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An Introduction to Weighted Wiener Amalgams

Christopher Heil

ABSTRACT Wiener amalgam spaces are a class of spaces of functions or distributions defined by a norm which amalgamates a local criterion for membership in the space with a global criterion. Feichtinger has developed an extensive theory of amalgams allowing a wide range of Banach spaces to serve as local or global components in the amalgam. We present an introduction to a more limited case, namely, the weighted amalgams $W(L^p, L^q_w)$ over the real line, where local and global components are defined solely by integrability criteria. We derive some of the basic properties of these weighted amalgams, and give one application of amalgam spaces to time-frequency analysis, proving the Amalgam Balian-Low Theorem.[1]

11.1 Introduction

This article presents an introduction to a class of spaces of functions or distributions defined by a norm which amalgamates, or mixes, a local criterion for membership in the space with a global criterion. Or, it may be more precise to interpret the norm as giving a global criterion for a local property. Versions of such amalgam spaces have arisen independently many times in the literature, and often provide a natural or compelling context for formulating results. As one illustration of the shortcomings of the usual Lebesgue space $L^p(\mathbf{R})$ in regard to distinguishing between local and global properties of functions, note that all rearrangements of a function have identical L^p norms. Hence it is not possible to recognize from its norm whether a function is the characteristic function of an interval or the sum of many characteristic functions of small intervals spread widely over $\mathbf{R}$.

The first appearance of amalgam spaces can be traced to Norbert Wiener in his development of the theory of generalized harmonic analysis. In particular, Wiener defined in [58] the spaces that we will call $W(L^1, L^2)$ and $W(L^2, L^1)$, and in [59], [60] he defined the spaces $W(L^1, L^\infty)$ and $W(L^\infty, L^1)$, using what we will

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refer to as a discrete norm for these spaces, namely,

$$\|f\|_{W(L^p,L^q)} \;=\; \biggl(\sum_{n\in\mathbf{Z}} \biggl(\int_n^{n+1} |f(t)|^p\,dt\biggr)^{q/p}\biggr)^{1/q}, \tag{11.1}$$

with the usual adjustments if p or q is infinity. We will not attempt to provide a complete historical discussion of papers on amalgam spaces, but mention here only some specific instances in which amalgams were introduced or studied. The first systematic study of these spaces appears to be the paper by Holland [45], which included discussion of dual spaces, multipliers, and interpolation properties, with some indications of earlier related works such as [8], [61]. In the next few years, there were several important independent studies of amalgams, including the papers of Bertrandias, Datry, and Dupuis [12], Stewart [55], and Busby and Smith [13]. Among other results, amalgams were generalized from functions on $\mathbf{R}$ or $\mathbf{R}^n$ to functions on locally compact groups. The paper of Busby and Smith was the first to derive convolution theorems for amalgams. The survey paper by Fournier and Stewart [35] is a very useful review dealing with the case of locally compact abelian groups, presenting a number of applications and containing a large bibliography.

Much, though not all, of the focus in the papers listed above is on the "standard" amalgams $W(L^p, L^q)$, where local and global criteria consist only of integrability or summability conditions. Even using only such integrability criteria, we can see the essential differences between local and global behavior which lead naturally to the formulation of amalgam spaces. To mention one property, unlike the Lebesgue spaces $L^p(\mathbf{R})$, the amalgam spaces $W(L^p, L^q)$ are ordered appropriately with respect to inclusion principles for local and global components, cf. equation (11.2) below.

Our introduction to the topic of amalgams was through the papers of Feichtinger, who, influenced by [45], [12], [55], [13] (preprint version), developed a comprehensive notion of amalgams which allow an extremely wide range of Banach spaces of functions or distributions defined on a locally compact group to be used as local or global components, resulting in a deep and powerful theory. In recognition of Wiener's first use of amalgams, Feichtinger initially called these spaces "Wiener-type spaces," and then, following a suggestion of Benedetto, adopted the name "Wiener amalgam spaces." We refer to [20], [21], [22], [23] for the introduction and early development of Wiener amalgams, and to [28], [24], [26], [27] for continued developments and for the source of the arguments that we will present here. Further developments and much additional material can be found in [29], [30], [38], [31]. Amalgam spaces are so useful that it is not surprising that some specific instances of Wiener amalgams with components defined not just by integrability criteria can be found prior to Feichtinger's work. For example, the paper of Essen [19] in renewal theory used what we would call the space $W(M, L^\infty_w)$, where the local component is the space of locally bounded measures and the global component is a weighted L^∞ space (using a weight that is a particular case of the moderate weights we define later).

Although we will not define Wiener amalgams in full generality, to illustrate their importance let us mention one specific example of a Wiener amalgam space that today plays a central role in the theory of time-frequency analysis. This is the space $W(\mathcal{F}L^1, L^1)$, consisting of functions that are locally the Fourier transform of an L^1 function and have a global L^1 behavior. $W(\mathcal{F}L^1, L^1)$ is often denoted S_0 to indicate that it is the smallest Segal algebra on which both time-shifts (by definition) and frequency-shifts act isometrically. We refer to [25] for an introduction to S_0 and its properties, and to [53] for more information on Segal algebras, including S_0 in particular. $W(\mathcal{F}L^1, L^1)$ is also known as the modulation space M^1, and in this form is in many respects the "correct" space of window functions for time-frequency analysis. We refer to [33], [39] for recent treatments of S_0 and the modulation spaces in the context of time-frequency analysis, and mention [42] for an exposition of one application of modulation spaces to pseudodifferential operators. The Wiener amalgam $W(\mathcal{F}L^1, L^1) = S_0 = M^1$ has many useful properties and characterizations. In particular, it is an algebra under both pointwise multiplication and under convolution, it is invariant under the Fourier transform, and it is the smallest Banach space that is isometrically invariant under both translations and modulations (and hence is a subspace of the Fourier algebra $\mathcal{F}L^1$).

Without presenting details or giving (important!) precise hypotheses, we list below a few of the basic properties of general Wiener amalgams $W(B, C)$ on a locally compact group G which, in addition to their control of local and global behavior, contribute to their utility in applications:

- *Inclusions*. If $B_1 \subset B_2$ and $C_1 \subset C_2$, then $W(B_1, C_1) \subset W(B_2, C_2)$. Moreover, the inclusion of B_1 into B_2 need only hold "locally" and the inclusion of C_1 into C_2 need only hold "globally" in order to obtain an inclusion of the corresponding amalgams. In particular, for $G = \mathbf{R}$ we have

$$p_1 \ge p_2 \text{ and } q_1 \le q_2 \quad \Longrightarrow \quad W(L^{p_1}, L^{q_1}) \subset W(L^{p_2}, L^{q_2}). \tag{11.2}$$

- *Duality*. If a space of test functions (e.g., for $G = \mathbf{R}$ this might be the Schwartz space of smooth, rapidly decreasing functions) is dense in B and C, then $W(B, C)' = W(B', C')$.

- *Complex interpolation*. Complex interpolation can be carried out in each component of $W(B, C)$ separately.

- *Pointwise products*. If $B_1 \cdot B_2 \subset B_3$ and $C_1 \cdot C_2 \subset C_3$, then

$$W(B_1, C_1) \cdot W(B_2, C_2) \;\subset\; W(B_3, C_3).$$

- *Convolutions*. If G is an IN-group and $B_1 * B_2 \subset B_3$ and $C_1 * C_2 \subset C_3$, then

$$W(B_1, C_1) * W(B_2, C_2) \;\subset\; W(B_3, C_3).$$

- *Hausdorff–Young*. If G is a locally compact abelian group and $1 \le p, q \le 2$, then $\mathcal{F}\big(W(L^p, L^q)\big) \subset W(L^{q'}, L^{p'})$, where $\mathcal{F}$ is the Fourier transform. In particular, local and global properties are interchanged on the Fourier side.

IN-groups are defined precisely at the end of Section 11.8. They include all abelian groups, and some non-abelian groups such as the reduced Heisenberg group (which is important for time-frequency analysis), but not the $ax + b$ group (which is important for wavelet theory). However, even for the $ax + b$ group there are interesting, but more complicated, convolution relations [29].

In summary, Wiener amalgams are an extremely useful class of spaces which allow a wide variety of local properties to be mixed with global properties. Our goal in this article is to present a brief introduction to a part of this rich theory. We will present only the "weighted Wiener amalgams" $W(L^p, L^q_w)$ consisting of functions which are locally in L^p and globally in a weighted L^q space. The weights we consider are the moderate weights, which are exactly those weights for which L^q_w is translation-invariant, including submultiplicative weights in particular. Additionally, we only consider the case of functions defined on the real line, and will not consider extensions to locally compact groups. We will provide proofs of some of the properties of amalgams listed above for the case of the weighted amalgams $W(L^p, L^q_w)$. Specifically, we consider inclusions, duality, pointwise products, and convolutions, as well as some other properties such as translation-invariance. We will also give one application of the amalgam space $W(C_0, L^1)$ to a problem in time-frequency analysis, deriving what is now known as the "Amalgam Balian–Low Theorem."

Even though they are only a special case, the weighted amalgams $W(L^p, L^q_w)$ have many applications. For example, the weighted amalgams, and in particular the convolution theorems for them, play an important role in time-frequency analysis, cf. the development that is presented by Gröchenig in [39]. Another area in which the weighted amalgams and their convolution theorems are important is the recent progress in sampling theory. Sampling is concerned with the problem of reconstructing a function f in a given function space from a set of sample values $\{f(t_n)\}_{n \in J}$. The Classical Sampling Theorem states that a bandlimited function f on the real line can be reconstructed from regularly spaced samples $\{f(nT)\}_{n \in \mathbf{Z}}$ if the sampling rate is high enough, cf. [9] for background. Sampling problems are ubiquitous in modern engineering, but the functions that must be reconstructed are often not bandlimited and the samples are often not uniformly spaced. Further, fast and robust algorithms for reconstruction are essential in applications. Amalgam spaces provide a useful context for developing sampling theory, and we refer to [1], [2], [3], [4], [32] for recent results in this direction.

Of course, in order that sample values are defined, sampling theory actually requires a local assumption of at least continuity. For this and many other applications, integrability criteria alone are not sufficient. We hope that our short introduction to the weighted amalgam spaces will inspire the reader to pursue the entire theory of Wiener amalgams in more detail. In particular, our proofs, while short and self-contained, often rely too much on the fact that our local and global components are defined only by integrability criteria, and that the underlying domain of the functions is the real line. Hence in many cases our approach fails to convey the full flavor and power of general Wiener amalgams, even for functions on $\mathbf{R}^n$ and not just for generalizations to locally compact groups. We therefore suggest that the reader follow this article with a more complete development of the theory, such as is presented in the thesis of Dobler in [17]. Even though we do not present the most general cases or proofs, the techniques presented here will be a good technical preparation for the reader who wishes to understand the techniques needed in the general case.

Finally, we emphasize that, with the exception of the Amalgam Balian-Low Theorem, *no* results presented here are original with the author! Our sole contribution has been to select and summarize some results drawn from the papers of Feichtinger listed in the references. Even if we do not credit each individual result, they can be directly traced to those papers.

As a historical note, we remark that this article grew out of an expository section in one part of the author's thesis [41]. That section was expanded to an unpublished set of notes in 1992, and this article is a revision, rewriting, and expansion of those notes.

This article is organized as follows. General preliminaries are included in Section 11.2, including a definition of moderate weights, which are the weights for which L^p_w is translation-invariant. The definition of $W(L^p, L^q_w)$ is given in Section 11.3 in terms of a "continuous"-type norm, which emphasizes much more clearly than equation (11.1) the way is which L^p is amalgamated with L^q_w to form $W(L^p, L^q_w)$. Translation-invariance properties of $W(L^p, L^q_w)$ are presented in Section 11.4, and some inclusion relations are given in Section 11.5. Section 11.6 derives equivalent "discrete"-type norms for $W(L^p, L^q_w)$, in the spirit of equation (11.1) but allowing much more flexibility, allowing partitions of unity instead of just disjoint partitionings of the line. The dual space of $W(L^p, L^q_w)$ is computed in Section 11.7, and a convolution theorem is proved in Section 11.8. Finally, the Amalgam Balian-Low Theorem is derived in Section 11.9.

11.2 Preliminaries

11.2.1 General notation

The Lebesgue measure of a subset E of the real line $\mathbf{R}$ is denoted $|E|$.

The characteristic function of a set E is χ_E.

The support of a function $f\colon \mathbf{R} \to \mathbf{C}$ is the closure in $\mathbf{R}$ of $\{x \in \mathbf{R} : f(x) \neq 0\}$ and is denoted $\mathrm{supp}(f)$.

Given $a \in \mathbf{R}$, the translation operator T_a is defined by $T_a f(x) = f(x-a)$.

The translation of a set $E \subset \mathbf{R}$ is $E + a = \{x + a : x \in E\}$.

Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on a space X are said to be equivalent if there exist constants A, $B > 0$ such that $A\,\|f\|_b \le \|f\|_a \le B\,\|f\|_b$ for all $f \in X$. In this case we write $\|\cdot\|_a \asymp \|\cdot\|_b$.

11.2.2 Weighted L^p spaces

The space L^1_{loc} of locally integrable functions on the real line $\mathbf{R}$ consists of all complex-valued functions $f\colon \mathbf{R} \to \mathbf{C}$ such that $\int_K |f(x)|\,dx < \infty$ for every compact subset K of $\mathbf{R}$.

A positive, measurable, locally integrable function $w\colon \mathbf{R} \to (0,\infty)$ is called a *weight*. The weighted L^p-space L^p_w is the set of all complex-valued functions on $\mathbf{R}$

for which $fw \in L^p$, with norm

$$\|f\|_{L^p_w} \;=\; \|fw\|_{L^p} \;=\; \left(\int_{\mathbf{R}} |f(x)\,w(x)|^p\,dx\right)^{1/p}, \qquad 1 \le p < \infty,$$

or

$$\|f\|_{L^\infty_w} \;=\; \|fw\|_{L^\infty} \;=\; \operatorname*{ess\,sup}_{x\in\mathbf{R}} |f(x)|\,w(x), \qquad p = \infty.$$

If $w \equiv 1$ we simply write L^p.

It can be shown that L^p_w is a Banach space for $1 \le p \le \infty$. The *dual index* to p is $p' = p/(p-1)$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. We have $(L^p_w)' = L^{p'}_{1/w}$ for $1 \le p < \infty$, where the prime denotes the Banach space dual and the duality is defined by $\langle f, g\rangle = \int_{\mathbf{R}} f(x)\,\overline{g(x)}\,dx$.

L^p_w is a *solid* space, i.e., if $g \in L^p_w$ is given and $f \in L^1_{\mathrm{loc}}$ satisfies $|f| \le |g|$ a.e., then $f \in L^p_w$ and $\|f\|_{L^p_w} \le \|g\|_{L^p_w}$.

Other Banach spaces that will arise in our discussion include the Banach space C_b consisting of bounded continuous functions on $\mathbf{R}$ under the L^∞ norm, and the Banach space C_0 consisting of continuous functions on $\mathbf{R}$ which vanish at infinity, also under the L^∞ norm. The subspace C_c consisting of continuous functions with compact support is dense in C_0 and in L^p_w for $1 \le p < \infty$.

11.2.3 Moderate weights

We collect some properties of those weights w for which L^p_w is translation-invariant. The definition we use follows [39, Def. 11.1.1].

Definition 11.2.1.

a. A *weight* is a positive and locally integrable function $w\colon \mathbf{R} \to (0,\infty)$.

b. A weight v is *submultiplicative* if

$$\forall\, x, y \in \mathbf{R}, \quad v(x+y) \;\le\; v(x)\,v(y). \tag{11.3}$$

c. A weight w is *moderate with respect to a submultiplicative function* v, or simply v-*moderate* for short, if there exists a constant $C_w > 0$ such that

$$\forall\, x, y \in \mathbf{R}, \quad w(x+y) \;\le\; C_w\,w(x)\,v(y). \tag{11.4}$$

If the associated submultiplicative function v is implicit, we may just write that "w is moderate."

d. Two weights w_1 and w_2 are *equivalent*, denoted $w_1 \asymp w_2$, if there exist constants A, $B > 0$ such that $Aw_1(x) \le w_2(x) \le Bw_1(x)$ for all $x \in \mathbf{R}$.

The prototypical examples of submultiplicative weights are the polynomially-growing weights $w_s(x) = (1+|x|)^s$ with $s \ge 0$. If $0 \le t \le s$, then both w_t and w_t^{-1} are w_s-moderate [39, Lem. 11.1.1].

If desired, we could relax the definition of submultiplicative weight and require only that there exists a constant C_v such that $v(x+y) \le C_v\, v(x)\, v(y)$ for all x, y. This introduces only technical complications, and we leave the details of this generalization to the reader. Also, note that we are implicitly assuming in Definition 11.2.1 that the weight functions v and w are defined at all points and not just almost everywhere, and that the inequalities (11.3) and (11.4) hold at all points. This allows us to perform calculations involving the value of the weight at single points and enables us to show in Lemma 11.2.2(a) below that all submultiplicative and moderate weights are locally bounded. An alternative might be to define submultiplicative or moderate weights to be locally bounded functions which are defined almost everywhere, and then only require (11.3) and (11.4) to hold almost everywhere.

Note that equivalent weights define the same weighted L^p spaces, i.e., if $w_1 \asymp w_2$ then $L^p_{w_1} = L^p_{w_2}$. In the results we give, one weight may be replaced with an equivalent weight with no change in the result. We show in Lemma 11.2.3 below that given a v-moderate weight w, we can always find a continuous weight $\tilde{w}$ that is equivalent to w. Moreover, that equivalent weight will necessarily be v-moderate as well, since if $A\, w(x) \le \tilde{w}(x) \le B\, w(x)$ for all x then

$$\tilde{w}(x+y) \;\le\; B\, w(x+y) \;\le\; BC_w\, w(x)\, v(y) \;\le\; \frac{BC_w}{A}\, \tilde{w}(x)\, v(y).$$

For this reason, many authors simply incorporate into the definition of submultiplicative or moderate weights the requirement that the weights be continuous (and hence are certainly locally bounded).

We have the following basic but important properties of moderate weights. Part a of the next lemma has been long known, at least for the case of submultiplicative weights (from which the extension to moderate weights is trivial). For example, it is observed in [53, Remark 1.6.6] that the local boundedness of submultiplicative weights is a consequence of [44, Thm. 7.13.1]. For completeness we give a direct proof, adapted from [57, Lem. 1.1.4] (which itself references [50]).

Lemma 11.2.2.

a. *A moderate weight w is locally bounded, i.e., $w \cdot \chi_K$ is bounded for every compact set K.*

b. *If $r \ge 0$ and w is v-moderate, then w^r is v^r-moderate.*

c. *If $r \ge 0$ and w is v-moderate, then w^{-r} is $\tilde{v}^r$-moderate, where $\tilde{v}(x) = v(-x)$.*

Proof. a. Suppose that w is a v-moderate weight, and suppose we knew that v was locally bounded. Then given a compact set $K \subset \mathbf{R}$, we would have for $x \in K$ that

$$w(x) \;=\; w(0+x) \;\le\; C_w\, w(0)\, v(x) \;\le\; C_w\, w(0)\, \|v \cdot \chi_K\|_\infty,$$

and therefore w would be bounded on K. Thus, it suffices to show that submultiplicative functions are locally bounded.

So, suppose that v is a submultiplicative function. We claim it is enough to show that v is bounded on the neighborhood $(-1,1)$ of the origin. For, once this is done, given any compact set $K \subset \mathbf{R}$ we can write $K \subset \cup_{k=1}^{N}(x_k - 1, x_k + 1)$ for some finitely many points $x_1, \dots, x_N \in \mathbf{R}$. Then, given $x \in K$ we have have $x \in (x_k - 1, x_k + 1)$ for some k, so

$$v(x) \;=\; v(x - x_k)\, v(x_k) \;\leq\; \|v \cdot \chi_{(-1,1)}\|_\infty \max_{k=1,\dots,N} v(x_k),$$

and hence v would be bounded on K.

So, suppose on the contrary that v was unbounded on $(-1,1)$. For each integer $N > 0$, define

$$A_N \;=\; \{x \in (-2,2) : v(x) \geq N\} \qquad \text{and} \qquad B_N \;=\; \{x \in (-2,2) : v(x) < N\}.$$

Note that $|A_N| + |B_N| = 4$. Since v takes only finite values, we have $\cap A_N = \emptyset$.

Now we estimate the measure of A_N. Since v is unbounded on $(-1,1)$, for each $N > 0$ we can find a point $x_N \in (-1,1)$ such that $v(x_N) \geq N^2$. We claim that

$$(x_N - B_N) \cap (-2,2) \;\subset\; A_N, \tag{11.5}$$

where $x_N - B_N = \{x_N - y : y \in B_N\}$. To prove (11.5), suppose $z = x_n - y \in (-2,2)$ where $y \in B_N$. Then

$$N^2 \;\leq\; v(x_N) \;=\; v(z+y) \;\leq\; v(z)\, v(y).$$

But $v(y) < N$ since $y \in B_N$, so we must have $v(z) \geq N$. Since $z \in (-2,2)$, we conclude that $z \in A_N$, and therefore (11.5) is valid.

Now,

$$x_N - B_N \subset x_N - (2,2) \;=\; (x_N - 2, x_N + 2).$$

Intersecting this set with $(-2,2)$ removes either the interval $[2, x_N + 2)$ if $x_N \geq 0$, or the interval $(x_N - 2, -2]$ if $x_N \leq 0$. In either case, an interval of length at most 2 is removed. Hence,

$$|A_N| \;\geq\; \bigl|(x_N - B_N) \cap (-2,2)\bigr| \;\geq\; \bigl|x_N - B_N\bigr| - 2 \;=\; |B_N| - 2 \;=\; 2 - |A_N|.$$

Thus $|A_N| \geq 1$ for all N, contradicting the fact that $\cap A_N = \emptyset$.

b. If $r \geq 0$ then we have

$$w(x+y)^r \;\leq\; (C_w\, w(x)\, v(y))^r \;=\; C_w^r\, w(x)^r\, v(y)^r,$$

so w^r is v^r-moderate.

c. For x, $y \in \mathbf{R}$ we have

$$w(x) \;=\; w(x+y-y) \;\leq\; C_w\, w(x+y)\, v(-y),$$

so

$$\frac{1}{w(x+y)} \;\leq\; C_w \frac{1}{w(x)}\, v(-y) \;=\; C_w \frac{1}{w(x)}\, \tilde{v}(y).$$

Thus w^{-1} is $\tilde{v}$-moderate, and then as in the proof of part b we can see that w^{-r} is $\tilde{v}^r$-moderate for each $r \geq 0$. □

In particular, by Lemma 11.2.2 parts (b) and (c), the class of moderate weights is closed under reciprocals, and consequently the class of spaces L^p_w using moderate weights w is closed under duality (with the usual exception for $p = \infty$). This would not be the case if we restricted to submultiplicative weights.

In the next lemma we will show that every moderate weight w is equivalent to some continuous moderate weight $\tilde{w}$. This new weight is easily obtained from w by convolving w with an appropriate continuous and compactly supported function.

Lemma 11.2.3. *Let w be a v-moderate weight. Then there exists a continuous, v-moderate weight $\tilde{w}$ such that $w \asymp \tilde{w}$.*

Proof. Let $k \in C_c$ be any function such that $k \geq 0$, $\mathrm{supp}(k) \subset [-1,1]$, and $\int k(x)\,dx = 1$. For example, we could let k be the hat or tent function on $[-1,1]$. Then since w is locally bounded, we can define

$$\tilde{w}(x) \;=\; (w * k)(x) \;=\; \int_{\mathbf{R}} w(t)\,k(x-t)\,dt.$$

Clearly $\tilde{w} > 0$, and by using the facts that w is locally bounded and k is uniformly continuous, we can readily see that $\tilde{w}$ is continuous.

So, it remains only to show that $\tilde{w} \asymp w$. Given $x \in \mathbf{R}$, we compute that

$$\begin{aligned}
\tilde{w}(x) \;=\; \int_{\mathbf{R}} w(x+t)\,k(-t)\,dt \;&\leq\; C_w \int_{\mathbf{R}} w(x)\,v(t)\,k(-t)\,dt \\
&\leq\; C_w\, w(x)\, \|v \cdot \chi_{[-1,1]}\|_\infty \int_{-1}^{1} k(-t)\,dt \\
&=\; Bw(x),
\end{aligned}$$

where $B = C_w\, \|v \cdot \chi_{[-1,1]}\|_\infty$. Similarly,

$$\begin{aligned}
w(x) \;=\; w(x) \int_{-1}^{1} k(t)\,dt \;&=\; \int_{-1}^{1} w(x-t+t)\,k(t)\,dt \\
&\leq\; C_w \int_{-1}^{1} w(x-t)\,v(t)\,k(t)\,dt \\
&\leq\; C_w\, \|v \cdot \chi_{[-1,1]}\|_\infty \int_{-1}^{1} w(x-t)\,k(t)\,dt \\
&=\; B\,\tilde{w}(x).
\end{aligned}$$

Thus $\frac{1}{B}\,\tilde{w}(x) \leq w(x) \leq B\,\tilde{w}(x)$ for every x, so $\tilde{w} \asymp w$. ☐

In light of the preceding lemma, we will assume throughout the rest of this article that all submultiplicative and moderate weights that we encounter are continuous.

Perhaps the main reason for dealing with moderate weights is revealed in the following proposition. Specifically, L^p_w is translation-invariant exactly when w is moderate [20], cf. also [18], [37]. The proof given below is adapted from [57, Thm. 1.1.6], [41, Thm. 2.1.6], and [39, Lem. 11.1.2].

Proposition 11.2.4. *Given a weight $w \in L^1_{\mathrm{loc}}$ and given $1 \le p \le \infty$, the following statements are equivalent.*

a. *w is moderate with respect to some submultiplicative weight v.*

b. *L^p_w is translation-invariant, i.e., the translation operator T_a is a continuous mapping of L^p_w into itself for each $a \in \mathbf{R}$.*

c. *Given any compact set $K \subset \mathbf{R}$ with $|K| > 0$, there exists a constant $C_1(K) > 0$ such that*

$$\sup_{t \in K+x} w(t) \ \le\ C_1(K) \inf_{t \in K+x} w(t) \quad \text{for every } x \in \mathbf{R}. \tag{11.6}$$

d. *Given any compact set $K \subset \mathbf{R}$ with $|K| > 0$, there exist constants $C_2(K)$, $C_3(K) > 0$ such that*

$$C_2(K)\, w(y) \ \le\ \int_{K+x} w(t)\, dt \ \le\ C_3(K)\, w(y) \quad \text{for all } y \in K+x. \tag{11.7}$$

In case these hold, then for each $a \in \mathbf{R}$ and $1 \le p \le \infty$ we have

$$\forall\, f \in L^p_w, \quad \|T_a f\|_{L^p_w} \ \le\ C_w\, v(a)\, \|f\|_{L^p_w}. \tag{11.8}$$

In particular, the operator norm of $T_a \colon L^p_w \to L^p_w$ satisfies

$$\|T_a\|_{L^p_w \to L^p_w} \ =\ \sup\bigl\{\|T_a f\|_{L^p_w} : \|f\|_{L^p_w} = 1\bigr\} \ \le\ C_w\, v(a).$$

Proof. a $\Rightarrow$ b. Assume that w is v-moderate. If $1 \le p < \infty$ and $a \in \mathbf{R}$, then

$$\begin{aligned}
\|T_a f\|^p_{L^p_w} &= \int_{\mathbf{R}} |f(x-a)|^p\, w(x)^p\, dx \\
&= \int_{\mathbf{R}} |f(x)|^p\, w(a+x)^p\, dx \\
&\le \int_{\mathbf{R}} |f(x)|^p\, C_w^p\, v(a)^p\, w(x)^p\, dx \ =\ C_w^p\, v(a)^p\, \|f\|^p_{L^p_w}.
\end{aligned}$$

Thus T_a is a continuous map of L^p_w into itself, and we have also shown that equation (11.8) holds. The proof for the case $p = \infty$ is similar.

b $\Rightarrow$ a. Assume that L^p_w is translation-invariant, and fix $1 \le p < \infty$ (the case $p = \infty$ is similar). For each $a \in \mathbf{R}$ define $v(a) = \|T_a\|_{L^p_w \to L^p_w}$, the operator norm of T_a on L^p_w. Then for every $f \in L^p_w$ we have

$$\begin{aligned}
\int_{\mathbf{R}} |f(x)|^p\, w(x+a)^p\, dx \ &=\ \int_{\mathbf{R}} |f(x-a)|^p\, w(x)^p\, dx \\
&=\ \|T_a f\|^p_{L^p_w}
\end{aligned}$$

$$
\begin{aligned}
&\leq \|T_a\|^p_{L^p_w \to L^p_w} \, \|f\|^p_{L^p_w} \\
&= v(a)^p \int_{\mathbf{R}} |f(x)|^p \, w(x)^p \, dx.
\end{aligned}
$$

Consequently $\int |f(x)|^p \big(v(a)^p \, w(x)^p - w(x+a)^p\big) \, dx \geq 0$ for every $f \in L^p_w$. Since we can assume that v, w are continuous, this implies that $v(a)^p \, w(x)^p - w(x+a)^p \geq 0$ for every $x \in \mathbf{R}$. Thus w is v-moderate.

a $\Rightarrow$ c. Suppose that w is v-moderate, and fix any compact set $K \subset \mathbf{R}$ with positive measure and any $x \in \mathbf{R}$. Let $t = r + x$ and $u = s + x$ be any points in $K + x$. Then

$$
w(t) \;=\; w(x+r) \;\leq\; C_w \, w(x) \, v(r) \;\leq\; C_w \, \|v \cdot \chi_K\|_\infty \, w(x)
$$

and

$$
w(x) \;=\; w(u-s) \;\leq\; C_w \, w(u) \, v(-s) \;\leq\; C_w \, \|v \cdot \chi_{-K}\|_\infty \, w(u).
$$

Thus if we set $C_1(K) = C_w^2 \, \|v \cdot \chi_K\|_\infty \, \|v \cdot \chi_{-K}\|_\infty$ then we have $w(t) \leq C_1(K) \, w(u)$ for every t, $u \in K + x$, from which (11.6) follows.

c $\Rightarrow$ d. Suppose that statement c holds, fix any compact $K \subset \mathbf{R}$ with positive measure, and fix any $x \in \mathbf{R}$. Then given any $y \in K + x$ we have

$$
w(y) \;\leq\; \sup_{t \in K+x} w(t) \;\leq\; C_1(K) \inf_{t \in K+x} w(t) \;\leq\; \frac{C_1(K)}{|K|} \int_{K+x} w(t) \, dt
$$

and

$$
\int_{K+x} w(t) \, dt \;\leq\; |K| \sup_{t \in K+x} w(t) \;\leq\; |K| \, C_1(K) \inf_{t \in K+x} w(t) \;\leq\; |K| \, C_1(K) \, w(y).
$$

Hence (11.7) holds using $C_2(K) = |K|/C_1(K)$ and $C_3(K) = |K| \, C_1(K)$.

d $\Rightarrow$ c. This implication follows easily.

c $\Rightarrow$ b. Assume that statement c holds, and fix any $a \in \mathbf{R}$. Let K be any compact set with positive measure which contains both a and 0. Then for any $x \in \mathbf{R}$ we have

$$
w(x+a) \;\leq\; \sup_{t \in K+x} w(t) \;\leq\; C_1(K) \inf_{t \in K+x} w(t) \;\leq\; C_1(K) \, w(x).
$$

Hence for any $1 \leq p < \infty$ and any $f \in L^p_w$ we have

$$
\begin{aligned}
\|T_a f\|^p_{L^p_w} &= \int_{\mathbf{R}} |f(x-a)|^p \, w(x)^p \, dx \\
&= \int_{\mathbf{R}} |f(x)|^p \, w(x+a)^p \, dx \\
&\leq C_1(K)^p \int_{\mathbf{R}} |f(x)|^p \, w(x)^p \, dx \\
&= C_1(K)^p \, \|f\|^p_{L^p_w}.
\end{aligned}
$$

Thus T_a is continuous on L^p_w. The case $p = \infty$ is similar. $\square$

11.3 Definition of Weighted Amalgams

We will now define the amalgam $W(L^p, L^q_w)$ where w is a moderate weight, and derive some basic properties. We can always assume a moderate weight is continuous, so when restricted to any compact set, the weight is bounded above and below. As a consequence, there is no gain in generality by allowing the local component to also be a weighted L^p space. Or, equivalently, it can be shown that $W(L^p_v, L^q_w) = W(L^p, L^q_{vw})$, so it suffices to only consider the case where the global component is a weighted space.

We define the amalgam spaces $W(L^p, L^q_w)$ in this section by a "continuous"-type norm. An equivalent "discrete" norm in the style of equation (11.1) will be given in Section 11.6. The continuous norm reveals much more clearly than equation (11.1) the sense in which L^p is amalgamated with L^q_w, and emphasizes that general Wiener amalgams $W(B, C)$ are not simply "piecings together of disjoint local pieces." In particular, overlaps are essential when moving to more general amalgams determined by continuity, smoothness, or other criteria—the simple cutoff function χ_Q used in the following definition is clearly inadequate in that setting. To define general Wiener amalgams $W(B, C)$, an appropriate smooth cutoff function needs to be employed in place of the sharp cutoff χ_Q that we use.

Definition 11.3.1. Let $1 \le p, q \le \infty$ and a weight w be given, and fix a compact $Q \subset \mathbf{R}$ with nonempty interior. Then the amalgam space $W(L^p, L^q_w)$ consists of all functions $f \colon \mathbf{R} \to \mathbf{C}$ such that $f \cdot \chi_K \in L^p$ for each compact $K \subset \mathbf{R}$, and for which the *control function*

$$F_f(x) \;=\; F_f^Q(x) \;=\; \|f \cdot \chi_{Q+x}\|_{L^p} \;=\; \|f \cdot T_x\chi_Q\|_{L^p}, \qquad x \in \mathbf{R},$$

lies in L^q_w. The norm on $W(L^p, L^q_w)$ is

$$\begin{aligned} \|f\|_{W(L^p,L^q_w)} \;=\; \|F_f\|_{L^q_w} \;&=\; \big\| \|f \cdot \chi_{Q+x}\|_{L^p} \big\|_{L^q_w} \\ &=\; \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}} |f(t)|^p\, \chi_Q(t-x)\, dt \right)^{q/p} w(x)^q\, dx \right)^{1/q} \\ &=\; \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}} |f(t)|^p\, \chi_Q(t-x)\, w(x)^p\, dt\, dx \right)^{q/p} \right)^{1/q}, \end{aligned}$$

with the appropriate adjustments if p or q is infinity. When p, q, and w are understood, we will just write $W = W(L^p, L^q_w)$ and $\|f\|_W = \|f\|_{W(L^p,L^q_w)}$.

We see in this definition that we first view f "locally" through translations $T_x\chi_Q$ of the cutoff function χ_Q, and measure those local pieces in the L^p norm, then measure the global behavior of those local pieces according to the L^q_w norm. Note that the "window" through which we view f locally need not be a unit interval—*any* compact Q with nonempty interior is allowed.

The appropriate definition for a general Wiener amalgam $W(B, C)$ is similar but must employ translations of a better cutoff if we hope to measure any local property other than just integrability. Translations $T_x\phi$ of a smooth window function ϕ should be used in place of the translations $T_x\chi_Q$ of the sharp cutoff function χ_Q. The theorems and proofs that we present require some modifications to handle such smooth windows; the modifications are usually small and stem mostly from the fact that we can no longer use the fact that $\chi_Q^2 = \chi_Q$. Instead, the argument is based on control over the overlaps of the supports of the translations of ϕ.

Proposition 11.3.2. *Let w be a v-moderate weight. Then $W(L^p, L^q_w)$ is a Banach space, and the definition of $W(L^p, L^q_w)$ is independent of the choice of Q, i.e., different choices of Q define the same space with equivalent norms.*

Proof. a. To show that $W = W(L^p, L^q_w)$ is a Banach space, we must show it is complete. For this, it suffices to show that if $\{f_n\}_{n=1}^\infty$ is a sequence of functions in W with $\sum \|f_n\|_W < \infty$, then $\sum f_n$ converges to an element of W.

Now, $\sum \|f_n\|_W = \sum \|F_{f_n}\|_{L^q_w}$ and L^q_w is complete, so $\sum F_{f_n}$ must converge in L^q_w. In particular, this function is finite almost everywhere, i.e.,

$$\sum_{n=1}^\infty F_{f_n}(x) \;=\; \sum_{n=1}^\infty \|f_n \cdot \chi_{Q+x}\|_{L^p} \;<\; \infty, \quad \text{a.e. } x \in \mathbf{R}.$$

Since L^p is also complete, $\sum f_n \cdot \chi_{Q+x}$ must converge to an element $g_x \in L^p$. Clearly $g_x = g_y$ a.e. on $(Q+x) \cap (Q+y)$, so we can define a function g a.e. by $g(t) = g_x(t)$ for $t \in Q+x$, i.e., $g \cdot \chi_{Q+x} = g_x$. Then

$$\begin{aligned}
\|g\|_W \;&=\; \big\| \, \|g \cdot \chi_{Q+x}\|_{L^p} \big\|_{L^q_w} \\
&=\; \Big\| \, \Big\| \sum_{n=1}^\infty f_n \cdot \chi_{Q+x} \Big\|_{L^p} \Big\|_{L^q_w} \\
&\le\; \sum_{n=1}^\infty \big\| \, \|f_n \cdot \chi_{Q+x}\|_{L^p} \big\|_{L^q_w} \;=\; \sum_{n=1}^\infty \|f_n\|_W \;<\; \infty,
\end{aligned}$$

so $g \in W$. A similar computation then shows that $\sum f_n = g$ a.e., which completes the proof that W is complete.

b. Now we show that each different choice of Q yields an equivalent norm for W. Let Q_1, Q_2 be two compact subsets of $\mathbf{R}$ with nonempty interiors. Then we can find $x_1, \dots, x_N \in \mathbf{R}$ such that $Q_2 \subset \bigcup_{k=1}^N (Q_1 + x_k)$. Hence, using the fact that L^p is a solid space,

$$\begin{aligned}
F_f^{Q_2}(x) \;=\; \big\|f \cdot \chi_{Q_2+x}\big\|_{L^p} \;&\le\; \Big\|f \cdot \sum_{k=1}^N \chi_{Q_1+x_k+x}\Big\|_{L^p} \\
&\le\; \sum_{k=1}^N \|f \cdot \chi_{Q_1+x_k+x}\|_{L^p} \\
&\le\; \sum_{k=1}^N F_f^{Q_1}(x_k + x) \\
&=\; \sum_{k=1}^N (T_{-x_k} F_f^{Q_1})(x).
\end{aligned}$$

Since both $F_f^{Q_1}$ and $\sum T_{-x_k} F_f^{Q_1}$ are elements of L^q_w and L^q_w is solid, we can therefore compute, by making use of the estimate (11.8), that

$$\|F_f^{Q_2}\|_{L^q_w} \;\le\; \Big\|\sum_{k=1}^{N} T_{-x_k} F_f^{Q_1}\Big\|_{L^q_w} \;\le\; C_w \sum_{k=1}^{N} v(-x_k)\, \|F_f^{Q_1}\|_{L^q_w} \;\le\; C_w M N\, \|F_f^{Q_1}\|_{L^q_w},$$

where $M = \max\{v(-x_1), \dots, v(-x_N)\}$. A symmetric argument gives the reverse inequality. □

As an application, we obtain closure under pointwise products.

Theorem 11.3.3. *Let the indices p_i, q_i and moderate weights w_i be such that there exist constants C_1, $C_2 > 0$ so that*

$$\forall\, h \in L^{p_1}, \quad \forall\, k \in L^{p_2}, \quad \|hk\|_{L^{p_3}} \;\le\; C_1\, \|h\|_{L^{p_1}}\, \|k\|_{L^{p_2}}$$

and

$$\forall\, h \in L^{q_1}_{w_1}, \quad \forall\, k \in L^{q_2}_{w_2}, \quad \|hk\|_{L^{q_3}_{w_3}} \;\le\; C_2\, \|h\|_{L^{q_1}_{w_1}}\, \|k\|_{L^{q_2}_{w_2}}.$$

Then there exists a constant $C > 0$ such that for all $f \in W(L^{p_1}, L^{q_1}_{w_1})$ and $g \in W(L^{p_2}, L^{q_2}_{w_2})$ we have

$$\|fg\|_{W(L^{p_3}, L^{q_3}_{w_3})} \;\le\; C\, \|f\|_{W(L^{p_1}, L^{q_1}_{w_1})}\, \|g\|_{W(L^{p_2}, L^{q_2}_{w_2})}.$$

In other words, if $L^{p_1} \cdot L^{p_2} \subset L^{p_3}$ and $L^{q_1}_{w_1} \cdot L^{q_2}_{w_2} \subset L^{q_3}_{w_3}$, then $W(L^{p_1}, L^{q_1}_{w_1}) \cdot W(L^{p_2}, L^{q_2}_{w_2}) \subset W(L^{p_3}, L^{q_3}_{w_3})$.

Proof. If $f \in W(L^{p_1}, L^{q_1}_{w_1})$ and $g \in W(L^{p_2}, L^{q_2}_{w_2})$, then

$$\begin{aligned}
\|fg\|_{W(L^{p_3}, L^{q_3}_{w_3})} &= \big\|\, \|fg \cdot \chi_{Q+x}\|_{L^{p_3}} \,\big\|_{L^{q_3}_{w_3}} \\
&= \big\|\, \|(f \cdot \chi_{Q+x}) \cdot (g \cdot \chi_{Q+x})\|_{L^{p_3}} \,\big\|_{L^{q_3}_{w_3}} \\
&\le C_1 \big\|\, \|f \cdot \chi_{Q+x}\|_{L^{p_1}}\, \|g \cdot \chi_{Q+x}\|_{L^{p_2}} \,\big\|_{L^{q_3}_{w_3}} \\
&= C_1\, \|F_f \cdot F_g\|_{L^{q_3}_{w_3}} \\
&\le C_1 C_2\, \|F_f\|_{L^{q_1}_{w_1}}\, \|F_g\|_{L^{q_2}_{w_2}} \\
&= C_1 C_2\, \|f\|_{W(L^{p_1}, L^{q_1}_{w_1})}\, \|g\|_{W(L^{p_2}, L^{q_2}_{w_2})}.
\end{aligned}$$

□

Note that the above proof we have made use of the simplifying fact that multiplication by χ_{Q+x} is idempotent, i.e., $\chi^2_{Q+x} = \chi_{Q+x}$.

11.4 Translation Invariance

We show now that the translation-invariance properties of the local and global components carry over to the weighted amalgam.

Proposition 11.4.1. *If w is a v-moderate weight, then $W = W(L^p, L^q_w)$ is translation-invariant, with $\|T_a f\|_W \le C_w\, v(a)\, \|f\|_W$ for $f \in W$. If $w \equiv 1$, then translation is an isometry on W.*

Proof. Assume $f \in W$. Since L^p is translation-isometric, we have $T_a f \in L^p$ and

$$F_{T_a f}(x) \;=\; \|T_a f \cdot \chi_{Q+x}\|_{L^p} \;=\; \|f \cdot \chi_{Q+x-a}\|_{L^p} \;=\; F_f(x-a) \;=\; (T_a F_f)(x). \tag{11.9}$$

Therefore, since L^q_w is translation-invariant, we have $F_{T_a f} = T_a F_f \in L^q_w$. Finally, by using (11.8), we find that

$$\|T_a f\|_W \;=\; \|F_{T_a f}\|_{L^q_w} \;=\; \|T_a F_f\|_{L^q_w} \;\le\; C_w\, v(a)\, \|F_f\|_{L^q_w} \;=\; C_w\, v(a)\, \|f\|_W.$$

If $w \equiv 1$ then we use the fact that translation is an isometry on L^q_w. □

When p and q are finite, translation is strongly continuous in L^p and L^q_w, and this fact carries over to the amalgam as well.

Proposition 11.4.2. *Let w be v-moderate, and assume $1 \le p, q < \infty$. Then translation is strongly continuous on $W = W(L^p, L^q_w)$, i.e.,*

$$\forall\, b \in \mathbf{R}, \quad \forall\, f \in W, \quad \lim_{a \to b} \|T_a f - T_b f\|_W \;=\; 0.$$

Proof. It suffices to consider the case $b = 0$. Let $f \in W = W(L^p, L^q_w)$, and fix $\varepsilon > 0$. Since the submultiplicative weight v is locally bounded, we can find an $M \ge 1$ such that $C_w\, v(a) \le M$ for all $|a| < 1$.

Since C_c is dense in L^q_w, there exists a function $k \in C_c$ such that

$$\|F_f \cdot (1-k)\|_{L^q_w} \;<\; \frac{\varepsilon}{M}. \tag{11.10}$$

Since $k \in C_c$, there is a $\delta < 1$ such that

$$|a| < \delta \quad \Longrightarrow \quad \|T_{-a}k - k\|_{L^\infty} \;<\; \frac{\varepsilon}{M\|f\|_W} \;=\; \frac{\varepsilon}{M\|F_f\|_{L^q_w}}.$$

Now, since $0 \le F_{T_a f - f} \le F_{T_a f} + F_f = T_a F_f + F_f$, we have

$$\begin{aligned} \|T_a f - f\|_W \;&=\; \|F_{T_a f - f}\|_{L^q_w} \\ &\le\; \|F_{T_a f - f} \cdot (1-k)\|_{L^q_w} \;+\; \|F_{T_a f - f} \cdot k\|_{L^q_w} \\ &\le\; \|T_a F_f \cdot (1-k)\|_{L^q_w} \;+\; \|F_f \cdot (1-k)\|_{L^q_w} \;+\; \|F_{T_a f - f} \cdot k\|_{L^q_w}. \end{aligned} \tag{11.11}$$

We will estimate the size of each of these last three terms separately.

To estimate the first term in (11.11), note that for $|a| < \delta$ we have, by using the triangle inequality and the estimate (11.8), that

$$\begin{aligned} \|T_a F_f \cdot (1-k)\|_{L^q_w} \;&=\; \Big\| T_a\big(F_f \cdot (1-k)\big) \;+\; T_a\big(F_f \cdot (k - T_{-a}k)\big) \Big\|_{L^q_w} \\ &\le\; C_w\, v(a)\, \|F_f \cdot (1-k)\|_{L^q_w} \;+\; C_w\, v(a)\, \|F_f\|_{L^q_w}\, \|k - T_{-a}k\|_{L^\infty} \\ &\le\; M\, \frac{\varepsilon}{M} \;+\; M\, \|F_f\|_{L^q_w}\, \frac{\varepsilon}{M\, \|F_f\|_{L^q_w}} \;=\; 2\varepsilon. \end{aligned} \tag{11.12}$$

Note that by equation (11.10), we already know that the second term in (11.11) is bounded by $\varepsilon/M \le \varepsilon$.

We proceed to bound the third term in (11.11). Note that $\operatorname{supp}(k)$ and Q are both compact sets, so $\operatorname{supp}(k) \subset [-R, R]$ and $Q \subset [-S, S]$ for some R, $S > 0$. Let

$$I \;=\; [-R-S-1, R+S+1].$$

Then, since f is locally in L^p and I is a compact interval, we can find a function $g \in C_c$ such that

$$\|(f-g)\,\chi_I\|_{L^p} \;\le\; \frac{\varepsilon}{\|k\|_{L^q_w}}.$$

Further, there exists a $\delta' < 1$ such that

$$|a| < \delta' \qquad\Longrightarrow\qquad \|T_a g - g\|_{L^p} \;<\; \frac{\varepsilon}{\|k\|_{L^q_w}}.$$

Now, if $x \in \operatorname{supp}(k)$ and $|a| < \delta'$ then $Q + x \subset I$ and $Q + x - a \subset I$, and therefore

$$\begin{aligned}
|F_{T_af-f}(x)| \;&=\; \|T_af \cdot \chi_{Q+x} - f \cdot \chi_{Q+x}\|_{L^p} \\
&\le\; \|(T_af - T_ag) \cdot \chi_{Q+x}\|_{L^p} \;+\; \|(T_ag - g) \cdot \chi_{Q+x}\|_{L^p} \;+\; \|(g-f) \cdot \chi_{Q+x}\|_{L^p} \\
&\le\; \|(f-g) \cdot \chi_I\|_{L^p} \;+\; \|T_ag - g\|_{L^p} \;+\; \|(f-g)\,\chi_I\|_{L^p} \\
&\le\; \frac{3\varepsilon}{\|k\|_{L^q_w}}.
\end{aligned}$$

Hence, for $|a| < \delta'$ we have

$$\|F_{T_af-f} \cdot k\|_{L^q_w} \;\le\; \Biggl(\sup_{x \in \operatorname{supp}(k)} |F_{T_af-f}(x)|\Biggr)\, \|k\|_{L^q_w} \;\le\; 3\varepsilon. \tag{11.13}$$

Finally, by combining equations (11.10), (11.11), (11.12), and (11.13), we find that $\|T_af - f\|_W \le 6\varepsilon$ for $|a| < \min\{\delta, \delta'\}$. □

11.5 Some Inclusion Relations

We show now that inclusion relations on the local components imply inclusion relations for amalgams. Later, after deriving equivalent discrete norms for amalgams, we show that there are corresponding inclusions based on the global components. The local inclusions that we are alluding to are simply the inclusions for L^p spaces on compact sets, namely that if K is compact, then $p_1 \le p_2$ implies $L^{p_1}(K) \supset L^{p_2}(K)$.

Proposition 11.5.1. *Let w be a moderate weight. If $1 \le p_1 \le p_2 \le \infty$ then $W(L^{p_1}, L^q_w) \supset W(L^{p_2}, L^q_w)$.*

Proof. Since Q is compact, we have

$$\|f \cdot \chi_{Q+x}\|_{L^{p_1}} \;\leq\; |Q+x|^{\frac{1}{p_1}-\frac{1}{p_2}}\, \|f \cdot \chi_{Q+x}\|_{L^{p_2}} \;=\; |Q|^{\frac{1}{p_1}-\frac{1}{p_2}}\, \|f \cdot \chi_{Q+x}\|_{L^{p_2}}.$$

The result then follows from the fact that L^q_w is a solid space. □

In the next result, we show that if $p = q$ then $W(L^p, L^p_w) = L^p_w$, so in particular, $W(L^p, L^p) = L^p$. However, this result is misleading in the context of general Wiener amalgams—in general it is *not* always true that $W(B, B) = B$. A specific example is that

$$W(L^1 \cap L^\infty, L^1 \cap L^\infty) \;=\; W(L^\infty, L^1) \;\subsetneq\; L^1 \cap L^\infty. \tag{11.14}$$

In fact this is natural— the "stronger" of the local components on the left side of equation (11.14) is L^∞, while for the global component it is L^1. However, for the particular case of the weighted amalgams $W(L^p, L^q_w)$, we have the following result.

Proposition 11.5.2. *If w is a moderate weight, then $W(L^p, L^p_w) = L^p_w$.*

Proof. If $1 \leq p < \infty$ then

$$\|f\|^p_{W(L^p,L^p_w)} \;=\; \int_{\mathbf{R}} \int_{\mathbf{R}} |f(t)|^p\, \chi_{Q+x}(t)\, w(x)^p\, dt\, dx \;=\; \int_{\mathbf{R}} |f(t)|^p \int_{t-Q} w(x)^p\, dx\, dt.$$

Since w^p is moderate and $-Q$ is compact, by Proposition 11.2.4 there exist constants A, $B > 0$ such that

$$A\, w(t)^p \;\leq\; \int_{t-Q} w(x)^p\, dx \;\leq\; B\, w(t)^p. \tag{11.15}$$

Thus,

$$A\, \|f\|^p_{L^p_w} \;\leq\; A \int_{\mathbf{R}} |f(t)|^p\, w(t)^p\, dt \;\leq\; \|f\|^p_{W(L^p,L^p_w)} \;\leq\; B \int_{\mathbf{R}} |f(t)|^p\, w(t)^p\, dt \;=\; B\, \|f\|^p_{L^p_w}.$$

The case $p = \infty$ is similar. □

Again for the special case of the weighted amalgams, we can derive some inclusions between the amalgam and its local or global components.

Proposition 11.5.3. *Let w be a moderate weight.*

a. If $1 \leq p \leq q \leq \infty$ then $W(L^p, L^q_w) \supset L^p \cup L^q_w$.

b. If $1 \leq q \leq p \leq \infty$ then $W(L^p, L^q_w) \subset L^p \cup L^q_w$.

Proof. We will only prove part a for the case $1 \le p \le q < \infty$, as the remaining cases are similar. By Proposition 11.5.1, we have

$$\|f\|_{W(L^p,L^q_w)} \;\le\; C_1\, \|f\|_{W(L^q,L^q_w)} \;\le\; C_2\, \|f\|_{L^q_w},$$

so $W(L^p, L^q_w) \supset L^q_w$. The inclusion $W(L^p, L^q_w) \supset L^p$ can be easily proved using the discrete norms we develop later. Alternatively, we can apply the integral version of Minkowski's inequality and an inequality analogous to equation (11.15) with q in place of p to compute

$$\begin{aligned}
\|f\|^p_{W(L^p,L^q_w)} &= \biggl(\int_{\mathbf{R}} \biggl(\int_{\mathbf{R}} |f(t)|^p\, \chi_{Q+x}(t)\, w(x)^p\, dt\, dx\biggr)^{q/p}\biggr)^{p/q} \\
&\le \int_{\mathbf{R}} \biggl(\int_{\mathbf{R}} |f(t)|^q\, \chi_{Q+x}(t)\, w(x)^q\, dx\biggr)^{p/q} dt \\
&= \int_{\mathbf{R}} |f(t)|^p \biggl(\int_{\mathbf{R}} \chi_{t-Q}(x)\, w(x)^q\, dx\biggr)^{p/q} dt \\
&\le \int_{\mathbf{R}} |f(t)|^p \bigl(B\, w(t)^q\bigr)^{p/q}\, dt \\
&= B^{p/q} \int_{\mathbf{R}} |f(t)|^p\, w(t)^p\, dt \;=\; B^{p/q}\, \|f\|_{L^p}. \qquad \square
\end{aligned}$$

The following Hölder inequality can be extended to a duality theorem. However, we delay consideration of duality until after we develop equivalent discrete norms.

Proposition 11.5.4. *Let w be a moderate weight. If $1 \le p,\, q < \infty$ then*

$$\|fg\|_{L^1} \;\asymp\; \|fg\|_{W(L^1,L^1)} \;\le\; \|f\|_{W(L^p,L^q_w)}\, \|g\|_{W(L^{p'},L^{q'}_{1/w})}.$$

Proof. The norm equivalence follows from Proposition 11.5.2, and the inequality follows from Theorem 11.3.3. $\square$

11.6 Discrete Norms

In this section we will derive equivalent "discrete" norms for the weighted amalgams, analogous to equation (11.1) but allowing much more flexibility in how we divide up the real line. In particular, we do not need to divide into equal pieces or have a lattice structure in the partition. Moreover, we do not even need to divide the line into *disjoint* pieces, but rather can use any partition of unity which has control over the overlaps of the functions in the partition. The discrete norm in the setting of general Wiener amalgams is one of the main results in [21].

While BUPUs and discrete norms are very useful, we emphasize again that the continuous norm provides our fundamental definition of amalgam spaces, and in particular leads much more naturally to the definition of general Wiener amalgams.

Definition 11.6.1. A set of functions $\{\psi_i\}_{i \in J}$ on $\mathbf{R}$ is a *bounded uniform partition of unity*, or *BUPU*, if

a. $\sum \psi_i \equiv 1$,

b. $\sup \|\psi_i\|_{L^\infty} < \infty$,

c. there exists a compact set $U \subset \mathbf{R}$ with nonempty interior and points $y_i \in \mathbf{R}$ such that $\operatorname{supp}(\psi_i) \subset U + y_i$ for all i, and

d. for each compact $K \subset \mathbf{R}$,

$$\sup_{x \in \mathbf{R}} \#\{i \in J : x \in K + y_i\} \;=\; \sup_{i \in J} \#\{j \in J : K + y_i \cap K + y_j \neq \emptyset\} \;<\; \infty.$$

We will implicitly associate the set U and points y_i with the BUPU $\{\psi_i\}_{i \in J}$.

On the real line it is very easy to create BUPUs, we can even create them so that each ψ_i is a translate of a given function ψ. Even in the general group setting, it is possible to find BUPUs of prescribed size in any homogeneous Banach space [23], although it is *not* always possible (or desirable) in the general case to choose the functions ψ_i to be characteristic functions.

The assumption in the following result that V contains U is for convenience and is not necessary.

Theorem 11.6.2. *Let w be a moderate weight. If $\{\psi_i\}_{i \in J}$ is a BUPU and V is a compact set containing U, then*

$$\|f\|_{W(L^p, L^q_w)} \;\asymp\; \Big\|\sum_{i \in J} \|f\psi_i\|_{L^p}\, \chi_{V+y_i}\Big\|_{L^q_w}. \tag{11.16}$$

Proof. a. For simplicity, denote the right-side norm of equation (11.16) by $\|\cdot\|_V$. We first show this is independent of V in the sense that different choices of V give equivalent norms.

Fix f, and let $V_1,\, V_2 \supset U$ be compact sets with nonempty interiors. Then we can find $x_1, \dots, x_N$ such that $V_2 \subset \bigcup_{k=1}^N (V_1 + x_k)$. For a generic compact set V, define $G_V(x) = \sum \|f\psi_i\|_{L^p}\, \chi_{V+y_i}(x)$, so that $\|f\|_V = \|G_V\|_{L^q_w}$. We compute that

$$\begin{aligned} G_{V_2}(x) \;&=\; \sum_{i \in J} \|f\psi_i\|_{L^p}\, \chi_{V_2+y_i}(x) \;\le\; \sum_{i \in J} \|f\psi_i\|_{L^p} \sum_{k=1}^N \chi_{V_1+x_k+y_i}(x) \\ &=\; \sum_{k=1}^N G_{V_1}(x - x_k) \\ &=\; \sum_{k=1}^N (T_{x_k} G_{V_1})(x). \end{aligned}$$

Since L^q_w is solid and translation-invariant, this implies

$$\|f\|_{V_2} \;=\; \|G_{V_2}\|_{L^q_w} \;\le\; \Big\|\sum_{k=1}^N T_{x_k} G_{V_1}\Big\|_{L^q_w} \;\le\; \sum_{k=1}^N \|T_{x_k}\|_{L^q_w}\, \|G_{V_1}\|_{L^q_w} \;\le\; MN\, \|f\|_{V_1},$$

where $M = \max\{\|T_{x_k}\|_{L^q_w}\}_{k=1}^N < \infty$. A symmetric argument gives the reverse inequality and completes the claim.

b. Now we show that the left and right sides of equation (11.16) are equivalent norms. Fix Q large enough that $Q \supset U - U$. If $x \in U + y_i$ then $y_i \in x - U$, so $U + y_i \subset U + x - U \subset Q + x$. Therefore for such x we have $f\psi_i = f\psi_i\chi_{Q+x}$ since $\mathrm{supp}(\psi_i) \subset U + y_i \subset Q + x$, and so

$$\|f\psi_i\|_{L^p} = \|f\psi_i\chi_{Q+x}\|_{L^p} \le R\,\|f \cdot \chi_{Q+x}\|_{L^p}, \qquad x \in U + y_i,$$

where $R = \sup \|\psi_i\|_{L^\infty} < \infty$. Hence,

$$G_U(x) = \sum_{i\in J} \|f\psi_i\|_{L^p}\; \chi_{U+y_i}(x) = \sum_{\{i\,:\,x\in U+y_i\}} \|f\psi_i\|_{L^p} \le C_U R\,\|f \cdot \chi_{Q+x}\|_{L^p},$$

where $C_U = \sup_{x\in\mathbf{R}} \#\{i \in J : x \in U + y_i\} < \infty$. Since L^q_w is solid, this implies

$$\|f\|_U = \|G_U\|_{L^q_w} \le C_U R \Big\| \|f \cdot \chi_{Q+x}\|_{L^p} \Big\|_{L^q_w} = C_U R\,\|f\|_{W(L^p,L^q_w)}. \tag{11.17}$$

To prove the opposite inequality, let $V \supset U$ be such that $V \supset U - Q$. Given $x \in \mathbf{R}$, define

$$M_x = \{i \in J : (U + y_i) \cap (Q + x) \ne \emptyset\}.$$

If $i \in M_x$, then $u + y_i = q + x$ for some $u \in U$ and $q \in Q$, so $x = u + y_i - q \in V + y_i$. Therefore,

$$\begin{aligned}
\|f \cdot \chi_{Q+x}\|_{L^p} &= \Big\|\sum_{i\in J} (f \cdot \chi_{Q+x})\,\psi_i\Big\|_{L^p} \le \sum_{i\in M_x} \|f\psi_i\chi_{Q+x}\|_{L^p} \\
&\le \sum_{i\in M_x} \|f\psi_i\|_{L^p} \\
&= \sum_{i\in M_x} \|f\psi_i\|_{L^p}\; \chi_{V+y_i}(x) \le G_V(x).
\end{aligned}$$

Since L^q_w is solid, we therefore have

$$\|f\|_{W(L^p,L^q_w)} = \Big\| \|f \cdot \chi_{Q+x}\|_{L^p} \Big\|_{L^q_w} \le \|G_V\|_{L^q_w} = \|f\|_V. \tag{11.18}$$

From (11.17), (11.18), and the fact that $\|\cdot\|_V$ is independent of V, we conclude $\|\cdot\|_V$ and $\|\cdot\|_{W(L^p,L^q_w)}$ are equivalent norms. □

Example 11.6.3. Suppose that $\{y_i\}_{i\in J}$ and U are such that $\{U + y_i\}_{i\in J}$ is a partition of $\mathbf{R}$. Then $\{\chi_{U+y_i}\}_{i\in J}$ is a BUPU, so

$$\begin{aligned}
\|f\|_{W(L^p,L^q_w)} &\asymp \Big\|\sum_{i\in J} \|f \cdot \chi_{U+y_i}\|_{L^p}\; \chi_{U+y_i}\Big\|_{L^q_w} \\
&= \left(\int_{\mathbf{R}} \Big|\sum_{i\in J} \|f \cdot \chi_{U+y_i}\|_{L^p}\; \chi_{U+y_i}(x)\Big|^q\, w(x)^q\, dx\right)^{1/q} \\
&= \left(\sum_{i\in J} \|f \cdot \chi_{U+y_i}\|_{L^p}^q \int_{U+y_i} w(x)^q\, dx\right)^{1/q},
\end{aligned}$$

where the interchange of summation and integration is justified by the fact that $\{U + y_i\}_{i\in J}$ is a partition of $\mathbf{R}$. Since w^q is moderate, we know that the values $\int_{U+y_i} w^q$ are uniformly equivalent to the values of w^q at any point $z_i \in U + y_i$. Thus,

$$\|f\|_{W(L^p,L^q_w)} \asymp \left(\sum_{i\in J} \|f \cdot \chi_{U+y_i}\|^q_{L^p}\, w(z_i)^q\right)^{1/q} = \big\| \{\|f \cdot \chi_{U+y_i}\|_{L^p}\}_{i\in J} \big\|_{\ell^q_\omega},$$

where ω is the weight on the index set J defined by $\omega(i) = w(z_i)$.

In particular, consider $U = [0, 1]$ and $y_n = n$. Then one equivalent norm for $W(L^p, L^q)$ is

$$\|f\|_{W(L^p,L^q)} \asymp \left(\sum_{n\in\mathbf{Z}} \|f \cdot \chi_{[n,n+1]}\|^q_{L^p}\right)^{1/q},$$

which is exactly the norm given in equation (11.1). However, we have also produced in Theorem 11.6.2 many other norms for this space, and in particular, we do not need to assume a "regular" or "lattice" distribution of the points y_n.

More general BUPUs not corresponding to disjoint partitions of $\mathbf{R}$, and whose elements need not be characteristic functions of sets, are often necessary. Even though the supports of the ψ_i will overlap in that case, the definition ensures that the number of overlaps is limited, namely,

$$\sup_{i\in J} \#\{j \in J : \operatorname{supp}(\psi_i) \cap \operatorname{supp}(\psi_i) \neq \emptyset\} < \infty.$$

This fact will allow us to prove that $W(L^p, L^q_w)$ has an equivalent discrete-type norm based on any BUPU. First, we require some preliminary results.

Definition 11.6.4. We say that a family $\{E_i\}_{i\in J}$ of subsets of a set X has a *maximum of K overlaps* if

$$K = \sup_{i\in J} \#\{j \in J : E_i \cap E_j \neq \emptyset\} = \sup_{x\in X} \sum_{i\in J} \chi_{E_i}(x) < \infty.$$

The following result is proved in [28] and is known as the *disjointization principle.*

Lemma 11.6.5. *Let $\{E_i\}_{i\in J}$ be a family of subsets of a set X with a maximum of K overlaps. Then there is a partition of J into finitely many subsets $\{J_r\}_{r=1}^K$ so that*

$$i \neq j \in J_r \implies E_i \cap E_j = \emptyset. \tag{11.19}$$

Proof. Let J_1 be a maximal subset of J with respect to (11.19) for $r = 1$. Inductively define J_r for $r \geq 2$ as a maximal subset of $J \setminus \bigcup_{s=1}^{r-1} J_s$ having property (11.19). Suppose $i \in J \setminus \bigcup_{r=1}^{K} J_r$. Then given $1 \leq r \leq K$, we have $i \in J \setminus \bigcup_{s=1}^{r-1} J_s$ and $i \notin J_r$. Since J_r is maximal in $J \setminus \bigcup_{s=1}^{r-1} J_s$ with respect to (11.19), $J_r \cup \{i\}$ cannot satisfy (11.19). That means there is a $j_r \in J_r$ such that $E_i \cap E_{j_r} \neq \emptyset$. Hence, for each $l \in \{i, j_1, \ldots, j_K\}$ we have $E_i \cap E_l \neq \emptyset$. However, the J_r are disjoint, so $i, j_1, \ldots, j_K$ are distinct, which contradicts the definition of K. Therefore $J = \bigcup_{r=1}^{K} J_r$. $\square$

Proposition 11.6.6. *Let (X,μ) be a measure space, and let $1 \le p \le \infty$. Assume $\{f_n\}_{n\in J} \subset L^p(X,d\mu)$ are nonnegative functions such that $\{\mathrm{supp}(f_n)\}_{n\in J}$ has a maximum of K overlaps. If $1 \le p < \infty$ then for each finite $F \subset J$ we have*

$$\biggl(\sum_{n\in F} \|f_n\|_{L^p}^p\biggr)^{1/p} \;\le\; \Bigl\|\sum_{n\in F} f_n\Bigr\|_{L^p} \;\le\; K^{1/p'} \biggl(\sum_{n\in F} \|f_n\|_{L^p}^p\biggr)^{1/p}. \tag{11.20}$$

Therefore, $\sum \|f_n\|_{L^p}^p < \infty$ if and only if $\sum f_n$ converges in $L^p(X,d\mu)$. In this case the convergence is unconditional, and we can replace F by J in equation (11.20). An analogous result holds for $p = \infty$.

Proof. We will only prove the case $p < \infty$. By Lemma 11.6.5, we can partition J as $J = \bigcup_1^K J_r$, with the property that $\mathrm{supp}(f_m) \cap \mathrm{supp}(f_n) = \emptyset$ whenever $m \ne n \in J_r$. Therefore, since each $f_n \ge 0$,

$$\begin{aligned}
\Bigl\|\sum_{n\in F} f_n\Bigr\|_{L^p}^p \;=\; \int_X \biggl|\sum_{r=1}^K \sum_{n\in F\cap J_r} f_n\biggr|^p d\mu \;&\le\; K^{p/p'} \sum_{r=1}^N \int_X \biggl|\sum_{n\in F\cap J_r} f_n\biggr|^p d\mu \\
&=\; K^{p/p'} \sum_{r=1}^N \sum_{n\in F\cap J_r} \int_X |f_n|^p \, d\mu \\
&=\; K^{p/p'} \sum_{n\in F} \|f_n\|_{L^p}^p,
\end{aligned}$$

where the next-to-last equality follows from the fact that the supports of the f_n for $n \in F \cap J_r$ are all disjoint. The opposite inequality is similar, and the statements about convergence follow from standard arguments. $\square$

Now we obtain an equivalent discrete norm for the weighted amalgams using any BUPU. The following result is proved in [24] for general Wiener amalgams $W(B,C)$ on locally compact groups.

Theorem 11.6.7. *Let w be a moderate weight, and let $1 \le p, q \le \infty$. Let $\{\psi_i\}_{i\in J}$ be a BUPU, fix any $z_i \in U + y_i$, and set $\omega(i) = w(z_i)$. Then*

$$\|f\|_{W(L^p,L^q_w)} \;\asymp\; \biggl(\sum_{i\in J} \|f\psi_i\|_{L^p}^q \, w(z_i)^q\biggr)^{1/q} \;=\; \bigl\| \bigl\{\|f\psi_i\|_{L^p}\bigr\}_{i\in J} \bigr\|_{\ell^q_\omega}.$$

Proof. We will only prove the case $1 \le q < \infty$. By Theorem 11.6.2 and Proposition 11.6.6,

$$\begin{aligned}
\|f\|_{W(L^p,L^q_w)} \;&\asymp\; \Bigl\|\sum_{i\in J} \|f\psi_i\|_{L^p} \; \chi_{U+y_i}\Bigr\|_{L^q_w} \\
&\asymp\; \biggl(\sum_{i\in J} \bigl\| \, \|f\psi_i\|_{L^p} \; \chi_{U+y_i}\bigr\|_{L^q_w}^q\biggr)^{1/q} \\
&=\; \biggl(\sum_{i\in J} \|f\psi_i\|_{L^p}^q \; \|\chi_{U+y_i}\|_{L^q_w}^q\biggr)^{1/q}.
\end{aligned}$$

The proof then follows since w is a moderate weight, so in particular, $\|\chi_{U+y_i}\|_{L^q_w}^q = \int_{U+y_i} w^q \asymp w(z_i)^q$ by Proposition 11.2.4. □

As a corollary, we obtain a result for global inclusions of weighted amalgams.

Corollary 11.6.8. *Let w be a moderate weight. If $1 \le p \le q \le \infty$ then $W(L^p, L^p_w) \subset W(L^p, L^q_w)$.*

Proof. Fixing any BUPU $\{\psi_i\}_{i\in J}$ and using the discrete-type norm of Theorem 11.6.7, we have

$$\|f\|_{W(L^p,L^p_w)} \asymp \left\| \left\{ \|f\psi_i\|_{L^p} \right\}_{i\in J} \right\|_{\ell^p_w} \ge \left\| \left\{ \|f\psi_i\|_{L^p} \right\}_{i\in J} \right\|_{\ell^q_w} \asymp \|f\|_{W(L^p,L^q_w)}.$$

□

11.7 Duality

In this section we will compute the dual space of the weighted amalgams. Note that the simplifying assumption in the beginning of the proof of the following result that the BUPU is a partition is not achievable for general locally compact groups; here again we are being oversimplistic in our presentation in regard to pointing to generalizations to general Wiener amalgams. Instead, a BUPU that is not simply a partition should be constructed and used to prove the result in that setting.

Theorem 11.7.1. *Let w be a moderate weight, and assume that $1 \le p,\, q < \infty$. Then $W(L^p, L^q_w)' = W(L^{p'}, L^{q'}_{1/w})$.*

Proof. We assume for simplicity that $\{\chi_{U+y_i}\}_{i\in J}$ is a BUPU, so we can take

$$\|f\|_{W(L^p,L^q_w)} = \left(\sum_{i\in J} \|f \cdot \chi_{U+y_i}\|_{L^p}^q \; \omega(i)^q \right)^{1/q}$$

and

$$\|g\|_{W(L^{p'},L^{q'}_{1/w})} = \left(\sum_{i\in J} \|g \cdot \chi_{U+y_i}\|_{L^{p'}}^{q'} \; \omega(i)^{-q'} \right)^{1/q'},$$

where $z_i \in U + y_i$ is any fixed choice of points and $\omega(i) = w(z_i)$.

a. Given $f \in W(L^p, L^q_w)$ and $g \in W(L^{p'}, L^{q'}_{1/w})$, we have by Hölder's inequality that

$$\begin{aligned} \int_{\mathbf{R}} |f(t)\, g(t)|\, dt &= \sum_{i\in J} \int_{U+y_i} |f(t)\, g(t)|\, dt \\ &\le \sum_{i\in J} \|f \cdot \chi_{U+y_i}\|_{L^p} \; \|g \cdot \chi_{U+y_i}\|_{L^{p'}}\, \omega(i)\, \omega(i)^{-1} \end{aligned}$$

$$\le \left(\sum_{i\in J} \|f\cdot\chi_{U+y_i}\|_{L^p}^q\ \omega(i)^q\right)^{1/q} \left(\sum_{i\in J} \|g\cdot\chi_{U+y_i}\|_{L^{p'}}^{q'}\ \omega(i)^{-q'}\right)^{1/q'}$$
$$= \|f\|_{W(L^p,L^q_w)}\ \|g\|_{W(L^{p'},L^{q'}_{1/w})}.$$

Therefore $\langle f,g\rangle = \int_{\mathbf{R}} f\bar{g}$ is well-defined, g determines a continuous linear functional on $W(L^p,L^q_w)$, and

$$\sup\bigl\{|\langle f,g\rangle| : \|f\|_{W(L^p,L^q_w)} = 1\bigr\} \ \le\ \|g\|_{W(L^{p'},L^{q'}_{1/w})}. \tag{11.21}$$

b. We will show now that equality holds in equation (11.21). To do this, we construct an f with $\|f\|_{W(L^p,L^q_w)} = 1$ such that $\|g\|_{W(L^{p'},L^{q'}_{1/w})} = \langle f,g\rangle$.

For simplicity, let us consider only the case that $1 < p,q < \infty$. Let $g \in W(L^{p'},L^{q'}_{1/w})$ be given, and set $g_i = g\cdot\chi_{U+y_i}$ for $i \in J$. Define

$$f_i(t) \ =\ \begin{cases} |g_i(t)|^{p'}/\overline{g_i(t)}, & g_i(t)\ne 0,\\ 0, & g_i(t) = 0.\end{cases}$$

Then $\mathrm{supp}(f_i) \subset U + y_i$ and $|f_i(t)|^p = |g_i(t)|^{p'}$, so $\|f_i\|_{L^p}^p = \|g_i\|_{L^{p'}}^{p'} < \infty$. Moreover, since $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\begin{aligned}\langle f_i,g_i\rangle &= \int_{\mathbf{R}} f_i(t)\,\overline{g_i(t)}\,dt\\ &= \left(\int_{\mathbf{R}} |f_i(t)|^p\,dt\right)^{1/p} \left(\int_{\mathbf{R}} |g_i(t)|^{p'}\,dt\right)^{1/p'} \ =\ \|f_i\|_{L^p}\ \|g_i\|_{L^{p'}} \ =\ a_ib_i.\end{aligned}$$

Define

$$c_i \ =\ \begin{cases} \bigl(b_i\,\omega(i)^{-1}\bigr)^{q'}/(a_ib_i), & a_ib_i \ne 0,\\ 0, & a_ib_i = 0.\end{cases}$$

Then $\bigl(c_ia_i\omega(i)\bigr)^q = \bigl(b_i\,\omega^{-1}(i)\bigr)^{q'}$, so

$$\begin{aligned}\sum_{i\in J} c_ia_ib_i &= \left(\sum_{i\in J}\bigl(c_ia_i\omega(i)\bigr)^q\right)^{1/q}\left(\sum_{i\in J}\bigl(b_i\omega(i)^{-1}\bigr)^{q'}\right)^{1/q'}\\ &= \bigl\|\{c_ia_i\}_{i\in J}\bigr\|_{\ell^q_\omega}\ \bigl\|\{b_i\}_{i\in J}\bigr\|_{\ell^{q'}_{1/\omega}}\\ &= \bigl\|\{c_ia_i\}_{i\in J}\bigr\|_{\ell^q_\omega}\ \|g\|_{W(L^{p'},L^{q'}_{1/w})}.\end{aligned}$$

Since $\{\chi_{U+y_i}\}_{i\in J}$ is a BUPU we can define $f = \sum c_if_i$. Then

$$\|f\|_{W(L^p,L^q_w)}^q \ =\ \bigl\|\{c_ia_i\}_{i\in J}\bigr\|_{\ell^q_\omega}^q \ =\ \bigl\|\{b_i\}_{i\in J}\bigr\|_{\ell^{q'}_{1/\omega}}^{q'} \ <\ \infty,$$

and

$$\langle f, g\rangle \;=\; \sum_{i\in J} c_i\langle f_i, g_i\rangle \;=\; \sum_{i\in J} c_i a_i b_i \;=\; \|f\|_{W(L^p,L^q_w)}\,\|g\|_{W(L^{p'},L^{q'}_{1/w})},$$

completing the claim.

c. Finally, assume that $\mu \in W(L^p, L^q_w)'$ is given. Fix i. Then $L^p(U+y_i)$, the space of L^p functions supported in $U+y_i$, is contained in $W(L^p, L^q_w)$, and μ restricted to $L^p(U+y_i)$ is a continuous linear functional. Therefore, there exist $g_i \in L^p(U+y_i)' = L^{p'}(U+y_i)$ such that $\langle h, \mu\rangle = \langle h, g_i\rangle$ for $h \in L^p(U+y_i)$. Since $\operatorname{supp}(g_i) \subset U+y_i$ and $\{U+y_i\}_{i\in J}$ is a partition of $\mathbf{R}$, we can define $g = \sum g_i$.

To show $g \in W(L^{p'}, L^{q'}_{1/w})$, we first claim that $\big\{\|g_i\|_{L^{p'}}\big\}_{i\in J} \in \ell^{q'}_{1/\omega}$. Given $\{c_i\}_{i\in J} \in \ell^q_\omega$ and $\varepsilon > 0$, choose $f_i \in L^p(U+y_i)$ such that $\|f_i\|_{L^p} \le 1$ and

$$|\langle f_i, g_i\rangle| \;\ge\; \|g_i\|_{L^{p'}} \;-\; \frac{\varepsilon}{2^i |c_i|}.$$

Note that $f = \sum c_i f_i \in W(L^p, L^q_w)$ since $\|f\|_{W(L^p,L^q_w)} \le \|\{c_i\}_{i\in J}\|_{\ell^q_\omega} < \infty$. Hence,

$$\Big|\sum_{i\in J} c_i\langle f_i, g_i\rangle\Big| \;=\; \Big|\sum_{i\in J} c_i\langle f_i, \mu\rangle\Big| \;=\; |\langle f, \mu\rangle| \;\le\; \big\|\{c_i\}_{i\in J}\big\|_{\ell^q_\omega}\,\|\mu\|.$$

Without loss of generality, fix the phase of c_i so that $c_i\langle f_i, g_i\rangle \ge 0$. Then,

$$\sum_{i\in J} |c_i|\,\|g_i\|_{L^{p'}} \;\le\; \sum_{i\in J} |c_i| \left(|\langle f_i, g_i\rangle| + \frac{\varepsilon}{2^i|c_i|}\right) \;=\; \Big|\sum_{i\in J} c_i\,\langle f_i, g_i\rangle\Big| + \varepsilon \;\le\; \big\|\{c_i\}_{i\in J}\big\|_{\ell^q_\omega}\,\|\mu\| + \varepsilon.$$

Thus $\big\{\|g_i\|_{L^{p'}}\big\}_{i\in J} \in (\ell^q_\omega)' = \ell^{q'}_{1/\omega}$, as claimed. Hence,

$$\|g\|_{W(L^{p'},L^{q'}_{1/w})} \;=\; \left(\sum_{i\in J} \|g\cdot\chi_{U+y_i}\|^{q'}_{L^{p'}}\;\omega(i)^{-q'}\right)^{1/q'} \;=\; \big\|\,\big\{\|g_i\|_{L^{p'}}\big\}_{i\in J}\,\big\|_{\ell^{q'}_{1/\omega}} \;<\; \infty,$$

so $\mu = g \in W(L^{p'}, L^{q'}_{1/w})$. □

11.8 Convolution

We will prove a convolution theorem for weighted amalgams in this section. Recall that the convolution of two functions f, g on $\mathbf{R}$ is the function $f * g$ defined by

$$(f*g)(x) \;=\; \int_{\mathbf{R}} f(t)\,g(x-t)\,dt,$$

when this exists. Likewise, the convolution of two sequences $a = \{a(n)\}_{n\in\mathbf{Z}}$ and $b = \{b(n)\}_{n\in\mathbf{Z}}$ indexed by the integers $\mathbf{Z}$ is the sequence $a * b$ defined by

$$(a*b)(n) \;=\; \sum_{m\in\mathbf{Z}} a(m)\,b(n-m).$$

Young's convolution inequality states that if $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ then

$$\|f * g\|_{L^r} \ \le\ \|f\|_{L^p}\, \|g\|_{L^r},$$

or, in other words, $L^p * L^q \subset L^r$, and similarly for convolution of sequences. Various extensions of this inequality can be formulated for weighted L^p spaces. For example, we take the proof of the following result from [4, Thm. 2.2], and another proof is given in [39, Prop. 11.1.3].

Lemma 11.8.1. *Let w be v-moderate. Then for $1 \le p \le \infty$, we have*

$$\forall\, f \in L^p_w, \quad \forall\, g \in L^1_v, \quad \|f * g\|_{L^p_w} \ \le\ C_w\, \|f\|_{L^p_w}\, \|g\|_{L^1_v}.$$

In other words, $L^p_w * L^1_v \subset L^p_w$.

Proof. Let $f \in L^p_w$ and $g \in L^1_v$. Then, using the definition of moderate weight,

$$\begin{aligned} |(f * g)(x)|\, w(x) \ &=\ \left| \int_{\mathbf{R}} f(x-t)\, g(t)\, dt \right| w(x) \\ &\le\ C_w \int_{\mathbf{R}} |f(x-t)|\, |g(t)|\, w(x-t)\, v(t)\, dt \ =\ C_w \big((|f| \cdot w) * (|g| \cdot v) \big)(x). \end{aligned}$$

Hence, by Young's inequality applied to the convolution of the L^p function $|f| \cdot w$ with the L^1 function $|g| \cdot v$, we have

$$\|f * g\|_{L^p_w} \ =\ \|(f * g) \cdot w\|_{L^p} \ \le\ C_w\, \big\||f| \cdot w\big\|_{L^p}\, \big\||g| \cdot v\big\|_{L^1} \ =\ C_w\, \|f\|_{L^p_w}\, \|g\|_{L^1_v}. \quad \square$$

Next we state without proof another example of a convolution result for weighted L^p spaces, showing that L^∞_w is a convolution algebra if an appropriate assumption is made on the weight [20]; see also [39, Prop. 11.1.3] for a proof.

Lemma 11.8.2. *If w is a moderate weight for which there exists a constant $C > 0$ such that $(1/w) * (1/w) \le C/w$, then $L^\infty_w * L^\infty_w \subset L^\infty_w$. In particular, this is the case for $w_s(x) = (1 + |x|)^s$ when $s > 1$.*

Now we can give a convolution result for weighted amalgams.

Theorem 11.8.3. *Assume the indices p_i, q_i and the moderate weights w_i are such that there exist constants C_1, $C_2 > 0$ so that*

$$\forall\, h \in L^{p_1}, \quad \forall\, k \in L^{p_2}, \quad \|h * k\|_{L^{p_3}} \ \le\ C_1\, \|h\|_{L^{p_1}}\, \|k\|_{L^{p_2}}$$

and

$$\forall\, h \in L^{q_1}_{w_1}, \quad \forall\, k \in L^{q_2}_{w_2}, \quad \|h * k\|_{L^{q_3}_{w_3}} \ \le\ C_2\, \|h\|_{L^{q_1}_{w_1}}\, \|k\|_{L^{q_2}_{w_2}}.$$

Then there is a constant $C > 0$ such that for all $f \in W(L^{p_1}, L^{q_1}_{w_1})$ and $g \in W(L^{p_2}, L^{q_2}_{w_2})$ we have

$$\|f * g\|_{L^{q_3}_{w_3}} \ \le\ C\, \|f\|_{W(L^{p_1}, L^{q_1}_{w_1})}\, \|g\|_{W(L^{p_2}, L^{q_2}_{w_2})}.$$

In other words, if $L^{p_1} * L^{p_2} \subset L^{p_3}$ and $L^{q_1}_{w_1} * L^{q_2}_{w_2} \subset L^{q_3}_{w_3}$, then

$$W(L^{p_1}, L^{q_1}_{w_1}) * W(L^{p_2}, L^{q_2}_{w_2}) \ \subset\ W(L^{p_3}, L^{q_3}_{w_3}).$$

Proof. Let $\chi_n = \chi_{[n,n+1)} = \chi_{[0,1)+n}$. We will use the discrete norm induced by the BUPU $\{\chi_n\}_{n=1}^{\infty}$, which is the norm defined by equation (11.1). Given a generic function f, write $f_n = f \cdot \chi_n$, and define a p-norm "discrete control function" by

$$F_{f,p}(n) \;=\; \|f \cdot \chi_n\|_{L^p} \;=\; \|f_n\|_{L^p}, \qquad n \in \mathbf{Z}.$$

Then the $W(L^p, L^q_w)$ norm of f is the ℓ^q_ω norm of $F_{f,p}$, i.e.,

$$\|f\|_{W(L^p,L^q_w)} \;=\; \big\|F_{f,p}\big\|_{\ell^q_\omega} \;=\; \biggl(\sum_{n\in\mathbf{Z}} |F_{f,p}(n)|^q\, \omega(n)^q\biggr)^{1/q},$$

where ω is the weight on $\mathbf{Z}$ defined by $\omega(n) = w(n)$.

Fix $f \in W(L^{p_1}, L^{q_1}_{w_1})$ and $g \in W(L^{p_2}, L^{q_2}_{w_2})$, and set $f_m = f \cdot \chi_m$ and $g_n = g \cdot \chi_n$. Then

$$\operatorname{supp}(f_m * g_n) \;\subset\; [m, m+1] + [n, n+1] \;=\; [m+n, m+n+2] \;=\; [0,2] + m + n.$$

Hence $(f_m * g_n) \cdot \chi_k$ can only be nonzero when $k = m+n$ or $k = m+n+1$. Therefore,

$$\begin{aligned}
F_{f*g,p_3}(k) \;&=\; \big\|(f*g)\cdot\chi_k\big\|_{L^{p_3}} \\
&=\; \biggl\|\sum_{m\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} (f_m * g_n)\cdot\chi_k\biggr\|_{L^{p_3}} \\
&=\; \biggl\|\sum_{n\in\mathbf{Z}} (f_{k-n} * g_n)\cdot\chi_k + (f_{k-n+1} * g_n)\cdot\chi_k\biggr\|_{L^{p_3}} \\
&\le\; \sum_{n\in\mathbf{Z}} \big\|(f_{k-n} * g_n)\cdot\chi_k\big\|_{L^{p_3}} \;+\; \sum_{n\in\mathbf{Z}} \big\|(f_{k-n+1} * g_n)\cdot\chi_k\big\|_{L^{p_3}} \\
&\le\; \sum_{n\in\mathbf{Z}} \big\|f_{k-n} * g_n\big\|_{L^{p_3}} \;+\; \sum_{n\in\mathbf{Z}} \big\|f_{k-n+1} * g_n\big\|_{L^{p_3}} \\
&\le\; C_1 \sum_{n\in\mathbf{Z}} \|f_{k-n}\|_{L^{p_1}}\, \|g_n\|_{L^{p_2}} \;+\; C_1 \sum_{n\in\mathbf{Z}} \big\|f_{k-n+1}\big\|_{L^{p_1}}\, \|g_n\|_{L^{p_2}} \\
&=\; C_1 \sum_{n\in\mathbf{Z}} F_{f,p_1}(k-n)\, F_{g,p_2}(n) \;+\; C_1 \sum_{n\in\mathbf{Z}} F_{f,p_1}(k-n+1)\, F_{g,p_2}(n) \\
&=\; C_1\big(F_{f,p_1} * F_{g,p_2}\big)(k) \;+\; C_1\big(F_{f,p_1} * F_{g,p_2}\big)(k+1).
\end{aligned}$$

Thus the discrete control function of $f*g$ is bounded by a finite sum of translates of the convolution of the discrete control functions of f and g. Therefore, letting

T_n denote the translation operator, we have

$$\begin{aligned}
\|f*g\|_{W(L^{p_3},L^{q_3}_{w_3})} &= \big\|F_{f*g,p_3}\big\|_{\ell^{q_3}_{\omega_3}} \\
&\le C_1\,\big\|F_{f,p_1}*F_{g,p_2}\big\|_{\ell^{q_3}_{\omega_3}} + C_1\,\big\|T_{-1}\big(F_{f,p_1}*F_{g,p_2}\big)\big\|_{\ell^{q_3}_{\omega_3}} \\
&\le C_1C_2\,\big\|F_{f,p_1}\big\|_{\ell^{q_1}_{\omega_1}}\,\big\|F_{g,p_2}\big\|_{\ell^{q_2}_{\omega_2}} \\
&\qquad + C_1C_2\,\|T_{-1}\|_{\ell^{q_3}_{\omega_3}\to\ell^{q_3}_{\omega_3}}\,\big\|F_{f,p_1}\big\|_{\ell^{q_1}_{\omega_1}}\,\big\|F_{g,p_2}\big\|_{\ell^{q_2}_{\omega_2}} \\
&= C\,\|f\|_{W(L^{p_1},L^{q_1}_{w_1})}\,\|g\|_{W(L^{p_2},L^{q_2}_{w_2})}. \qquad \square
\end{aligned}$$

As one corollary, we obtain a convolution result that is useful in time-frequency analysis and sampling theory, cf. [4, Thm. 2.2], [39, Thm.11.1.5].

Corollary 11.8.4. *Let w be v-moderate. Then for $1 \le p \le \infty$, we have $L^p_w * W(L^\infty, L^1_v) \subset W(L^1, L^p_w) * W(L^\infty, L^1_v) \subset W(L^\infty, L^p_w)$.*

Proof. By Propositions 11.5.2 and 11.5.1, we have $L^p_w = W(L^p, L^p_w) \subset W(L^1, L^p_w)$. This gives the inclusion $L^p_w * W(L^\infty, L^1_v) \subset W(L^1, L^p_w) * W(L^\infty, L^1_v)$. The second inclusion follows from Theorem 11.8.3 since $L^1 * L^\infty \subset L^\infty$ (trivially) and $L^p_w * L^1_v \subset L^p_w$ (Lemma 11.8.1). □

Let us make some remarks on the extension of these convolution results to the setting of locally compact groups. The main point in the proof of Theorem 11.8.3 is that there is a neighborhood Q of the origin with compact closure such that $(x+Q)+(y+Q) = (x+y)+Q'$ (with $Q' = Q+Q$). This point in the argument is valid for abelian groups, and for non-abelian groups as long as we may choose Q in such a way that (writing multiplicatively now) $xQ = Qx$ for all x in the group. In particular, the reduced Heisenberg group $\mathbf{R}\times\mathbf{R}\times\mathbf{T}$ (which is important for time-frequency analysis) has this property using $Q = [-\varepsilon,\varepsilon]\times[-\varepsilon,\varepsilon]\times\mathbf{T}$. However, the $ax+b$ group (which is important for wavelet theory), does not.

In general, we say that a locally compact group G is an IN-group ("IN" for "invariant neighborhoods") if there exists a neighborhood Q of the identity with compact closure that is invariant under all inner automorphisms. That is, we must have $xQx^{-1} = Q$ for all $x \in G$. Theorem 11.8.3 can be be extended IN-groups, see for example [17, Thm. 8.2]. Even for non-IN-groups, it is sometimes possible to derive more complicated convolution theorems, cf. [29].

11.9 The Amalgam Balian-Low Theorem

In this section we will present one application of amalgam spaces, deriving a result in time-frequency analysis that is known as the Amalgam Balian-Low Theorem. This theorem is the only result of this article that is original with the author.

Given $g \in L^2(\mathbf{R})$ and $a,\, b > 0$, define

$$g_{mn}(x) = e^{2\pi i m b x} g(x - na). \tag{11.22}$$

Then $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ is an example of a *Gabor system*, named in honor of Dennis Gabor, who proposed using such a system in communications theory, specifically with g being the Gaussian function and with $a = b = 1$ [36]. We refer to [39] for a textbook development of time-frequency analysis, and to [43], [16], or [14] for additional background on Gabor systems and related topics.

One basic problem is to determine when $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ will be a basis or a frame for $L^2(\mathbf{R})$. A frame provides robust, basis-like expansions in terms of the frame elements, but allows redundancy in the system. We will consider only orthonormal bases and Riesz bases in this section. A Riesz basis is the image of an orthonormal basis under an invertible mapping, and thus includes orthonormal bases as a special case. It can be shown that a Gabor system $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ can only be a frame for $L^2(\mathbf{R})$ when $ab \le 1$, and can only be a Riesz basis for $L^2(\mathbf{R})$ when $ab = 1$. The first statement is a consequence of deep results on C^* algebras by Rieffel in [54]. Ramanathan and Steger introduced an elegant technique in [52] that proved both statements, and in fact they proved their results in the much more general setting of "irregular" Gabor systems of the form $\{e^{2\pi i b_k x} g(x - a_k)\}_{k\in\mathbf{N}}$. An extension of their results to higher dimensions and multiple generators can be found in [15], and many related results can be found in [16], [51], [48], [49], [5], [6], [7].

For our purposes in this section, we will only consider Gabor systems $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ of the form in (11.22) which are Riesz bases for $L^2(\mathbf{R})$. In this case we must have $ab = 1$, and by a change of variables we may reduce further to the case $a = b = 1$. Therefore, we set $a = b = 1$ for the remainder of this section.

The classical Balian-Low Theorem, or BLT, is a "no-go" result, stating that all Gabor systems which are Riesz bases for $L^2(\mathbf{R})$ must have poor time-frequency localization. Consequently, Gabor Riesz bases are not suitable for time-frequency analysis, and instead redundant Gabor frames are usually preferred. Letting $\hat{g}(\omega) = \int g(x)\, e^{-2\pi i\omega x}\, dx$ denote the Fourier transform of g, we can state the BLT precisely as follows.

Theorem 11.9.1 (Balian-Low Theorem). *Fix $g \in L^2(\mathbf{R})$ and set $a = b = 1$. If $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ is a Riesz basis for $L^2(\mathbf{R})$, then*

$$\left(\int_{\mathbf{R}} |xg(x)|^2\, dx\right)\left(\int_{\mathbf{R}} |\omega\hat{g}(\omega)|^2\, d\omega\right) \;=\; \infty. \tag{11.23}$$

Equation (11.23) says that g must maximize uncertainty in the sense of yielding infinity in the Heisenberg uncertainty inequality. In particular, either g must have poor decay at infinity or $\hat{g}$ must have poor decay (which implies that g cannot be very smooth). There is considerable history and literature on the BLT. We refer to [11] for discussion, history, proofs, and references, to [10] for recent results on the sharpness of the BLT, and to [40] for issues in higher-dimensional versions of the BLT.

We will prove the following related no-go result, in which the Wiener amalgam $W(C_0, L^1)$ plays a central role. This space consists of functions which are locally in C_0 and globally in L^1, and is sometimes called the *Wiener algebra* [53]. It can be shown that

$$W(C_0, L^1) \;=\; C_0 \cap W(L^\infty, L^1),$$

but observe that this result *cannot* be shown if $W(C_0, L^1)$ was defined using only a discrete norm based on a disjoint partition of the real line! Overlaps are essential in order for local continuity to imply global continuity.

Theorem 11.9.2 (Amalgam Balian-Low Theorem). *Fix $g \in L^2(\mathbf{R})$ and set $a = b = 1$. If $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ is a Riesz basis for $L^2(\mathbf{R})$, then $g, \hat{g} \notin W(C_0, L^1)$.*

Thus, the BLT and the Amalgam BLT both state that if $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ is an orthonormal or Riesz basis for $L^2(\mathbf{R})$, then either g has poor decay or g is not smooth, but the two theorems quantify this statement in different ways. In particular, neither result implies the other. The Amalgam BLT was first proved in [41], and the proof was first published in [11].

To prove the Amalgam BLT, we define the *Zak transform* of a function $g \in L^2(\mathbf{R})$ to be the function Zg with domain $\mathbf{R}^2$ defined by

$$Zg(t,\omega) \;=\; \sum_{k\in\mathbf{Z}} g(t+k)\, e^{2\pi i k\omega}, \qquad (t,\omega) \in \mathbf{R}^2.$$

We refer to [46], [47], [43], [16], or [39] for background information on the Zak transform, which also plays a key role in the proof of the classical BLT. The following lemma summarizes some of the basic facts about the Zak transform.

Lemma 11.9.3. *Let Q be any closed unit square in $\mathbf{R}^2$, and let $g \in L^2(\mathbf{R})$.*

a. *The series defining Zg converges in the norm of $L^2(Q)$.*

b. *Z is a unitary mapping of $L^2(\mathbf{R})$ onto $L^2(Q)$.*

c. *Zg satisfies the* quasiperiodicity relations

$$Zg(t+1,\omega) \;=\; e^{-2\pi i\omega} Zg(t,\omega) \quad \text{and} \quad Zg(t,\omega+1) \;=\; Zg(t,\omega), \qquad (t,\omega) \in \mathbf{R}^2.$$

In particular, the values of Zg on $\mathbf{R}^2$ are determined by its values on Q.

d. *If Zg is continuous on $\mathbf{R}^2$, then Zg has a zero in Q.*

e. *With $a = b = 1$, we have*

$$Z(g_{mn})(x,\omega) \;=\; e^{2\pi i m x} e^{2\pi i n\omega} Zg(t,\omega), \qquad m,n \in \mathbf{Z}. \tag{11.24}$$

As a special case of equation (11.24), we see that Z is the unitary mapping which maps the elements of the orthonormal basis $\{e^{2\pi i m x}\chi_{[0,1]}(x-n)\}_{m,n\in\mathbf{Z}}$ for $L^2(\mathbf{R})$ to the elements of the orthonormal basis $\{e^{2\pi i m x}e^{2\pi i n\omega}\}_{m,n\in\mathbf{Z}}$ for $L^2(Q)$.

Since Z is unitary, given $g \in L^2(\mathbf{R})$ we know that $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ is a Riesz basis for $L^2(\mathbf{R})$ if and only if $\{Z(g_{mn})\}_{m,n\in\mathbf{Z}}$ is a Riesz basis for $L^2(Q)$. However, by equation (11.24), we see that $\{Z(g_{mn})\}_{m,n\in\mathbf{Z}}$ is obtained by multiplying each element of the orthonormal basis $\{e^{2\pi i m x}e^{2\pi i n\omega}\}_{m,n\in\mathbf{Z}}$ for $L^2(Q)$ by the single function $Zg(x,\omega)$. Since multiplication by a function is an invertible mapping only when the function is bounded above and below, we see that the Gabor system is a Riesz basis if and only if Zg is bounded above and below. More precisely, we have the following result.

Lemma 11.9.4. *Let $g \in L^2(\mathbf{R})$ be fixed, and set $a = b = 1$.*

a. $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ *is an orthonormal basis for $L^2(\mathbf{R})$ if and only if $|Zg(x,\omega)| = 1$ a.e.*

b. $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ *is a Riesz basis for $L^2(\mathbf{R})$ if and only if there exist A, B such that $0 < A \le |Zg(x,\omega)| \le B < \infty$ a.e.*

Combining Lemma 11.9.3 part d with Lemma 11.9.4, we see that if Zg is continuous then the Gabor system $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ *cannot* be a Riesz basis for $L^2(\mathbf{R})$. We will use the following result, on the mapping properties of the Zak transform on the amalgam spaces $W(L^p, L^q)$, to derive a sufficient condition under which Zg is continuous. The following proposition was proved in [41, Prop. 7.5.1].

Proposition 11.9.5. *Let $1 \le p \le \infty$ be given, and let Q be a unit cube in $\mathbf{R}^2$. Then the Zak transform is a continuous, linear, and injective map of $W(L^p, L^1)$ into $L^p(Q)$.*

Proof. Fix $f \in W(L^p, L^1)$. For $k \in \mathbf{Z}$, define $F_k(t,\omega) = f(t+k)\, e^{2\pi i k\omega}$. Then $F_k \in L^p(Q)$, and

$$\sum_{k\in\mathbf{Z}} \|F_k\|_{L^p(Q)} \;=\; \sum_{k\in\mathbf{Z}} \|f \cdot \chi_{[k,k+1]}\|_{L^p(Q)} \;=\; \|f\|_{W(L^p,L^1)} \;<\; \infty.$$

Thus the series $Zf = \sum F_k$ converges absolutely in $L^p(Q)$. Further, $\|Zf\|_{L^p} = \big\|\sum F_k\big\|_{L^p} \le \|f\|_{W(L^p,L^1)}$, so Z is a continuous mapping of $W(L^p, L^1)$ into L^p, and it is easy to show that it is an injective mapping. □

Corollary 11.9.6. *If $f \in W(C_0, L^1)$ then Zf is continuous on $\mathbf{R}^2$.*

Proof. Since $W(C_0, L^1) \subset W(L^\infty, L^1)$, if $f \in W(C_0, L^1)$ then by Proposition 11.9.5 we have that the series defining Zf converges in the norm of $L^\infty(Q)$. In other words, the series defining Zf converges uniformly on Q. Since each term $f(t+k)\, e^{2\pi i k\omega}$ in this series is continuous, it follows that the series must converge to a function which is continuous on Q. Since this is true for every unit cube Q in $\mathbf{R}^2$, we conclude that Zg is continuous on all of $\mathbf{R}^2$. □

The proof of the Amalgam BLT is now immediate, for if $g \in W(C_0, L^1)$ then Zg is continuous by Corollary 11.9.6, hence has a zero by Lemma 11.9.3, and therefore by Lemma 11.9.4 the Gabor system $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ cannot be a Riesz basis for $L^2(\mathbf{R})$. Further, this argument also applies to $\hat{g}$ because by applying the Fourier transform we see that $\{\hat{g}_{mn}\}_{m,n\in\mathbf{Z}}$ is a Riesz basis for $L^2(\mathbf{R})$ if and only if $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ is a Riesz basis for $L^2(\mathbf{R})$.

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ERRATA

Note: This errata listing is not included in the published version of this paper.

The inequality on line 2 of page 208 should read:

$$\|f * g\|_{L^r} \ \leq \ \|f\|_{L^p} \, \|g\|_{L^q}.$$