CONVERGENCE OF FRAME SERIES

CHRISTOPHER HEIL AND PU-TING YU

Abstract. If $\{x_n\}_{n\in\mathbb{N}}$ is a frame for a Hilbert space H, then there exists a canonical dual frame $\{\widetilde{x}_n\}_{n\in\mathbb{N}}$ such that for every $x \in H$ we have $x = \sum \langle x, \widetilde{x}_n\rangle\, x_n$, with unconditional convergence of this series. However, if the frame is not a Riesz basis, then there exist alternative duals $\{y_n\}_{n\in\mathbb{N}}$ and sythensis pseudo-duals $\{z_n\}_{n\in\mathbb{N}}$ such that $x = \sum \langle x, y_n\rangle\, x_n$ and $x = \sum \langle x, x_n\rangle\, z_n$ for every x. We characterize the frames for which the *frame series* $x = \sum \langle x, y_n\rangle\, x_n$ converges unconditionally for every x for every alternative dual, and similarly for sythensis pseudo-duals. In particular, we prove that if $\{x_n\}_{n\in\mathbb{N}}$ does not contain infinitely many zeros then the frame series converge unconditionally for every alternative dual (or sythensis pseudo-dual) if and only if $\{x_n\}_{n\in\mathbb{N}}$ is a near-Riesz basis. We also prove that all alternative duals and sythensis pseudo-duals have the same excess as their associated frame.

1. Introduction

Frames were first introduced by Duffin and Schaeffer in [7] in their study of non-harmonic Fourier series, and interest resurged with the paper [6] by Daubechies et al. which applied frames to wavelet and Gabor systems. We refer to [5, 9] for relatively recent textbook recountings of the mathematics of frames.

We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a separable Hilbert space H is a frame if there exist positive constants $A \le B$, called *frame bounds*, such that

$$A\,\|x\|^2 \;\le\; \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \;\le\; B\,\|x\|^2, \qquad \text{for all } x \in H.$$

A frame possesses basis-like properties, as there exist sequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ such that each x in H can be represented as

$$x \;=\; \sum_{n=1}^{\infty} \langle x, y_n\rangle\, x_n \tag{1.1}$$

and

$$x \;=\; \sum_{n=1}^{\infty} \langle x, x_n\rangle\, z_n. \tag{1.2}$$

Such a sequence $\{y_n\}_{n\in\mathbb{N}}$ (respectively $\{z_n\}_{n\in\mathbb{N}}$) is said to be an *alternative dual* (respectively *synthesis pseudo-dual*) of $\{x_n\}_{n\in\mathbb{N}}$. If in addition $\{y_n\}_{n\in\mathbb{N}}$ (respectively $\{z_n\}_{n\in\mathbb{N}}$) is a frame then it is called an *alternative dual frame* (respectively *synthesis pseudo-dual frame*).

In general, an alternative dual or synthesis pseudo-dual of a frame need not be unique. In some sense, frames exchange uniqueness of frame expansions for flexibility in the choice of

Key words and phrases. Frames, Unconditional convergence, Alternative duals, Near-Riesz bases.

coefficients. The *canonical dual frame*, which we will define below, is one alternative dual frame and it is also a synthesis pseudo-dual frame. If $\{y_n\}_{n\in\mathbb{N}}$ is the canonical dual frame then the series in Eq. (1.1) converges unconditionally for all $x \in H$, and similarly for Eq. (1.2) if $\{z_n\}_{n\in\mathbb{N}}$ is the canonical dual frame.

The study of the convergence of frame series has a long history. In [8], Heil gave sufficient and necessary conditions for a series $\sum c_n x_n$ to converge unconditionally, where $\{x_n\}_{n\in\mathbb{N}}$ is a frame that is norm-bounded below. In [10], Holub established relations between types of frames that he termed *Besselian frames*, *unconditional frames*, and *near-Riesz bases*. Convergence of Weyl–Heisenberg frame series was also studied in [15], and unconditional constants were discussed in [3]. Stoeva and Balazs [11, 12, 13] studied the convergence of more general types of series related to frames, called frame multipliers.

We study the unconditional convergence of the series $\sum \langle x, y_n\rangle\, x_n$ and $\sum \langle x, x_n\rangle\, z_n$ where $\{y_n\}_{n\in\mathbb{N}}$ is an alternative dual and $\{z_n\}_{n\in\mathbb{N}}$ is a synthesis pseudo-dual of $\{x_n\}_{n\in\mathbb{N}}$. A natural question is whether this series must converge unconditionally for every alternative dual and every x, and similarly for synthesis pseudo-duals. Barring the case where $\{x_n\}_{n\in\mathbb{N}}$ contains infinitely many zeros, we prove that this holds true if and only if $\{x_n\}_{n\in\mathbb{N}}$ is a near-Riesz basis. We also study the excess of alternative duals and synthesis pseudo-duals, showing that the excess of these duals must be the same as the excess of their associated frames.

Notation, terminology, and preliminary results will be presented in Sect. 2. In Sect. 3, we address the convergence of $\sum \langle x, y_n\rangle\, x_n$ and $\sum \langle x, x_n\rangle\, z_n$, obtaining several equivalent characterizations of near-Riesz bases. Finally, Sect. 4 studies the excess of alternative duals, sythensis pseudo-duals, and their associated frames.

2. Preliminaries

Throughout this paper, H denotes an infinite-dimensional separable Hilbert space with inner product $\langle \cdot, \cdot\rangle$ and corresponding norm $\|\cdot\|$, and ℓ^2 is the space of square-summable sequences indexed by the natural numbers $\mathbb{N}$.

We list some basic results required for this paper, and refer to [5] and [9] for more details and proofs.

If $\{x_n\}_{n\in\mathbb{N}}$ is a frame for H, then its *frame operator* $Sx = \sum \langle x, x_n\rangle\, x_n$ is a bounded linear invertible map of H onto itself. The *canonical dual frame* is $\{\widetilde{x}_n\}_{n\in\mathbb{N}}$, where $\widetilde{x}_n = S^{-1}x_n$. We have

$$x = \sum_{n=1}^{\infty} \langle x, \widetilde{x}_n\rangle\, x_n = \sum_{n=1}^{\infty} \langle x, x_n\rangle\, \widetilde{x}_n, \qquad \text{for all } x \in H.$$

In particular, $\{\widetilde{x}_n\}_{n\in\mathbb{N}}$ is an alternative dual frame and also a synthesis pseudo-dual frame.

We say that $\{x_n\}_{n\in\mathbb{N}}$ is a *Bessel sequence* if it satisfies at least the upper inequality in the definition of a frame. That is, there exists some $B > 0$, called an *upper frame bound*, such that $\sum |\langle x, x_n\rangle|^2 \le B\,\|x\|^2$ for every x. In this case, the series $\sum c_n x_n$ converges unconditionally whenever $(c_n) \in \ell^2$.

We recall the definition of a Riesz basis and define unconditional frames and near-Riesz basis, which were first introduced in [10]. A variety of equivalent characterizations of Riesz bases can be found in [9, Thm. 8.32].

Definition 2.1. Let $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ be a frame for H.

(a) $\mathcal{F}$ is a *Riesz basis* for H if it is the image of an orthonormal basis for H under a continuous linear bijection of H onto itself.

(b) $\mathcal{F}$ is a *near-Riesz basis* if there exists a finite set $F \subseteq \mathbb{N}$ such that $\{x_n\}_{n \notin F}$ is a Riesz basis.

(c) $\mathcal{F}$ is an *unconditional frame* if $\sum c_n x_n$ converges unconditionally whenever it converges. $\diamondsuit$

The notion of excess was introduced in [1] to address questions regarding overcompleteness of frames. Letting $|\mathcal{G}|$ denote the cardinality of $\mathcal{G}$ if $\mathcal{G}$ is finite and $|\mathcal{G}| = \infty$ otherwise, the excess of a sequence $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ in H is

$$e(\mathcal{F}) \;=\; \sup\bigl\{|\mathcal{G}| : \mathcal{G} \subseteq \mathcal{F} \text{ and } \overline{\operatorname{span}}(\mathcal{F}\backslash\mathcal{G}) = \overline{\operatorname{span}}(\mathcal{F})\bigr\}.$$

A frame has finite excess if and only if it is a near-Riesz basis. Holub [10] established that unconditional frames and near-Riesz bases are equivalent when the frame is norm-bounded below. Casazza and Christensen [4] generalized this to any frame that does not contain infinitely many zeros. We summarize those results as follows.

Theorem 2.2. If $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ is a sequence in H, then the following two statements are equivalent.

(a) $\mathcal{F}$ is a near-Riesz basis.

(b) $\mathcal{F}$ is a frame that has finite excess.

Moreover, if $\mathcal{F}$ does not contain infinitely many zeros then statements (a) and (b) are also equivalent to the following statement.

(c) $\mathcal{F}$ is an unconditional frame. $\diamondsuit$

The next result is Lemma 6.3.2 in [5] (also see Theorem 8.18 in [9]).

Lemma 2.3. Let $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ be a frame for H.

(a) If an alternative dual $\{y_n\}_{n\in\mathbb{N}}$ is a Bessel sequence, then $\{y_n\}_{n\in\mathbb{N}}$ is a frame for H. In particular, an alternative dual frame is also a synthesis pseudo-dual frame.

(b) If a synthesis pseudo-dual $\{z_n\}_{n\in\mathbb{N}}$ is a Bessel sequence, then $\{z_n\}_{n\in\mathbb{N}}$ is a frame for H. In particular, a synthesis pseudo-dual frame is also an alternative dual frame. $\diamondsuit$

The following is Theorem 2.2 in [2].

Theorem 2.4. If $= \{x_n\}_{n\in\mathbb{N}}$ is a frame for H and $\{y_n\}_{n\in\mathbb{N}}$ is an alternative dual frame, then $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ have the same excess. $\diamondsuit$

It is known that, in at least some cases, convergence of the series $\sum \langle x, y_n\rangle\, x_n$ implies the convergence of $\sum \langle x, x_n\rangle\, y_n$. For example, the following is a special case of [12, Lem. 3.1].

Lemma 2.5. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences in H. Then $\sum \langle x, y_n\rangle\, x_n$ converges unconditionally for every x if and only if $\sum \langle x, x_n\rangle\, y_n$ converges unconditionally for every x. In particular, if $\sum \langle x, x_n\rangle\, y_n$ and $\sum \langle x, y_n\rangle\, x_n$ converge for every x, then $\sum \langle x_0, y_n\rangle\, x_n$ converges conditionally for some $x_0 \in H$ if and only if $\sum \langle y_0, x_n\rangle\, y_n$ converges conditionally for some $y_0 \in H$. $\diamondsuit$

Finally, a sequence $\{f_n\}_{n\in\mathbb{N}}$ in H is *minimal* if $f_m \notin \overline{\text{span}}\{f_n\}_{n\neq m}$ for every $m \in \mathbb{N}$. This occurs if and only if there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ that is *biorthogonal* to $\{f_n\}_{n\in\mathbb{N}}$; that is, $\langle f_m, g_n\rangle = \delta_{mn}$ for m, $n \in \mathbb{N}$ (for one proof, see [9, Lem. 5.4]).

3. Convergence of Frame Series

If $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ is a frame then $\sum c_n x_n$ converges unconditionally whenever $(c_n)_{n\in\mathbb{N}}$ belongs to ℓ^2. Therefore $\sum \langle x, y_n\rangle\, x_n$ (respectively $\sum \langle x, x_n\rangle\, y_n$) converges unconditionally whenever $\{y_n\}_{n\in\mathbb{N}}$ is an alternative dual frame (respectively sythensis pseudo-dual frame). Therefore, if every alternative dual (respectively sythensis pseudo-dual) of $\mathcal{F}$ is a frame, then unconditional convergence of $\sum \langle x, y_n\rangle\, x_n$ (respectively $\sum \langle x, x_n\rangle\, z_n$) is ensured. However, we will construct a frame that possesses non-frame alternative duals and non-frame sythensis pseudo-duals for which $\sum \langle u, y_n\rangle\, x_n$ and $\sum \langle v, x_n\rangle\, z_n$ converge conditionally for some u and v. The frame $\mathcal{F}$ in this construction is norm-bounded below.

Example 3.1. Let $\mathcal{E} = \{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for H, and consider the frame

$$\mathcal{F} = \{x_n\}_{n\in\mathbb{N}} = \{e_1, e_1, e_1, e_2, e_2, e_3, e_3, e_4, e_4, \dots\}.$$

Define $y_1 = e_1$, $y_2 = e_1$, and $y_3 = -e_1$, and for $n \geq 2$ set

$$y_{2n} = \frac{1}{\sqrt{n}}\, e_1 + \frac{1}{2}\, e_n \qquad \text{and} \qquad y_{2n+1} = -\frac{1}{\sqrt{n}}\, e_1 + \frac{1}{2}\, e_n.$$

If $x \in H$, then for each integer $K \geq 4$ we have that

$$\sum_{n=1}^{K} \langle x, y_n\rangle\, x_n = \begin{cases} \displaystyle\sum_{n=1}^{(K-1)/2} \langle x, e_n\rangle\, e_n, & \text{if } K \text{ is odd,} \\ \displaystyle\sum_{n=1}^{(K/2)-1} \langle x, e_n\rangle\, e_n + \sqrt{\frac{2}{K}}\, \langle x, e_1\rangle\, e_{K/2} + \frac{1}{2}\, \langle x, e_{K/2}\rangle\, e_{K/2}, & \text{if } K \text{ is even,} \end{cases}$$

and

$$\sum_{n=1}^{K} \langle x, x_n\rangle\, y_n = \begin{cases} \displaystyle\sum_{n=1}^{(K-1)/2} \langle x, e_n\rangle\, e_n, & \text{if } K \text{ is odd,} \\ \displaystyle\sum_{n=1}^{(K/2)-1} \langle x, e_n\rangle\, e_n + \sqrt{\frac{2}{K}}\, \langle x, e_{K/2}\rangle\, e_1 + \frac{1}{2}\, \langle x, e_{K/2}\rangle\, e_{K/2}, & \text{if } K \text{ is even.} \end{cases}$$

Since both $\sqrt{2/K}$ and $\langle x, e_{K/2}\rangle$ tend to zero as K increases, it follows that

$$\sum_{n=1}^{\infty} \langle x, y_n\rangle\, x_n = x = \sum_{n=1}^{\infty} \langle x, x_n\rangle\, y_n. \tag{3.1}$$

Therefore $\mathcal{Y} = \{y_n\}_{n\in\mathbb{N}}$ is both an alternative dual and a sythensis pseudo-dual of $\mathcal{F}$. However, for $x = e_1$ the first representation $e_1 = \sum_{n=1}^{\infty} \langle e_1, y_n\rangle\, x_n$ from Eq. (3.1) is

$$e_1 = e_1 + e_1 - e_1 + \frac{1}{\sqrt{2}}\, e_2 - \frac{1}{\sqrt{2}}\, e_2 + \cdots + \frac{1}{\sqrt{n}}\, e_n - \frac{1}{\sqrt{n}}\, e_n + \cdots,$$

which converges conditionally. Consequently $\mathcal{Y}$ cannot be a frame. Moreover, Lemma 2.5 implies that there exists some v such that the second representation $v = \sum_{n=1}^{\infty} \langle v, x_n \rangle \, y_n$ in Eq. (3.1) must converges conditionally. $\diamondsuit$

Several different examples illustrating convergence of frame series are given in [11].

The frame $\mathcal{F}$ in Example 3.1 has infinite excess. We prove next that if a frame $\{x_n\}_{n\in\mathbb{N}}$ has finite excess, then for every x the frame series $\sum \langle x, y_n \rangle \, x_n$ and $\sum \langle x, x_n \rangle \, z_n$ converge unconditionally for every alternative dual $\{y_n\}_{n\in\mathbb{N}}$ and synthesis pseudo-dual $\{z_n\}_{n\in\mathbb{N}}$. We remark that a different proof can be given by using [11, Prop. 5.11].

Theorem 3.2. If $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ is a near-Riesz basis for H, then every alternative dual and sythensis pseudo-dual of $\mathcal{F}$ are near-Riesz bases and hence are frames.

Proof. By Lemma 2.3 and Theorem 2.4, it suffices to show that every alternative dual $\mathcal{Y} = \{y_n\}_{n\in\mathbb{N}}$ and every synthesis pseudo-dual $\mathcal{Z} = \{z_n\}_{n\in\mathbb{N}}$ is a Bessel sequence.

Let $A = \{n_j\}_{j=1}^{N}$ be a subset of $\mathbb{N}$ such that $\{x_n\}_{n\notin A}$ is a Riesz basis for H. Then $\{x_n\}_{n\notin A}$ has a biorthogonal sequence $\{\widetilde{x}_n\}_{n\notin A}$, and consequently

$$x \;=\; \sum_{n\notin A} \langle x, \widetilde{x}_n \rangle \, x_n \tag{3.2}$$

is the unique representation of x in terms of $\{x_n\}_{n\notin A}$.

Fix $x \in H$. Since $\{y_n\}_{n\in\mathbb{N}}$ is an alternative dual of $\{x_n\}_{n\in\mathbb{N}}$,

$$x \;=\; \sum_{n=1}^{\infty} \langle x, y_n \rangle \, x_n \;=\; \sum_{n\notin A} \langle x, y_n \rangle \, x_n \;+\; \sum_{j=1}^{N} \langle x, y_{n_j} \rangle \, x_{n_j}. \tag{3.3}$$

By Eq. (3.2), for each $j = 1, \dots, N$ we have $x_{n_j} = \sum_{n\notin A} \langle x_{n_j}, \widetilde{x}_n \rangle \, x_n$. Substituting this into Eq. (3.3) gives

$$x \;=\; \sum_{n\notin A} \biggl(\langle x, y_n \rangle \;+\; \sum_{j=1}^{N} \langle x, y_{n_j} \rangle \, \langle x_{n_j}, \widetilde{x}_n \rangle \biggr) \, x_n. \tag{3.4}$$

Since Eq. (3.2) is the unique representation of x in terms of $\{x_n\}_{n\notin A}$, we must therefore have

$$\langle x, y_n \rangle \;+\; \sum_{j=1}^{N} \langle x, y_{n_j} \rangle \, \langle x_{n_j}, \widetilde{x}_n \rangle \;=\; \langle x, \widetilde{x}_n \rangle \qquad \text{for } n \notin A.$$

Consequently, as sequences in $\ell^2(\mathbb{N}\backslash A)$,

$$\bigl\{ \langle x, y_n \rangle \bigr\}_{n\notin A} \;=\; \bigl\{ \langle x, \widetilde{x}_n \rangle \bigr\}_{n\notin A} \;-\; \sum_{j=1}^{N} \bigl\{ \langle x, y_{n_j} \rangle \, \langle x_{n_j}, \widetilde{x}_n \rangle \bigr\}_{n\notin A}.$$

Let L denote an upper frame bound for $\{\widetilde{x}_n\}_{n\notin A}$. Then by applying the Triangle and Cauchy–Bunyakowski–Schwarz Inequalities, we compute that

$$\begin{aligned}\Big\|\big\{\langle x, y_n\rangle\big\}_{n\notin A}\Big\|_{\ell^2(\mathbb{N}\setminus A)} &\le \Big\|\big\{\langle x, \widetilde{x}_n\rangle\big\}_{n\notin A}\Big\|_{\ell^2(\mathbb{N}\setminus A)} + \sum_{j=1}^{N} \Big\|\big\{\langle x, y_{n_j}\rangle\,\langle x_{n_j}, \widetilde{x}_n\rangle\big\}_{n\notin A}\Big\|_{\ell^2(\mathbb{N}\setminus A)} \\ &\le L^{1/2}\,\|x\| + \sum_{j=1}^{N} \|x\|\,\|y_{n_j}\|\,\Big\|\big\{\langle x_{n_j}, \widetilde{x}_n\rangle\big\}_{n\notin A}\Big\|_{\ell^2(\mathbb{N}\setminus A)} \\ &\le L^{1/2}\,\|x\| + \sum_{j=1}^{N} \|x\|\,\|y_{n_j}\|\,L^{1/2}\,\|x_{n_j}\| \\ &= K\,\|x\|,\end{aligned}$$

where K is a constant independent of x. Therefore

$$\sum_{n=1}^{\infty} |\langle x, y_n\rangle|^2 = \sum_{n\notin A} |\langle x, y_n\rangle|^2 + \sum_{n\in A} |\langle x, y_n\rangle|^2 \le K^2\,\|x\|^2 + \sum_{n\in A} \|x\|^2\,\|y_n\|^2 = C\,\|x\|^2,$$

where C is a constant independent of x. Thus $\{y_n\}_{n\in\mathbb{N}}$ is a Bessel sequence in H.

On the other hand, if $k \notin A$ then

$$\widetilde{x}_k = \sum_{n=1}^{\infty} \langle \widetilde{x}_k, x_n\rangle\, z_n = z_k + \sum_{j=1}^{N} \langle \widetilde{x}_k, x_{n_j}\rangle\, z_{n_j}. \tag{3.5}$$

Consequently, if $x \in H$ then

$$\big\{\langle x, z_k\rangle\big\}_{k\notin A} = \big\{\langle x, \widetilde{x}_k\rangle\big\}_{k\notin A} - \sum_{j=1}^{N} \big\{\langle x, z_{n_j}\rangle\,\langle x_{n_j}, \widetilde{x}_k\rangle\big\}_{k\notin A}.$$

Arguing as before, it follows that $\{z_n\}_{n\in\mathbb{N}}$ is a Bessel sequence. □

Corollary 3.3. If $\{x_n\}_{n\in\mathbb{N}}$ is a near-Riesz basis for H, then $\sum \langle x, y_n\rangle\, x_n$ (respectively $\sum \langle x, x_n\rangle\, z_n$) converges unconditionally for every $x \in H$ and every alternative dual $\{y_n\}_{n\in\mathbb{N}}$ (respectively every sythensis pseudo-dual $\{z_n\}_{n\in\mathbb{N}}$). ♢

We will prove a converse to Theorem 3.2 and Corollary 3.3 below (see Theorem 3.10). To this end, we first prove that if c_n are scalars such that the series $x = \sum c_n x_n$ converges to a nonzero element in H, then there exists an alternative dual $\{y_n\}_{n\in\mathbb{N}}$ such that $c_n = \langle x, y_n\rangle$ for every n.

Theorem 3.4. Let $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ be a frame for H and let $(c_n)_{n\in\mathbb{N}}$ be a sequence of scalars. If $\sum c_n x_n$ converges to a nonzero vector $x_0 \in H$, then there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ that is both an alternative dual and a synthesis pseudo-dual of $\mathcal{F}$ such that $c_n = \langle x_0, y_n\rangle$ for every n.

Proof. Assume that $\sum c_n x_n = x_0 \neq 0$ and let $\{\widetilde{x}_n\}_{n\in\mathbb{N}}$ be the canonical dual frame of $\mathcal{F}$. Set $M = \overline{\operatorname{span}}\{x_0\}$, and let $P_{M^\perp}$ denote the orthogonal projection of H onto $M^\perp$. Define

$$y_n \;=\; \frac{\overline{c_n}}{\|x_0\|^2}\, x_0 \;+\; P_{M^\perp}\widetilde{x}_n, \qquad \text{for } n \in \mathbb{N}.$$

We will show that $\{y_n\}_{n\in\mathbb{N}}$ has the required properties.

Fix $x \in H$, and write x as $x = x_M + x_{M^\perp}$ where $x_M \in M$ and $x_{M^\perp} \in M^\perp$. Since M is one-dimensional, there exists a scalar α such that $x_M = \alpha x_0$. Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \langle x, y_n\rangle\, x_n \;&=\; \sum_{n=1}^{\infty} \Big\langle x_M \,+\, x_{M^\perp},\, \frac{\overline{c_n}}{\|x_0\|^2}\, x_0 \,+\, P_{M^\perp}\widetilde{x}_n \Big\rangle\, x_n \\
&=\; \sum_{n=1}^{\infty} \Big\langle \alpha x_0,\, \frac{\overline{c_n}}{\|x_0\|^2}\, x_0 \Big\rangle\, x_n \,+\, \sum_{n=1}^{\infty} \big\langle x_{M^\perp},\, P_{M^\perp}\widetilde{x}_n \big\rangle\, x_n \qquad \text{(cross terms vanish)} \\
&=\; \sum_{n=1}^{\infty} \alpha c_n x_n \,+\, \sum_{n=1}^{\infty} \big\langle x_{M^\perp},\, P_{M^\perp}\widetilde{x}_n \big\rangle\, x_n \\
&=\; \alpha x_0 \,+\, \sum_{n=1}^{\infty} \langle x_{M^\perp}, \widetilde{x}_n\rangle\, x_n \;=\; x_M \,+\, x_{M^\perp} \;=\; x.
\end{aligned}$$

This shows that $\{y_n\}_{n\in\mathbb{N}}$ is an alternative dual of $\mathcal{F}$. Further, for every n we have

$$\langle x_0, y_n\rangle \;=\; \Big\langle x_0,\, \frac{\overline{c_n}}{\|x_0\|^2}\, x_0 \,+\, P_{M^\perp}\widetilde{x}_n \Big\rangle \;=\; c_n.$$

In order to show that $\{y_n\}_{n\in\mathbb{N}}$ is a sythensis pseudo-dual of $\{x_n\}_{n\in\mathbb{N}}$, fix any $x \in H$. Then, since $\{\widetilde{x}_n\}_{n\in\mathbb{N}}$ is a frame, Lemma 2.3 implies that

$$\sum_{n=1}^{\infty} \langle x, x_n\rangle\, P_{M^\perp}\widetilde{x}_n \;=\; P_{M^\perp}\bigg(\sum_{n=1}^{\infty} \langle x, x_n\rangle\, \widetilde{x}_n\bigg) \;=\; P_{M^\perp}\bigg(\sum_{n=1}^{\infty} \langle x, \widetilde{x}_n\rangle\, x_n\bigg) \;=\; x_{M^\perp}.$$

On the other hand,

$$\sum_{n=1}^{\infty} \langle x, x_n\rangle\, \frac{\overline{c_n}}{\|x_0\|^2}\, x_0 \;=\; \Big\langle x,\, \sum_{n=1}^{\infty} \frac{c_n}{\|x_0\|}\, x_n \Big\rangle\, \frac{x_0}{\|x_0\|} \;=\; \Big\langle x,\, \frac{x_0}{\|x_0\|} \Big\rangle\, \frac{x_0}{\|x_0\|} \;=\; x_M,$$

the last equality following from the fact that $x_0/\|x_0\|$ is a unit vector. Consequently,

$$\sum_{n=1}^{\infty} \langle x, x_n\rangle\, y_n \;=\; \sum_{n=1}^{\infty} \langle x, x_n\rangle \bigg(\frac{\overline{c_n}}{\|x_0\|^2}\, x_0 + P_{M^\perp}\widetilde{x}_n\bigg) \;=\; x_M \,+\, x_{M^\perp} \;=\; x. \qquad \square$$

We obtain some corollaries of Theorem 3.4.

Definition 3.5. Let $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ be a frame for H. We say that a sequence of scalars $(c_n)_{n\in\mathbb{N}}$ is *realizable* with respect to $\mathcal{F}$ if there exists an $x \in H$ and an alternative dual $\{y_n\}_{n\in\mathbb{N}}$ such that $c_n = \langle x, y_n\rangle$ for all n. $\diamondsuit$

Corollary 3.6. Let $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ be a frame for H. Then a sequence $(c_n)_{n\in\mathbb{N}}$ is realizable with respect to $\mathcal{F}$ if and only if either $\sum c_n x_n$ converges to a nonzero $x \in H$ or $c_n = 0$ for every n.

Proof. ($\Rightarrow$) Assume that $(c_n)_{n\in\mathbb{N}}$ is realizable. Then there is some $x \in H$ and alternative dual $\{y_n\}_{n\in\mathbb{N}}$ such that $c_n = \langle x, y_n\rangle$ for every n. Then $x = \sum \langle x, y_n\rangle x_n = \sum c_n x_n$. If $x = 0$, then $c_n = \langle x, y_n\rangle = 0$ for every n.

($\Leftarrow$) If $c_n = 0$ for every n then take $x = 0$ and let $\{y_n\}_{n\in\mathbb{N}}$ be any alternative dual. The case $\sum c_n x_n = x_0 \neq 0$ follows from Theorem 3.4. □

Corollary 3.7. Let $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ be a frame for H. If there exist scalars c_n such that $\sum c_n x_n$ converges conditionally, then there exists an alternative dual $\{y_n\}_{n\in\mathbb{N}}$ that is also a synthesis pseudo-dual and some vectors x, $y \in H$ such that the series $\sum \langle x, y_n\rangle x_n$ and $\sum \langle y, x_n\rangle y_n$ converge conditionally.

Proof. If $\sum c_n x_n \neq 0$, then this follows from Theorem 3.4 and Lemma 2.5. Therefore, suppose that $\sum c_n x_n$ converges to zero conditionally. Then $c_{n_0} x_{n_0} \neq 0$ for some $n_0 \in \mathbb{N}$. Let $d_n = c_n$ for $n \neq n_0$ and set $d_{n_0} = 2c_{n_0}$. Then $\sum d_n x_n$ converges conditionally to $c_{n_0} x_{n_0}$, which is not zero. Hence we can apply Theorem 3.4 and Lemma 2.5 again. □

Corollary 3.8. Assume that $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ is a frame for H that contains at most finitely many zeros. Then the following statements are equivalent.

(a) $\mathcal{F}$ is a near-Riesz basis.

(b) $\sum \langle x, y_n\rangle x_n$ converges unconditionally for all alternative duals $\{y_n\}_{n\in\mathbb{N}}$ and every $x \in H$.

(c) $\sum \langle x, x_n\rangle z_n$ converges unconditionally for all sythensis pseudo-duals $\{z_n\}_{n\in\mathbb{N}}$ and every $x \in H$.

Proof. The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) follow from Theorem 3.2.

For the converse directions, suppose that $\mathcal{F}$ is not a near-Riesz basis. Then, by Theorem 2.2, $\mathcal{F}$ is not an unconditional frame. Therefore there are some scalars c_n such that $\sum c_n x_n$ converges conditionally. The result therefore follows by Corollary 3.7. □

Remark 3.9. The key point in the proof of the implication (b) $\Rightarrow$ (a) in Corollary 3.8 is that we can realize c_n as $\langle x, y_n\rangle$ for some alternative dual $\{y_n\}_{n\in\mathbb{N}}$ and $x \in H$ if $\sum c_n x_n$ converges to some nonzero element. We can generalize this as follows. The proof of Theorem 3.4 for the case of alternative duals does not utilize all of the properties of frame. In fact, we do not need the existence of a canonical dual; instead, we only require the existence of at least one sequence $\{\widetilde{y}_n\}_{n\in\mathbb{N}}$ such that $x = \sum \langle x, \widetilde{y}_n\rangle x_n$ for every x. Hence, whenever sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\widetilde{y}_n\}_{n\in\mathbb{N}}$ have this property, it will be the case that a sequence of scalars $(c_n)_{n\in\mathbb{N}}$ can be expressed as $\{\langle x, y_n\rangle\}_{n\in\mathbb{N}}$ for some x and alternative dual $\{y_n\}_{n\in\mathbb{N}}$ if $\sum c_n x_n$ converges to a nonzero element. Here, "alternative dual" simply means a sequence such that $x = \sum \langle x, y_n\rangle x_n$ for every $x \in H$.

Based on this observation, we can give a more general characterization of sequences for which $x = \sum \langle x, y_n\rangle x_n$ converges unconditionally for every x and alternative dual $\{y_n\}_{n\in\mathbb{N}}$. In [4], Casazza and Christensen showed that if a sequence $\{x_n\}_{n\in\mathbb{N}}$ containing at most finitely many zeros is such that $\sum c_n x_n$ converges unconditionally whenever it converges, then $\{x_n\}_{n\in\mathbb{N}}$ is an unconditional basis plus at most finitely many elements. Therefore, if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence that contains at most finitely many zeros and it is the case that whenever $\{y_n\}_{n\in\mathbb{N}}$ is a sequence such that $x = \sum \langle x, y_n\rangle x_n$ for every x then this series converges unconditionally for every x, then a proof similar to that of Corollary 3.8 shows that $\{x_n\}_{n\in\mathbb{N}}$ must be an unconditional basis plus at most finitely many elements. $\diamondsuit$

Now we prove that the converse of Theorem 3.2 holds.

Theorem 3.10. Let $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ be a frame for H. Then the following statements are equivalent.

(a) $\mathcal{F}$ is a near-Riesz basis.

(b) Every alternative dual of $\mathcal{F}$ is a frame.

(c) Every sythensis pseudo-dual of $\mathcal{F}$ is a frame.

Proof. The implication (a) $\Rightarrow$ (b) was proved in Theorem 3.2.

For the implication (b) $\Rightarrow$ (a), assume that $\mathcal{F}$ is not a near-Riesz basis. We must show that there is an alternative dual of $\mathcal{F}$ that is not a frame.

First consider the case where $\mathcal{F}$ contains infinitely many zeros. Since it is a frame it must also contain infinitely many nonzero elements, so by reindexing we may assume that $x_{2n} = 0$ for every n. Let $\{y_n\}_{n\in\mathbb{N}}$ be any alternative dual. Choose any $x_0 \neq 0$ in H, and define a new sequence $\{w_n\}_{n\in\mathbb{N}}$ by $w_{2n-1} = y_{2n-1}$ and $w_{2n} = x_0$. Then $\{w_n\}_{n\in\mathbb{N}}$ is an alternative dual of $\mathcal{F}$, but it is not a frame.

On the other hand, if $\mathcal{F}$ does not contain infinitely many zeros, then Corollary 3.8 implies that there exists an alternative dual $\{y_n\}_{n\in\mathbb{N}}$ and some $x \in H$ such that $\sum \langle x, y_n\rangle\, x_n$ converges conditionally. But then $\{y_n\}_{n\in\mathbb{N}}$ is not a frame.

A similar argument shows the equivalence of (a) and (c). $\square$

Remark 3.11. We have seen that excess of a frame is related to the unconditional convergence of frame series. Consequently, if we use unconditional convergence of frame series as a criterion to distinguish "good" and "bad" frames, then only near-Riesz bases can be "good" frames. A natural follow-up question is: For which alternative dual $\{y_n\}_{n\in\mathbb{N}}$ (respectively sythensis pseudo-dual) can the corresponding frame expansion $\sum \langle x, y_n\rangle\, x_n$ (respectively $\sum \langle x, x_n\rangle\, y_n$) converge unconditionally for every $x \in H$? If the frame $\{x_n\}_{n\in\mathbb{N}}$ is norm-bounded below then we know that an alternative dual (respectively sythensis pseudo-dual) can be "good" in this sense if it is a frame. For two arbitrary sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in H, it was conjectured in [12] that the series $\sum \langle x, y_n\rangle\, x_n$ converges unconditionally for every $x \in H$ if and only if there exist sequences of scalars $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ such that $c_n\overline{d_n} = 1$ for every n and $\{c_n x_n\}_{n\in\mathbb{N}}$ and $\{d_n y_n\}_{n\in\mathbb{N}}$ are Bessel sequences. $\diamondsuit$

A frame is a Riesz basis if and only if the range of the analysis operator $Cx = \{\langle x, x_n\rangle\}_{n\in\mathbb{N}}$ is ℓ^2 (see [9, Thm. 8.32]). We know that a frame that is not a Riesz basis possesses more than one alternative dual, in fact it has infinitely many (see [5, Lem. 6.3.1]). Further, in this case the range of the analysis operator is a proper closed subspace of ℓ^2. However, could it be that the union of the ranges of the analysis operators of all alternative duals is ℓ^2? We address this question next.

Definition 3.12. (a) The *moment space* associated with a sequence $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ in H is

$$m(\mathcal{F}) \;=\; \Bigl\{\{\langle x, f_n\rangle\}_{n\in\mathbb{N}} : x \in H\Bigr\}.$$

(b) The *extended moment space* of a frame $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ for H is the union of all moment spaces over all alternative duals:

$$\mathcal{M}(\mathcal{F}) = \bigcup \bigl\{m(\mathcal{Y}) : \mathcal{Y} \text{ is an alternative dual of } \mathcal{F}\bigr\}. \qquad \diamondsuit$$

For details on moment spaces, we refer to [14]. The following result is Theorem 7 in Chapter 4 of [14].

Lemma 3.13. If $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ and $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ are two complete sequences in H, then $m(\mathcal{F}) = m(\mathcal{G})$ if and only if there exists an bounded linear invertible operator $T\colon H \to H$ such that $T(f_n) = g_n$ for every $n \in \mathbb{N}$. $\diamondsuit$

We will need the following lemma.

Lemma 3.14. Let $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ be a frame for H, and let $\widetilde{\mathcal{F}} = \{\widetilde{x}_n\}_{n\in\mathbb{N}}$ be its canonical dual frame. Then for any alternative dual or sythensis pseudo-dual $\{y_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}$,

$$\{\langle y, y_n\rangle\}_{n\in\mathbb{N}} \in m(\widetilde{\mathcal{F}}\,)^\perp \iff y = 0.$$

Proof. Assume that $\{\langle y, y_n\rangle\}_{n\in\mathbb{N}}$ belongs to $m(\widetilde{\mathcal{F}}\,)^\perp$. The frame operator S for $\{x_n\}_{n\in\mathbb{N}}$ is a bounded linear invertible map of H onto itself and the canonical dual frame is given by $\widetilde{x}_n = S^{-1}x_n$. Therefore Lemma 3.13 implies that $m(\widetilde{\mathcal{F}}\,) = m(\mathcal{F})$. Choose any $x \in H$. Then $\{\langle x, x_n\rangle\}_{n\in\mathbb{N}}$ belongs to $m(\mathcal{F})$, so if $\{y_n\}_{n\in\mathbb{N}}$ is an alternative dual then

$$\langle x, y\rangle = \Bigl\langle x, \sum_{n=1}^\infty \langle y, y_n\rangle\, x_n \Bigr\rangle = \sum_{n=1}^\infty \langle x, x_n\rangle\, \overline{\langle y, y_n\rangle} = 0.$$

Similarly, if $\{y_n\}_{n\in\mathbb{N}}$ is a sythensis pseudo-dual, then

$$\langle x, y\rangle = \Bigl\langle \sum_{n=1}^\infty \langle x, x_n\rangle\, y_n,\, y \Bigr\rangle = \sum_{n=1}^\infty \langle x, x_n\rangle\, \overline{\langle y, y_n\rangle} = 0.$$

In either case we see that $\langle x, y\rangle = 0$ for every x, so $y = 0$. $\square$

We now characterize the relation between $\mathcal{M}(\mathcal{F})$ and ℓ^2.

Theorem 3.15. If $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ is a frame for H, then $\mathcal{M}(\mathcal{F}) \subseteq \ell^2$ if and only if $\mathcal{F}$ is a near-Riesz basis. Consequently, $\mathcal{M}(\mathcal{F}) = \ell^2$ if and only if $\mathcal{F}$ is a Riesz basis.

Proof. By Theorem 3.2, if $\mathcal{F}$ is a near-Riesz basis then $\mathcal{M}(\mathcal{F}) \subseteq \ell^2$

Conversely, if $\mathcal{M}(\mathcal{F}) \subseteq \ell^2$ then every alternative dual is a Bessel sequence and consequently a frame by Theorem 2.3. Therefore $\mathcal{F}$ is a near-Riesz basis by Theorem 3.10.

Now assume that $\mathcal{F}$ is a Riesz basis. In this case the canonical dual frame is also a Riesz basis, so $\mathcal{M}(\mathcal{F}) = \ell^2$.

Conversely, suppose that $\mathcal{M}(\mathcal{F}) = \ell^2$. Let $\widetilde{\mathcal{F}} = \{\widetilde{x}_n\}_{n\in\mathbb{N}}$ be the canonical dual frame of $\mathcal{F}$, but suppose that $m(\widetilde{\mathcal{F}}\,) \neq \ell^2$. Then $m(\widetilde{\mathcal{F}}\,)^\perp$ contains a nonzero sequence, so since $\mathcal{M}(\mathcal{F}) = \ell^2$ there is some nonzero $y \in H$ such that $\{\langle y, y_n\rangle\}_{n\in\mathbb{N}} \in m(\widetilde{\mathcal{F}}\,)^\perp$. But then $y = 0$ by Lemma 3.14, which is a contradiction. Therefore $m(\widetilde{\mathcal{F}}\,) = \ell^2$, which implies that $\widetilde{\mathcal{F}}$, and hence $\mathcal{F}$, is a Riesz basis. $\square$

We can also define the extended moment space for sythensis pseudo-duals. Since the canonical dual frame is a sythensis pseudo-dual frame, we can use a similar proof to obtain a sythensis pseudo-dual version of Theorem 3.15.

4. The Excess of Alternative Duals and Synthesis-Pseudo Duals

Now we consider the relation between the excess of a frame and the excess of its alternative duals and sythensis pseudo-duals. We know from Theorem 2.4 that every alternative dual that is a frame has the same excess as its associated frame. Moreover, we proved in Theorem 3.2 that every alternative dual (respectively sythensis pseudo-dual) of a frame with finite excess must be a frame. In particular, every alternative dual frame is also a sythensis pseudo-dual frame by Lemma 2.3. Consequently, if $\mathcal{F}$ is a frame with finite excess then $e(\mathcal{F}) = e(\mathcal{Y})$ for every alternative dual and synthesis pseudo-dual $\mathcal{Y}$. We will prove in this section that this same relation holds for frames with infinite excess.

First we need the following lemma.

Lemma 4.1. If $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ is a minimal sequence in H and $\mathcal{G} = \{g_n\}_{n=1}^N$ is a finite sequence, then $e(\mathcal{F}\cup\mathcal{G}) < \infty$.

Proof. Observe that if J is a finite subset of $\mathbb{N}$, then $\operatorname{codim}\big(\overline{\operatorname{span}}\{f_n\}_{n\notin J}\big) \geq |J|$ since $\{f_n\}_{n\in\mathbb{N}}$ is minimal.

By replacing H with the closed span of $\mathcal{F}\cup\mathcal{G}$, we may assume that $\mathcal{F}\cup\mathcal{G}$ is complete. Suppose that $e(\mathcal{F}\cup\mathcal{G}) = \infty$. Then we can find $3N$ elements of $\mathcal{F}\cup\mathcal{G}$ such that the removal of these $3N$ elements still leaves a complete sequence. Precisely, there exist sets $J_1 \subseteq \mathbb{N}$ and $J_2 \subseteq \{1,\dots,N\}$ such that $|J_1| + |J_2| = 3N$ and

$$\mathcal{A} = \{f_n\}_{n\in\mathbb{N}\setminus J_1} \cup \{g_n\}_{n\in\{1,\dots,N\}\setminus J_2} \quad \text{is complete.}$$

Since $|J_2| \leq N$, we must have $|J_1| \geq 2N$. Therefore $\operatorname{codim}\big(\overline{\operatorname{span}}\{f_n\}_{n\in\mathbb{N}\setminus J_1}\big) \geq 2N$. However, the completeness of $\mathcal{A}$ implies that the codimension of $\overline{\operatorname{span}}\{f_n\}_{n\in\mathbb{N}\setminus J_1}$ is at most N, which is a contradiction. □

Now we prove that the excess of a frame equals the excess of any alternative or sythensis pseudo-dual.

Theorem 4.2. If $\mathcal{F} = \{x_n\}_{n\in\mathbb{N}}$ is a frame for H then $e(\mathcal{F}) = e(\mathcal{Y}) = e(\mathcal{Z})$ for every alternative dual $\mathcal{Y}$ and sythensis pseudo-dual $\mathcal{Z}$.

Proof. As we pointed out earlier, it suffices to consider the case $e(\mathcal{F}) = \infty$.

Alternative Duals. Let $\mathcal{Y} = \{y_n\}_{n\in\mathbb{N}}$ be an alternative dual of $\mathcal{F}$, and suppose that $e(\mathcal{Y}) = 0$. In this case $\mathcal{Y}$ is minimal, so there exists a biorthogonal sequence $\{\widetilde{y}_n\}_{n\in\mathbb{N}}$. But since $\mathcal{Y}$ is an alternative dual, this implies that $\widetilde{y}_k = \sum \langle \widetilde{y}_k, y_n\rangle\, x_n = x_k$ for every k. Hence $\{x_n\}_{n\in\mathbb{N}}$ is minimal and so has zero excess, which is a contradiction.

Next, suppose that $0 < e(\mathcal{Y}) < \infty$, and let $N = e(\mathcal{Y})$. By reindexing, we may assume $\{y_n\}_{n>N}$ has zero excess and hence is minimal. Consequently it has a biorthogonal sequence $\{\widetilde{y}_n\}_{n>N}$. Therefore, if $k > N$ then

$$\widetilde{y}_k = \sum_{n=1}^{\infty} \langle \widetilde{y}_k, y_n\rangle\, x_n = x_k + \sum_{n=1}^{N} \langle \widetilde{y}_k, y_n\rangle\, x_n.$$

For each $k > N$ let

$$w_k \;=\; \sum_{n=1}^{N} \langle \widetilde{y}_k, y_n\rangle \, x_n,$$

so $x_k + w_k = \widetilde{y}_k$ for $k > N$. By Lemma 4.1,

$$K \;=\; e\Bigl(\{w_n + x_n\}_{n>N} \cup \{x_n\}_{n=1}^{N}\Bigr) \;<\; \infty.$$

Since $\mathcal{F}$ has infinite excess, it is possible to remove $N + 2K$ elements from $\mathcal{F}$ yet leave the closed span unchanged. Let $J_1 \subseteq \{1, \dots, N\}$ and $J_2 \subseteq \{N+1, N+2, \dots\}$ be such that $|J_1| + |J_2| = 2K + N$ and $\overline{\mathrm{span}}\{x_n\}_{n \notin J_1 \cup J_2} = H$. Since $w_n \in \overline{\mathrm{span}}\{x_n\}_{n=1}^{N}$, it follows that

$$H \;=\; \overline{\mathrm{span}}\bigl(\{x_n\}_{n \in \{1,\dots,N\} \setminus J_1} \cup \{x_n\}_{n>N,\, n \notin J_2}\bigr) \;\subseteq\; \overline{\mathrm{span}}\bigl(\{x_n\}_{n=1}^{N} \cup \{x_n + w_n\}_{n>N,\, n\notin J_2}\bigr).$$

Thus it is possible to remove $|J_2|$ elements from $\{x_n\}_{n=1}^{N} \cup \{w_n + x_n\}_{n>N}$ without changing its closed span. But $|J_2| \ge 2K > K$, so this is a contradiction.

We conclude that we must have $e(\mathcal{Y}) = \infty$.

Synthesis-Pseudo Duals. Let $\mathcal{Z} = \{z_n\}_{n\in\mathbb{N}}$ be a sythensis pseudo-dual. If $e(\mathcal{Z}) = 0$, then it has a biorthogonal sequence $\{\widetilde{z}_n\}_{n\in\mathbb{N}}$. Consequently, if $x \in H$ then for every k we have

$$\langle x, \widetilde{z}_k\rangle \;=\; \Bigl\langle \sum_{n=1}^{\infty} \langle x, x_n\rangle \, z_n,\, \widetilde{z}_k \Bigr\rangle \;=\; \langle x, x_k\rangle.$$

This implies that $x_k = \widetilde{z}_k$ for every k, contradicting our assumption that $e(\mathcal{F}) = \infty$.

Finally, suppose that $0 < e(\mathcal{Z}) < \infty$. We may assume then that $\{z_n\}_{n>N}$ is minimal and has a biorthogonal sequence $\{\widetilde{z}_n\}_{n>N}$. For $k > N$ and $x \in H$,

$$\begin{aligned}\langle x, \widetilde{z}_k\rangle \;&=\; \Bigl\langle \sum_{n=1}^{\infty} \langle x, x_n\rangle \, z_n,\, \widetilde{z}_k \Bigr\rangle \;=\; \langle x, x_k\rangle \;+\; \Bigl\langle \sum_{n=1}^{N} \langle x, x_n\rangle \, z_n,\, \widetilde{z}_k \Bigr\rangle \\ &=\; \Bigl\langle x,\, x_k \;+\; \sum_{n=1}^{N} \langle \widetilde{z}_k, z_n\rangle \, x_n \Bigr\rangle.\end{aligned}$$

Consequently, $\widetilde{z}_k = x_k \,+\, \sum_{n=1}^{N} \langle \widetilde{z}_k, z_n\rangle \, x_n$ for $k > N$. An argument similar to the one used for alternative duals then leads to a contradiction. □

Acknowledgments

We thank Hans Feichtinger for bringing Ref. [13] to our attention.

Funding

This research was partially supported by a grant from the Simons Foundation.

References

[1] R. Balan, P. G. Casazza, C. Heil and Z. Landau, *Deficits and excesses of frames*, Adv. Comput. Math., **18** (2003), 93–116.
[2] D. Bakíc and T. Berić, *On excesses of frames*, Glas. Mat. Ser. III, **50**(70) (2015), 415–427.
[3] T. Bemrose, P. G. Casazza, V. Kaftal, and R. G. Lynch, *The unconditional constants for Hilbert space frame expansions*, Linear Algebra Appl., **521** (2017), 1–18.
[4] P. G. Casazza and O. Christensen, *Hilbert space frames containing a Riesz basis and Banach spaces which have no subspace isomorphic to* c_0, J. Math. Anal. Appl., **202** (1996), 940–950.
[5] O. Christensen, *An Introduction to Frames and Riesz Bases*, Second edition, Birkhäuser, Boston, 2016.
[6] I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., **27** (1986), 1271–1283.
[7] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., **72** (1952), 341–366.
[8] C. Heil, *Wiener amalgam spaces in generalized harmonic analysis and wavelet theory*, Ph.D. Thesis, University of Maryland, College Park, MD, 1990.
[9] C. Heil, *A Basis Theory Primer*, Expanded Edition, Birkhüser, Boston, 2011.
[10] J. R. Holub, *Pre-frame operators, Besselian frames, and near-Riesz bases in Hilbert spaces*, Proc. Amer. Math. Soc., **122** (1994), 779–785.
[11] D. T. Stoeva, *Characterization of atomic decompositions, Banach frames, Xd-frames, duals and synthesis-pseudo-duals, with application to Hilbert frame theory*, preprint (2016).
[12] D. T. Stoeva and P. Balazs, *Canonical forms of unconditionally convergent multipliers.*, J. Math. Anal. Appl., **399** (2013), 252–259.
[13] D. T. Stoeva and P. Balazs, *A survey on the unconditional convergence and the invertibility of frame multipliers with implementation*, in: "Sampling: Theory and Applications—A Centennial Celebration of Claude Shannon", S. D. Casey et al., Eds., Birkhäuser/Springer, 2020, pp. 169–192
[14] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Revised First Edition, Academic Press, San Diego, 2001.
[15] Z. H. Zhang, *Pointwise convergence and uniform convergence of wavelet frame series*, Acta Math. Sin. (Engl. Ser.), **22** (2006), 653–658.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA
Email address: `heil@math.gatech.edu`
Email address: `pyu73@gatech.edu`