E^D \to E$ by

$$\mathcal{O}(x, \{e_d\}_{d\in D}) = \sum_{d\in D} \chi_{Q_d}(x) \cdot e_d. \tag{2.3.5}$$

That is, $\mathcal{O}(x, \{e_d\}_{d\in D}) = e_d$ if $x \in Q_d$. It is easy to see that this operator $\mathcal{O}$ is stable and satisfies the non-expansivity condition (2.2.4). Now define an operator T acting on vector functions $\mathbf{u} : Q \to E$ by

$$\begin{aligned} T\mathbf{u}(x) &= \mathcal{O}(x, \{\varphi_d(x, \mathbf{u}(w_d^{-1}(x)))\}_{d\in D}) \\ &= \sum_{d\in D} \chi_{Q_d}(x) \cdot T_d \mathbf{u}(Ax - d). \end{aligned} \tag{2.3.6}$$

Or, equivalently, T can be defined by

$$T\mathbf{u}(x) = T_d \mathbf{u}(Ax - d) \quad \text{if } x \in Q_d.$$

This operator T is connected to the refinement operator S defined by (2.3.2) as follows [5].

Proposition 2.3.1. *Let $\Omega \subset \mathbb{Z}^n$ be such that $K_\Lambda \subset Q + \Omega$. If $g : \mathbb{R}^n \to \mathbb{C}^r$ satisfies* $\mathrm{supp}(g) \subset K_\Lambda$*, then*

$$\Phi S g = T \Phi g \ \ a.e. \tag{2.3.7}$$

If the function g satisfies $\mathrm{supp}(g) \subset K_\Lambda$ and additionally vanishes on the boundary of K_Λ, then the equality in (2.3.7) holds everywhere and not merely almost everywhere. This is the case, for example, if g is continuous and supported in K_Λ.

In light of Proposition 2.3.1, in order to solve the refinement equation (2.3.1), we need to find a solution to the equation

$$\mathbf{u} = T\mathbf{u},$$

and this is precisely the type of generalized self-similarity that is defined in (2.2.2).

To do this, we apply Theorem 2.2.1. The operator $\mathcal{O}$ is non-expansive, the w_k are affine maps, and the functions φ_d are linear. Hence, if there exists a constant C with $0 < C < 1$ and a norm $\|\cdot\|$ on $(\mathbb{C}^r)^\Omega$ such that

$$\forall\, d \in D, \quad \forall\, x \in Q, \quad \forall\, e \in E, \quad \|\varphi_d(x,e)\| \le C\,\|e\|,$$

then T will have a unique fixed point. Considering the definition of φ_d, this means that there must exist a norm in $(\mathbb{C}^r)^\Omega$ such that

$$\forall\, d \in D, \quad \forall\, e \in E, \quad \|T_d e\| \le C\,\|e\|.$$

In other words, there must exist a norm under which all the matrices T_d are *simultaneously* contractive on some set. This leads naturally to the definition of the *joint spectral radius* of a set of matrices. Here we will focus only on the uniform joint spectral radius; it is possible to consider various generalizations as well. The uniform joint spectral radius was first introduced in [15] and was rediscovered and applied to refinement equations by Daubechies and Lagarias in [8].

If $\boldsymbol{\mathcal{M}} = \{M_1, \dots, M_m\}$ is a finite collection of $s \times s$ matrices, then the *uniform joint spectral radius* of $\boldsymbol{\mathcal{M}}$ is

$$\hat{\rho}(\boldsymbol{\mathcal{M}}) = \lim_{\ell\to\infty} \max_{\Pi \in \boldsymbol{\mathcal{P}}_\ell} \|\Pi\|^{1/\ell}, \tag{2.3.8}$$

where

$$\boldsymbol{\mathcal{P}}_0 = \{I\} \qquad \text{and} \qquad \boldsymbol{\mathcal{P}}_\ell = \{M_{j_1} \cdots M_{j_\ell} : 1 \le j_i \le m\}.$$

It is easy to see that the limit in (2.3.8) exists and is independent of the choice of norm $\|\cdot\|$ on $\mathbb{C}^{s\times s}$.

Note that if there is a norm such that $\max_j \|M_j\| \le \delta$, then $\hat{\rho}(\boldsymbol{\mathcal{M}}) \le \delta$. Rota and Strang [15] proved the following converse result.

Proposition 2.3.2. *Assume that $\boldsymbol{\mathcal{M}} = \{M_1, \dots, M_m\}$ is a finite collection of $s \times s$ matrices. If $\hat{\rho}(\boldsymbol{\mathcal{M}}) < \delta$, then there exists a vector norm $\|\cdot\|$ on $\mathbb{C}^s$ such that* $\max_j \|M_j\| \le \delta$.

Consequently, a given set of matrices is simultaneously contractive (i.e., there exists a norm such that $\max_j \|M_j\| < 1$) if and only if the uniform joint spectral radius of $\boldsymbol{\mathcal{M}}$ satisfies $\hat{\rho}(\boldsymbol{\mathcal{M}}) < 1$.

We can now state the main theorem relating generalized self-similarity to the existence of a continuous solution to the refinement equation.

Theorem 2.3.1. *Let $\Omega \subset \mathbb{Z}^n$ be a finite set such that $K_\Lambda \subset Q + \Omega$. Let E be a nonempty closed subset of $(\mathbb{C}^r)^\Omega$ such that $T_d(E) \subset E$ for each $d \in D$. Let V be a subspace of $(\mathbb{C}^r)^\Omega$ which contains $E - E$ and which is right-invariant under each T_d. Define*

$$\boldsymbol{\mathcal{F}} = \big\{ g \in L^\infty(\mathbb{R}^n, \mathbb{C}^r) \ : \ \operatorname{supp}(g) \subset K_\Lambda \ \ \text{and} \ \ \Phi g(Q) \subset E \big\}. \tag{2.3.9}$$

If $\boldsymbol{\mathcal{F}} \neq \emptyset$ and $\hat{\rho}(\{T_d|_V\}_{d\in D}) < 1$, then there exists a function $f \in \boldsymbol{\mathcal{F}}$ which is a solution to the refinement equation (2.3.1)*, and the cascade algorithm $f^{(i+1)} = Sf^{(i)}$ converges uniformly to f for each starting function $f^{(0)} \in \boldsymbol{\mathcal{F}}$. Furthermore, if there exists any continuous function $f^{(0)} \in \boldsymbol{\mathcal{F}}$, then f is continuous.*

Proof: We will apply Theorem 2.2.1 with $X = Q$ and with E the specified subset of $(\mathbb{C}^r)^\Omega$. We let w_d, φ_d, $\mathcal{O}$, and T be as defined above, specifically, by equations (2.3.3), (2.3.4), (2.3.5), and (2.3.6).

We will show that the hypotheses of Theorem 2.1 are satisfied. First, $w_d(x) = A^{-1}(x+d)$ is clearly injective and continuously differentiable.

Second, let δ be any number such that

$$\hat{\rho}(\{T_d|_V\}_{d\in D}) < \delta < 1.$$

Then by Proposition 2.3.2 applied to the matrices $T_d|_V$, there exists a vector norm $\|\cdot\|_V$ on V such that

$$\max_{d\in D} \|T_d w\|_V \le \delta \, \|w\|_V, \quad \text{all } w \in V.$$

Let $\|\cdot\|$ denote any extension of this norm to all of $(\mathbb{C}^r)^\Omega$. Recall that $\varphi_d(x,e) = T_d e$. Since $E - E \subset V$, we therefore have for each $x \in Q$ and u, $v \in E$ that

$$\max_{d\in D} \|\varphi_d(x,u) - \varphi_d(x,v)\| = \max_{d\in D} \|T_d(u-v)\| \le \delta \, \|u-v\|.$$

Therefore the functions φ_d satisfy the condition (2.2.3) with constant $C = \delta$. It is easy to check that each φ_d is stable.

Finally, $\mathcal{O}$ is non-expansive. Thus, the hypotheses of Theorem 2.2.1 are satisfied. Since $C < 1$, Theorem 2.2.1 implies that T maps $L^\infty(Q,E)$ into itself, and satisfies

$$\|T\mathbf{u} - T\mathbf{v}\| \le C \, \|\mathbf{u} - \mathbf{v}\|.$$

It follows that T is contractive on $L^\infty(Q,E)$ and there exists a unique function $\mathbf{v}^* \in L^\infty(Q,E)$ such that $T\mathbf{v}^* = \mathbf{v}^*$. Further, the iteration $\mathbf{v}^{(i+1)} = T\mathbf{v}^{(i)}$ converges in $L^\infty(Q,E)$ to $\mathbf{v}^*$ for each function $\mathbf{v}^{(0)} \in L^\infty(Q,E)$.

We want to relate now the fixed point $\mathbf{v}^*$ for T to a solution to the refinement equation. First, it can be shown that the space $\boldsymbol{\mathcal{F}}$ is invariant under the refinement operator S. Hence, by Proposition 2.3.1, the following diagram commutes, with T in particular being a contraction:

$$\begin{array}{ccc} \boldsymbol{\mathcal{F}} & \xrightarrow{\Phi} & L^\infty(Q,E) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle T} \\ \boldsymbol{\mathcal{F}} & \xrightarrow[\Phi]{} & L^\infty(Q,E). \end{array}$$

Now suppose that $f^{(0)}$ is any function in $\boldsymbol{\mathcal{F}}$, and define $f^{(i+1)} = Sf^{(i)}$. Then $f^{(i)} \in \boldsymbol{\mathcal{F}}$ for each i, and if we set $\mathbf{v}^{(i)} = \Phi f^{(i)}$, then

$$\mathbf{v}^{(i+1)} = \Phi f^{(i+1)} = \Phi S f^{(i)} = T\Phi f^{(i)} = T\mathbf{v}^{(i)},$$

so $\mathbf{v}^{(i)}$ must converge uniformly to $\mathbf{v}^*$. By choosing an appropriate choice of norm on $\boldsymbol{\mathcal{F}}$ (see [5]), it follows that $f^{(i)}$ converges uniformly to some function

$f \in L^\infty(\mathbb{R}^n, \mathbb{C}^r)$. We must have $f \in \boldsymbol{\mathcal{F}}$ since $\boldsymbol{\mathcal{F}}$ is a closed subset of $L^\infty(\mathbb{R}^n, \mathbb{C}^r)$. Further,

$$\Phi f = \mathbf{v}^* = T\mathbf{v}^* = T\Phi f = \Phi S f \quad \text{a.e.}$$

Therefore f satisfies the refinement equation (2.3.1) almost everywhere. Since $\mathbf{v}^*$ is unique, the cascade algorithm must converge to this particular f for any starting function $f^{(0)} \in \boldsymbol{\mathcal{F}}$. It only remains observe that if any $f^{(0)} \in \boldsymbol{\mathcal{F}}$ is continuous, then the iterates $f^{(i)}$ obtained from $f^{(0)}$ are continuous and converge uniformly to f, so f must itself be continuous. □

¿From the proof of the above theorem, it is clear that the rate of convergence of the cascade algorithm is geometric and can be specified explicitly if desired.

The preceding theorem immediately suggests two questions:

- Does there always exist a space E which is invariant for all T_d?
- Does $\boldsymbol{\mathcal{F}}$ always contain a continuous function?

The answer to both of these questions is yes, under some mild additional hypotheses.

To answer the question of the existence of the space E, let us recall the one-dimensional, single-function case. In this setting, if we impose the standard "minimal accuracy condition"

$$\sum_{k\in\mathbb{Z}} c_{2k} = \sum_{k\in\mathbb{Z}} c_{2k+1} = 1, \tag{2.3.10}$$

then E is the hyperplane through $(1, 0, \ldots, 0)$ that is orthogonal to the row vector $(1, 1, \ldots, 1)$. This vector is a common left eigenvector to all of the matrices T_d [8]. The minimal accuracy condition is so-called because it is directly related to the *accuracy* of the solution f. In n-dimensions with multiplicity r, i.e., with $f\colon \mathbb{R}^n \to \mathbb{C}^r$, the accuracy of f is defined to be the largest integer $\kappa > 0$ such that every polynomial $q(x) = q(x_1, \ldots, x_n)$ with $\deg(q) < \kappa$ can be written

$$q(x) = \sum_{k\in\mathbb{Z}^n} a_k f(x+k) = \sum_{k\in\mathbb{Z}^n} \sum_{i=1}^{r} a_{k,i} f_i(x+k) \text{ a.e.}, \quad x \in \mathbb{R}^n,$$

for some row vectors $a_k = (a_{k,1}, \ldots, a_{k,r}) \in \mathbb{C}^{1\times r}$. If no polynomials are reproducible from translates of f then we set $\kappa = 0$. We say that f has at least *minimal accuracy* if the constant polynomial is reproducible from translates of f, i.e., if $\kappa \geq 1$. We say that translates of f along $\mathbb{Z}^n$ are *linearly independent* if $\sum_{k\in\mathbb{Z}^n} a_k f(x+k) = 0$ implies $a_k = 0$ for each k. In one dimension, under the hypotheses of linear independence of translates, the minimal accuracy condition (2.3.10) implies that f has at least minimal accuracy. In the general setting of n dimensions and multiplicity r, the minimal accuracy condition is more complicated to formulate than (2.3.10). However, this condition is still the appropriate tool to construct an appropriate set E. We present here a weak form of the minimal accuracy condition, and refer to [4] for a general result.

Theorem 2.3.2. *Let $f:\mathbb{R}^n \to \mathbb{C}^r$ be an integrable, compactly supported solution of the refinement equation* (2.3.1)*, such that translates of f along $\mathbb{Z}^n$ are linearly independent. Then the following statements are equivalent.*

a) f has accuracy $\kappa \geq 1$.

b) There exists a row vector $u_0 \in \mathbb{C}^{1\times r}$ such that $u_0\hat{f}(0) \neq 0$ and

$$u_0 = \sum_{k\in\Gamma_d} u_0\, c_k \quad \textit{for each } d \in D.$$

In the case that either statement holds, we have

$$\sum_{k\in\Gamma} u_0 f(x+k) = 1 \ \textit{a.e.}$$

Assume now that the minimal accuracy condition given in Theorem 2.3.2 is satisfied, and let u_0 be the row vector such that $\sum_{k\in\mathbb{Z}^n} u_0 f(x+k) = 1$ a.e. It can be shown that the inclusions $\mathrm{supp}(f) \subset K_\Lambda \subset Q + \Omega$ imply that if $x \in Q$, then the only nonzero terms in the series $\sum_{k\in\mathbb{Z}^n} u_0 f(x+k) = 1$ occur when $k \in \Omega$. Hence, if we set $e_0 = (u_0)_{k\in\Omega}$, i.e., e_0 is the row vector obtained by repeating the block u_0 once for each $k \in \Omega$, then

$$e_0 \Phi f(x) = \sum_{k\in\Omega} u_0 f(x+k) = \sum_{k\in\mathbb{Z}^n} u_0 f(x+k) = 1 \text{ a.e.}, \quad \text{for } x \in Q.$$

Thus the values of $\Phi f(x)$ are constrained to lie in a particular hyperplane E_0 in $(\mathbb{C}^r)^\Omega$, namely, the collection of column vectors $v = [v_k]_{k\in\Omega}$ such that $e_0 v = \sum_{k\in\Omega} u_0 v_k = 1$. This hyperplane E_0 is a canonical choice for the set E appearing in the hypotheses of Theorem 2.3.1. In order to invoke Theorem 2.3.1, the starting functions $f^{(0)}$ for the cascade algorithm should therefore also have the property that $\Phi f^{(0)}(x)$ always lies in this hyperplane E_0. Note that with this definition of E_0, the set of differences $V_0 = E_0 - E_0$ is the subspace consisting of vectors $v = [v_k]_{k\in\Omega}$ such that $e_0 v = \sum_{k\in\Omega} u_0 v_k = 0$. Hence the minimal accuracy condition immediately provides an appropriate choice for the space E, namely, we take $E = E_0$.

Now, having defined $E = E_0$, we are ready to address the second question, whether the set $\boldsymbol{\mathcal{F}}$ defined by (2.3.9) always contains a continuous function. First we rewrite $\boldsymbol{\mathcal{F}}$ as

$$\boldsymbol{\mathcal{F}} = \Big\{ g \in L^\infty(\mathbb{R}^n, \mathbb{C}^r) \ : \ \mathrm{supp}(g) \subset K_\Lambda \ \text{ and } \sum_{k\in\mathbb{Z}^n} u_0 g(x+k) = 1 \Big\},$$

and note that this set is determined by two quantities: the set Λ and the row vector u_0. The set Λ is the support of the set of coefficients c_k in the refinement equation and is determined only by the location of the c_k and not their values. The vector u_0, on the other hand, is determined by the values of the c_k as well as their locations. However, it can be shown that, in fact, the question of whether $\boldsymbol{\mathcal{F}}$ contains a continuous function is determined solely by Λ and not by u_0. Thus only the location of the coefficients c_k is important for this question, and not their actual values. This is made precise in the following result [5].

Lemma 2.3.1. *Let $\Lambda \subset \mathbb{Z}^n$ be finite, and let u_0 be a nonzero row vector in $\mathbb{C}^{1\times r}$. Then the following statements are equivalent.*

a) $\mathcal{F} \neq \emptyset$.

b) $\mathcal{F}$ contains a continuous function.

c) $K_\Lambda^\circ + \mathbb{Z}^n = \mathbb{R}^n$, i.e., lattice translates of the interior K_Λ° of K_Λ cover $\mathbb{R}^n$.

Thus, in designing a multiwavelet system, after choosing the dilation matrix A and digit set D, the next step is to choose a set Λ which fulfills the requirements of Lemma 2.3.1. Small Λ are preferable, since the larger Λ is, the larger the matrices T_d will be, and the more computationally difficult the computation of the joint spectral radius becomes. While we expect that some "small" Λ may fail the requirement $K_\Lambda^\circ + \mathbb{Z}^n = \mathbb{R}^n$, it is not true that all "large" Λ will necessarily satisfy this requirement (see [5] for an example).

In summary, once we impose the minimal accuracy condition and choose an appropriate set Λ, in order to check for the existence of a continuous scaling function we must evaluate the uniform joint spectral radius $\hat{\rho}(\{T_d|_{V_0}\}_{d\in D})$. Unfortunately, this might involve the computation of products of large matrices. It can be shown that if the coefficients c_k satisfy the conditions for higher-order accuracy, then V_0 is only the largest of a decreasing chain of common invariant subspaces

$$V_0 \supset V_1 \supset \cdots \supset V_{\kappa-1}$$

of the matrices T_d, and that, as a consequence, the value of $\hat{\rho}(\{T_d|_{V_0}\}_{d\in D})$ is determined by the value of $\hat{\rho}(\{T_d|_{V_{\kappa-1}}\}_{d\in D})$ [5]. This reduction in dimension can ease the computational burden of approximating the joint spectral radius. Moreover, these invariant spaces V_s are directly determined from the coefficients c_k via the accuracy conditions, which are a system of linear equations. Hence it is a simple matter to compute the matrices $T_d|_{V_{\kappa-1}}$. Additionally, the fact that accuracy implies such specific structure in the matrices T_d suggests that this structure could potentially be used to develop theoretical design criteria for multiwavelet systems.

A final question concerns the converse of Theorem 2.3.1, namely, what can we say if after choosing coefficients c_k that satisfy the minimal accuracy condition, the joint spectral radius of $\hat{\rho}(\{T_d|_{V_0}\}_{d\in D})$ exceeds 1? The following theorem answers this question, and is somewhat surprising because it essentially says that if a given operator has a fixed point, then that operator must *necessarily* be contractive. This theorem is proved in this generality in [5], but is inspired by a one-dimensional theorem of Wang [17].

Theorem 2.3.3. *Let f be a continuous, compactly supported solution to the refinement equation* (2.3.1) *such that f has L^∞-stable translates* (*defined below*). *Assume that there exists a row vector $u_0 \in \mathbb{C}^{1\times r}$ such that*

$$u_0 \hat{f}(0) \neq 0 \qquad \text{and} \qquad u_0 = \sum_{k\in\Gamma_d} u_0\, c_k \text{ for } d \in D.$$

If $\Omega \subset \mathbb{Z}^n$ is any set such that

$$K_\Lambda \subset Q + \Omega \qquad \textit{and} \qquad A^{-1}(\Omega + \Lambda - D) \cap \mathbb{Z}^n \subset \Omega,$$

then

$$\hat{\rho}(\{T_d|_{V_0}\}_{d\in D}) < 1.$$

Here, we say that a vector function $g \in L^\infty(\mathbb{R}^n, \mathbb{C}^r)$ has *L^∞-stable translates* if there exist constants C_1, $C_2 > 0$ such that

$$C_1 \sup_{k\in\Gamma} \max_i |a_{k,i}| \le \left\| \sum_{k\in\Gamma} a_k\, g(x+k) \right\|_{L^\infty} \le C_2 \sup_{k\in\Gamma} \max_i |a_{k,i}|$$

for all sequences of row vectors $a_k = (a_{k,1}, \ldots, a_{k,r})$ with only finitely many a_k nonzero.

2.4 Existence of MRAs

In this section we turn to the problem of using the existence of a solution to the refinement equation to construct orthonormal multiwavelet bases for $L^2(\mathbb{R}^n)$. As in the classical one-dimensional, single-function theory, the key point is that a vector scaling function which has orthonormal lattice translates determines a multiresolution analysis for $\mathbb{R}^n$. The multiresolution analysis then, in turn, determines a wavelet basis for $L^2(\mathbb{R}^n)$.

The main novelty here, more than allowing more than one scaling function or working in arbitrary dimensions, is the result of having an arbitrary dilation matrix. The viewpoint of self-similarity and iterated function systems still leads naturally to the correct decompositions [5].

Definition 2.4.1. *A* multiresolution analysis (MRA) of multiplicity r *associated with a dilation matrix A is a sequence of closed subspaces $\{\boldsymbol{\mathcal{V}}_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^n)$ which satisfy:*

P1. $\boldsymbol{\mathcal{V}}_j \subset \boldsymbol{\mathcal{V}}_{j+1}$ *for each* $j \in \mathbb{Z}$,

P2. $g(x) \in \boldsymbol{\mathcal{V}}_j \iff g(Ax) \in \boldsymbol{\mathcal{V}}_{j+1}$ *for each* $j \in \mathbb{Z}$,

P3. $\bigcap_{j\in\mathbb{Z}} \boldsymbol{\mathcal{V}}_j = \{0\}$,

P4. $\bigcup_{j\in\mathbb{Z}} \boldsymbol{\mathcal{V}}_j$ *is dense in* $L^2(\mathbb{R}^n)$, *and*

P5. *there exist functions $\varphi_1, \ldots, \varphi_r \in L^2(\mathbb{R}^n)$ such that the collection of lattice translates*

$$\{\varphi_i(x-k)\}_{k\in\mathbb{Z}^n,\, i=1,\ldots,r}$$

forms an orthonormal basis for $\boldsymbol{\mathcal{V}}_0$.

If these conditions are satisfied, then the vector function $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ is referred to as a vector scaling function *for the MRA.*

The usual technique for constructing a multiresolution analysis is to start from a vector function $\varphi = (\varphi_1, \ldots, \varphi_r)^{\mathrm{T}}$ such that $\{\varphi_i(x-k)\}_{k\in\mathbb{Z}^n,\, i=1,\ldots,r}$

is an orthonormal system in $L^2(\mathbb{R}^n)$, and then to construct the subspaces $\mathcal{V}_j \subset L^2(\mathbb{R}^n)$ as follows. First, let $\mathcal{V}_0$ be the closed linear span of the translates of the component functions φ_i, i.e.,

$$\mathcal{V}_0 = \overline{\mathrm{span}}\{\varphi_i(x-k)\}_{k\in\mathbb{Z}^n,\, i=1,\dots,r}. \tag{2.4.1}$$

Then, for each $j \in \mathbb{Z}$, define $\mathcal{V}_j$ to be the set of all dilations of functions in $\mathcal{V}_0$ by A^j, i.e.,

$$\mathcal{V}_j = \{g(A^j x) : g \in \mathcal{V}_0\}. \tag{2.4.2}$$

If $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ defined in this way forms a multiresolution analysis for $L^2(\mathbb{R}^n)$ then we say that it is the *MRA generated by* φ.

Example 2.4.1. In one dimension, the box function $\varphi = \chi_{[0,1)}$ generates a multiresolution analysis for $L^2(\mathbb{R})$. This MRA is usually referred to as the *Haar multiresolution analysis*, because the wavelet basis it determines is the classical Haar system $\{2^{n/2}\psi(2^n x - k)\}_{n,k\in\mathbb{Z}}$, where $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$.

Gröchenig and Madych [10] proved that there is a Haar-like multiresolution analysis associated to each choice of dilation matrix A and digit set D for which the attractor $Q = K_D$ is a tile. In particular, they proved that if Q is a tile then the scalar-valued function χ_Q generates a multiresolution analysis of $L^2(\mathbb{R}^n)$ of multiplicity 1. By extension of the one-dimensional terminology, this MRA is called the *Haar MRA associated with A and D*. Note that the fact that $\{\chi_Q(x-k)\}_{k\in\Gamma}$ forms an orthonormal basis for $\mathcal{V}_0$ is a restatement of the assumption that the lattice translates of the tile Q have overlaps of measure zero. Further, χ_Q is refinable because Q is self-similar and because the lattice translates of Q have overlaps of measure zero.

We will characterize those φ which generate multiresolution analyses in the following theorem. To motivate this result, note that property P2 is achieved trivially when $\mathcal{V}_j$ is defined by (2.4.2). Moreover, property P5 is simply a statement that lattice translates of φ are orthonormal. It can be seen [5] that the fact that φ has orthonormal lattice translates implies that property P3 is also automatically satisfied. Thus, the main problem in determining whether φ generates a multiresolution analysis is the question of when properties P1 and P4 are satisfied. One necessary requirement for P1 is clear. If φ does generate a multiresolution analysis, then P1 implies that $\varphi_i \in \mathcal{V}_0 \subset \mathcal{V}_1$ for $i = 1, \dots, r$. Since P2 and P5 together imply that $\{m^{1/2}\,\varphi_j(Ax-k)\}_{k\in\mathbb{Z}^n,\, j=1,\dots,r}$ forms an orthonormal basis for $\mathcal{V}_1$, each function φ_i must therefore equal some (possibly infinite) linear combination of the functions $\varphi_j(Ax-k)$. Consequently, the vector function φ must satisfy a refinement equation of the form

$$\varphi(x) = \sum_{k\in\mathbb{Z}^n} c_k\,\varphi(Ax-k) \tag{2.4.3}$$

for some choice of $r \times r$ matrices c_k. Since we only consider the case where the functions φ_i have compact support and since φ has orthonormal lattice translates, this implies that only finitely many of the matrices c_k in (2.4.3) can be nonzero. Hence, in this case the refinement equation in (2.4.3) has the same form as the refinement equation (2.3.1).

Theorem 2.4.1. *Assume that* $\varphi = (\varphi_1, \dots, \varphi_r)^T \in L^2(\mathbb{R}^n, \mathbb{C}^r)$ *is compactly supported and has orthonormal lattice translates, i.e.,*

$$\langle \varphi_i(x-k), \varphi_j(x-\ell) \rangle = \int \varphi_i(x-k)\, \overline{\varphi_j(x-\ell)}\, dx = \delta_{i,j}\, \delta_{k,\ell}.$$

Let $\boldsymbol{\mathcal{V}}_j \subset L^2(\mathbb{R}^n)$ *for* $j \in \mathbb{Z}$ *be defined by* (2.4.1) *and* (2.4.2). *Then the following statements hold.*

a) *Properties P2, P3, and P5 are satisfied.*

b) *Property P1 is satisfied if and only if* φ *satisfies a refinement equation of the form*

$$\varphi(x) = \sum_{k \in \Lambda} c_k\, \varphi(Ax - k) \tag{2.4.4}$$

for some $r \times r$ *matrices* c_k *and some finite set* $\Lambda \subset \mathbb{Z}^n$.

c) *If*

$$\sum_{i=1}^{r} |\hat{\varphi}_i(0)|^2 = \sum_{i=1}^{r} \left| \int \varphi_i(x)\, dx \right|^2 = |Q| = 1, \tag{2.4.5}$$

then Property P4 is satisfied. If φ *is refinable, i.e., if* (2.4.4) *holds, then Property P4 is satisfied if and only if* (2.4.5) *holds.*

Note that the assumption that φ_i is square-integrable and compactly supported implies that $\varphi_i \in L^1(\mathbb{R}^n)$, so $\hat{\varphi}_i(0) = \int \varphi_i(x)\, dx$ is well-defined.

Theorem 2.4.1 generalizes a result of Cohen [7], which applied specifically to the case of multiplicity 1 and dilation $A = 2I$. Cohen's estimates used a decomposition of $\mathbb{R}^n$ into dyadic cubes, making essential use of the fact that the uniform dilation $A = 2I$ maps dyadic cubes into dyadic cubes. However, this need not be true for an arbitrary dilation matrix A, so this particular decomposition is no longer feasible. Instead, the proof in [5] uses a decomposition based on the tile Q and the associated Haar multiresolution analysis discussed in Example 2.4.1. One of the key observations lies in counting the number of lattice translates of Q which lie in the interior of a dilated tile $A^j Q$, $j \geq 1$. The fact that Q is self-similar combined with the fact that translates of Q tile $\mathbb{R}^n$ with overlaps with measure zero implies that $A^j Q$ is a union of exactly m^j translates of Q, with each such translate lying entirely inside $A^j Q$ (but not necessarily in the *interior* of $A^j Q$). It can be shown that the ratio of the number of those translates $Q + k$ that intersect the boundary of $A^j Q$ to the total number lying inside $A^j Q$ converges to zero.

We conclude by showing in Figure 2.2 a pair of wavelets associated to a MRA obtained by numerically solving the "accuracy 2" conditions given in [4] to obtain the coefficients c_k for a scaling vector $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^2$ with orthonormal lattice translates that is refinable with respect to a quincunx dilation matrix (these numerical estimates were obtained by A. Ruedin, see [16] for related results). Using the results outlined in this paper, one can prove that these coefficients yield a continuous scaling vector which generates a MRA whose "mother wavelets" are those pictured in Figure 2.2.

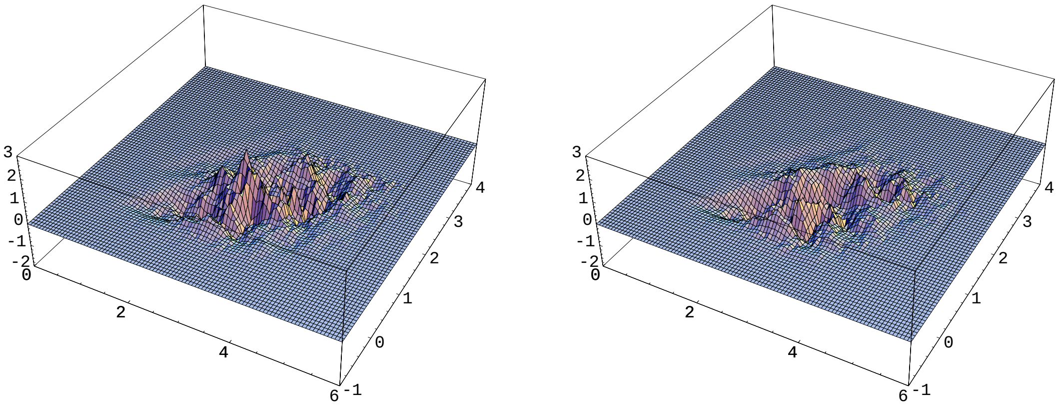

FIGURE 2.2. Wavelets ψ_1, ψ_2

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