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Complex-valued functions $f_1, \ldots, f_r$ on $\mathbf{R}^d$ are *refinable* if they are linear combinations of finitely many of the rescaled and translated functions $f_i(Ax-k)$, where the translates k are taken along a lattice $\Gamma \subset \mathbf{R}^d$ and A is a *dilation matrix* that expansively maps Γ into itself. Refinable functions satisfy a *refinement equation* $f(x) = \sum_{k\in\Lambda} c_k\, f(Ax-k)$, where Λ is a finite subset of Γ, the c_k are $r \times r$ matrices, and $f(x) = (f_1(x), \ldots, f_r(x))^{\mathrm{T}}$. The *accuracy* of f is the highest degree p such that all multivariate polynomials q with degree$(q) < p$ are exactly reproduced from linear combinations of translates of $f_1, \ldots, f_r$ along the lattice Γ. In this paper, we determine the accuracy p from the matrices c_k. Moreover, we determine explicitly the coefficients $y_{\alpha,i}(k)$ such that $x^\alpha = \sum_{i=1}^r \sum_{k\in\Gamma} y_{\alpha,i}(k)\, f_i(x+k)$. These coefficients are multivariate polynomials $y_{\alpha,i}(x)$ of degree $|\alpha|$ evaluated at lattice points $k \in \Gamma$.

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1. INTRODUCTION

Let Γ be a *lattice* in $\mathbf{R}^d$, i.e., $\Gamma = \{m_1 u_1 + \cdots + m_d u_d : m_i \in \mathbf{Z}\}$ is the collection of integer linear combinations of d independent vectors $u_1, \dots, u_d \in \mathbf{R}^d$. Equivalently, Γ is the image of $\mathbf{Z}^d$ under some nonsingular linear transformation.

A *dilation matrix* associated with Γ is a $d \times d$ matrix A such that

(a) $A(\Gamma) \subset \Gamma$, and

(b) A is *expansive*; i.e., all eigenvalues satisfy $|\lambda_k(A)| > 1$.

Since $\Gamma = W(\mathbf{Z}^d)$, where W is the invertible matrix with $u_1, \dots, u_d$ as columns, the matrix $W^{-1}AW$ maps $\mathbf{Z}^d$ into itself. Therefore $W^{-1}AW$ has integer entries and integer determinant. Hence A has integer determinant as well. We set $m = |\det(A)|$. By applying the similarity transform $W^{-1}AW$, it is always possible to take $\Gamma = \mathbf{Z}^d$ if desired.

Complex-valued functions $f_1, \dots, f_r$ on $\mathbf{R}^d$ are *refinable* with respect to A and Γ if they equal linear combinations of the rescaled and translated functions $f_i(Ax - k)$, where the translates k are taken along the lattice Γ. We shall only consider the case where each f_j is obtained as a finite linear combination of the $f_i(Ax - k)$. In this case, the vector-valued function $f \colon \mathbf{R}^d \to \mathbf{C}^r$ defined by $f(x) = (f_1(x), \dots, f_r(x))^{\mathrm{T}}$ satisfies a *refinement equation*, *dilation equation*, or *two-scale difference equation* of the form

$$f(x) \;=\; \sum_{k \in \Lambda} c_k\, f(Ax - k) \tag{1.1}$$

for some finite $\Lambda \subset \Gamma$ and some $r \times r$ matrices c_k. The one-dimensional case ($d = 1$) with $A = m$ integer and a single function ($r = 1$) leads to the familiar equation $f(x) = \sum_{k=0}^{N} c_k\, f(mx - k)$. This is the starting point for the construction of orthogonal or biorthogonal wavelet bases for $L^2(\mathbf{R})$ [Dau92] and for the analysis of subdivision schemes [CDM91], most often with $m = 2$. The multidimensional case ($d > 1$) with a single function ($r = 1$) leads to multidimensional wavelet bases for $L^2(\mathbf{R}^d)$ [GM92], [KV92], [Mey92], [CD93]. The one-dimensional case ($d = 1$) with multiple functions ($r > 1$) leads to multiwavelet bases for $L^2(\mathbf{R})$ [GLT93], [GHM94], [DGHM96], [SS94].

In this paper we study the general multidimensional, multifunction case ($d \geq 1$, $r \geq 1$) with an arbitrary dilation matrix A. We seek to determine one fundamental property of a refinable f based on the coefficients c_k. That property is the *accuracy* of f, the largest integer p such that all multivariate polynomials $q(x) = q(x_1, \dots, x_d)$ with $\deg(q) < p$ lie in the shift-invariant space

$$\begin{aligned} S(f) &= \left\{ \sum_{k\in\Gamma} \sum_{i=1}^{r} b_{k,i}\, f_i(x+k) : b_{k,i} \in \mathbf{C} \right\} \\ &= \left\{ \sum_{k\in\Gamma} b_k f(x+k) : b_k \in \mathbf{C}^{1\times r} \right\}, \end{aligned} \tag{1.2}$$

where $\mathbf{C}^{1\times r}$ is the space of row vectors of length r. As usual, equality of functions is interpreted as holding almost everywhere (a.e.). We shall deal only with compactly supported functions f_i, in which case each series in (1.2) is well-defined for all choices of $b_{k,i}$. There is a large literature on the connection between accuracy and order of approximation; we refer the reader to the survey papers [deB90], [Jia95] and the references therein.

The space $S(f)$ is called a principal shift-invariant (PSI) space if $r = 1$, and a finite shift-invariant (FSI) space if $r > 1$. We shall therefore refer to $r = 1$ as the PSI case, and to $r > 1$ as the FSI case. In wavelet theory, the space $V_0 = S(f) \cap L^2(\mathbf{R}^d)$ plays a special role. The dilated spaces $V_j = \{g(A^j x) : g \in V_0\}$ are nested, due to the refinement equation. With appropriate conditions on the matrices c_k, the spaces V_j together with the functions f_i form a multiresolution analysis, which leads then to a wavelet basis for $L^2(\mathbf{R}^d)$.

For arbitrary (not necessarily refinable) functions, the celebrated *Strang–Fix conditions* determine when polynomials are reproduced by translates [SF73]. It is known that for the case of a single, one-dimensional refinable function ($d = 1$, $r = 1$), the Strang–Fix conditions are computable from the scalars c_k. For example, if $A = 2$, $\Gamma = \mathbf{Z}$, and $\Lambda \subset \{0, \dots, N\}$, the requirement for f to have accuracy p is (assuming proper hypotheses) the following set of "sum rules":

$$\sum_{k=0}^{N} c_k = 2 \qquad \text{and} \qquad \sum_{k=0}^{N} (-1)^k\, k^j\, c_k = 0 \quad \text{for } j = 0, \dots, p-1. \tag{1.3}$$

The sum rules are often stated in an equivalent "zero at 1/2" form based on the *symbol* $M(\omega) = \frac{1}{2} \sum_{k=0}^{N} c_k\, e^{-2\pi i k\omega}$ of the refinement equation, namely,

$$M(0) = 1 \qquad \text{and} \qquad M^{(j)}(1/2) = 0 \quad \text{for } j = 0, \dots, p-1. \tag{1.4}$$

These sum rules imply that the symbol factorizes in the form $M(\omega) = (1 + e^{-2\pi i\omega})^p\, R(\omega)$.

Analogues of the sum rules for the one-dimensional FSI case ($d = 1$, $r \geq 1$) with $A = 2$ were recently derived independently by Heil, Strang, and Strela [HSS96], [SS94] and by Plonka [Plo97]. These "matrix sum rules" are recursive, and are much weaker than a literal extension of (1.3) from scalars to matrices. Working in the frequency domain, Plonka further obtained a fundamental factorization of the now matrix-valued symbol $M(\omega) = \frac{1}{2} \sum_{k=0}^{N} c_k \, e^{-2\pi i k \omega}$. This factorization has since led to new results on the construction of multiwavelets in one dimension [CDP97], [PS98].

Our primary goal in this paper is the elucidation of the conditions for accuracy of f in terms of a finite system of finite linear equations on the coefficients c_k, for the general higher-dimensional, multifunction case with an arbitrary dilation matrix, for the purpose of providing a base from which a future search for practical, nonseparable higher-dimensional multiwavelet systems for image analysis can be launched. A secondary goal is to present results which are interesting in the context of approximation theory. Of course, in this secondary context the reader will recognize that the classic counterexample of de Boor and Höllig shows that polynomial accuracy is only a weak concept. However, a complete discussion of the exact relations between our results on accuracy and analogous results on order of approximation would lengthen our paper to the point of unwieldiness. We therefore leave to the interested reader the pursuit of these connections. In particular, the reader who is expert in the literature of the de Boor school of approximation theory will recognize that a skillful extraction and combination of results from papers such as [BR92], [BDR94a], [BDR94b] can be used to construct alternative proofs of some of our results, and to formulate these results in terms of order of approximation. However, our results are distinct from those appearing in the literature, and we believe that our direct, straightforward, and self-contained proofs provide additional direct insight into the understanding of accuracy and the corresponding structure of translates.

The generalization of accuracy results from one to higher dimensions is nontrivial. We present now in this introduction a brief review of the one-dimensional theorems from [HSS96], [Plo97], and [JRZ97], in order to provide context and motivation for our results. When $d = 1$, the lattice Γ is simply a multiple of the integer lattice. It therefore suffices to consider $\Gamma = \mathbf{Z}$. In this case, A is an integer, and there is essentially no loss of insight by taking $A = 2$. Instead of dealing with the functions $f_1, \dots, f_r$ directly, it is usually much more convenient to consider the vector-valued function $f(x) = (f_1(x), \dots, f_r(x))^{\mathrm{T}}$, and to refer to properties of f rather than the individual f_i.

A key tool in the analysis of accuracy is the bi-infinite matrix L with block entries c_{2i-j}, i.e., $L = [c_{2i-j}]_{i,j \in \mathbf{Z}}$. Note that L is a "downsampled Toeplitz operator"—there is a double shift between rows. If we define the infinite column vector

$$F(x) \;=\; (\dots, f(x-1), f(x), f(x+1), \dots)^{\mathrm{T}},$$

then the refinement equation $f(x) = \sum_{k \in \mathbf{Z}} c_k\, f(2x-k)$ is equivalent to the equation $F(x) = L\, F(2x)$.

Suppose now that f has accuracy p. Then the p monomials $1, x, \dots, x^{p-1}$ can be exactly reproduced from integer translates of $f = (f_1, \dots, f_r)^{\mathrm{T}}$. Hence there exist $1 \times r$ row vectors $y_k^s = (y_{k,1}^s, \dots, y_{k,r}^s)$ such that

$$x^s \;=\; \sum_{k \in \mathbf{Z}} \sum_{i=1}^{r} y_{k,i}^s\, f_i(x+k) \;=\; \sum_{k \in \mathbf{Z}} y_k^s\, f(x+k), \qquad 0 \le s < p. \tag{1.5}$$

If we define the infinite row vector

$$Y_s \;=\; (\;\dots,\quad y_{-1}^s,\quad y_0^s,\quad y_1^s,\quad \dots\;),$$

then (1.5) reads

$$x^s \;=\; Y_s\, F(x), \qquad 0 \le s < p.$$

Applying the refinement equation $F(x) = LF(2x)$, we therefore have

$$Y_s\, L\, F(x) \;=\; Y_s\, F(x/2) \;=\; (x/2)^s \;=\; 2^{-s}\, x^s \;=\; 2^{-s}\, Y_s\, F(x). \tag{1.6}$$

With appropriate hypotheses on f (namely, that integer translates of f are independent), it follows from (1.6) that $Y_s\, L = 2^{-s}\, Y_s$, and therefore that Y_s is a left eigenvector for L for the eigenvalue 2^{-s}. Thus accuracy implies (with hypotheses) that the infinite matrix L has left eigenvalues $1, 2^{-1}, \dots, 2^{-(p-1)}$. It is shown in [HSS96] and [Plo97] that these eigenvectors Y_s have a special structure: there exist p row vectors $v_0, \dots, v_{p-1}$ that completely determine the vectors y_k^s via a simple formula. These p vectors, in turn, can be found by solving a finite system of linear equations. These equations are the "matrix sum rules." They have a block triangular form: equation s involves only $v_0, \dots, v_s$. It is also shown in [HSS96] and [Plo97] that the existence of eigenvalues $1, 2^{-1}, \dots, 2^{-(p-1)}$ for L with eigenvectors possessing the above-mentioned structure is essentially a necessary and sufficient condition for f to have accuracy p.

In [JRZ97], Jia, Riemenschneider, and Zhou realized that this structure was of a polynomial type, and that this implies further properties of the eigenvectors. Moreover, their results include consideration both of the case of independent translates of f and of the case of dependent translates of f. Assuming independence of translates, they showed that for each s there exist r polynomials $u_1^s, \dots, u_r^s$ of degree at most $p-1$ such that the eigenvectors Y_s satisfy $u_i^s(k) = y_{k,i}^s$. Comparing to the previous remarks, we see then that accuracy p is essentially equivalent to the existence of left 2^{-s}-eigenvectors Y_s for $s = 0, \dots, p-1$, each with a special polynomial structure. Surprisingly, it is shown in [JRZ97] that accuracy p is equivalent to the existence of a left eigenvector Y_{p-1} with polynomial structure, i.e., the existence of this structured eigenvector implies the existence of the other structured eigenvectors. They also showed that the existence of the eigenvalues $1, 2^{-1}, \dots, 2^{-(p-1)}$ alone is not sufficient to imply accuracy; the corresponding left eigenvectors must have the required polynomial structure. Some other important results in [JRZ97] include the fact that the nonzero left and right eigenvalues of L coincide and that there are only finitely many nonzero eigenvalues of L, and extensions of some of the results above to the case where f has dependent translates.

There are considerable difficulties involved in attempting to move the study of accuracy from one to higher dimensions. One purely technical problem is the explosion of indices: the functions $f_1, \dots, f_r$ are each functions of the variable $x = (x_1, \dots, x_d)^{\mathrm{T}} \in \mathbf{R}^d$, translates of these functions are indexed by lattice points $k = (k_1, \dots, k_d)^{\mathrm{T}} \in \Gamma$, and monomials $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ are indexed by multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$. Aside from this notational issue, there are more difficult theoretical obstacles, most importantly that $(Ax)^\alpha$ is not itself a monomial for all α except in the special case that $A = cI_d$. Instead, the dilation of x^α by A results in a new polynomial that is still homogeneous but can contain terms x^β for all β of degree $|\alpha|$. This prevents any trivial generalization of the one-dimensional results to higher dimensions.

One of the key insights of this paper, which overcomes this and other problems, is to consider together the monomials x^α of a given degree. Dilation and translation of the entire vector of polynomials

$$X_{[s]}(x) \;=\; [x^\alpha]_{|\alpha|=s}$$

leads to the relatively simple matrix equations

$$\begin{aligned} X_{[s]}(Ax) \;&=\; A_{[s]}\, X_{[s]}(x), \\ X_{[s]}(x-y) \;&=\; \sum_{t=0}^{s} Q_{[s,t]}(y)\, X_{[t]}(x), \end{aligned}$$

for some appropriate matrices $A_{[s]}$ and $Q_{[s,t]}(y)$. If we set $d_s = \binom{s+d-1}{d-1}$, the number of monomials x^α of degree $|\alpha| = s$, then $A_{[s]}$ is a $d_s \times d_s$ matrix completely determined by A and s. The entries of the $d_s \times d_t$ matrix $Q_{[s,t]}(y)$ are either 0 or are monomials in y of degree $s - t$. The matrix $A_{[s]}$ has a number of surprising properties. For example, if $\lambda = (\lambda_1, \dots, \lambda_d)^{\mathrm{T}}$ is the vector of eigenvalues of A, then the eigenvalues of $A_{[s]}$ are $[\lambda^\alpha]_{|\alpha|=s}$.

With this insight, we can now see how the one-dimensional results on accuracy presage the more complicated higher-dimensional results. Suppose that $f = (f_1, \dots, f_r)^{\mathrm{T}}$ satisfies the general multidimensional refinement equation (1.1). If we use a generalized matrix notation, allowing the matrix entries to be indexed by the lattice Γ, then the analogue of the matrix L for the general case is $L = [c_{Ai-j}]_{i,j\in\Gamma}$. Defining the "infinite column vector" $F(x) = [f(x+k)]_{k\in\Gamma}$ and using the obvious matrix/vector multiplication, we show that the refinement equation is equivalent to the equation $F(x) = L\,F(Ax)$.

If f has accuracy p, then each monomial x^α of degree less than p can be written $x^\alpha = \sum_{k\in\Gamma}\sum_{i=1}^{r} y_{\alpha,i}(k)\, f_i(x+k)$. Omitting the precise hypotheses and details (which are given in the statements of the theorems in Section 3), we prove in this paper that the coefficients $y_{\alpha,i}(k)$ are evaluations at lattice points of multivariate polynomials $y_{\alpha,i}$ of degree $|\alpha|$. Defining the matrix of polynomials

$$y_{[s]}(x) \;=\; \begin{bmatrix} y_{\alpha_1,1}(x) & \cdots & y_{\alpha_1,r}(x) \\ \vdots & \ddots & \vdots \\ y_{\alpha_{d_s},1}(x) & \cdots & y_{\alpha_{d_s},r}(x) \end{bmatrix}$$

and infinite row vectors $Y_{[s]} = \bigl(y_{[s]}(k)\bigr)_{k\in\Gamma}$ containing the evaluations of these polynomials at lattice points, it follows that if f has accuracy p then

$$X_{[s]}(x) \;=\; \sum_{k\in\Gamma} y_{[s]}(x)\, f(x+k) \;=\; Y_{[s]}\, F(x), \qquad 0 \le s < p.$$

We show that accuracy p holds if and only if

$$Y_{[s]}\, L \;=\; A_{[s]}^{-1}\, Y_{[s]} \tag{1.7}$$

for $s = 0, \dots, p-1$ with each $Y_{[s]}$ having the specified polynomial form. Moreover, we show that this occurs if and only if (1.7) is satisfied for $s = p-1$ with $Y_{[p-1]}$ having the required polynomial structure. Further, we show that this condition can be translated into a finite system of linear equations. The existence of a solution to this system, which has a block triangular structure, is equivalent to the accuracy of f. The solution of the system leads explicitly to the coefficients $y_{\alpha,i}(k)$ that are used to reproduce the monomial x^α from translates of f. For the case of a single refinable function ($r = 1$), this test for accuracy simplifies dramatically, to the following form similar to (1.3),

$$\sum_{k\in\Gamma} c_k = m \quad\text{and}\quad \sum_{k\in\Gamma_1} k^\alpha c_k \;=\; \cdots \;=\; \sum_{k\in\Gamma_m} k^\alpha c_k \quad \text{for } 0 \le |\alpha| < p,$$

where the Γ_i are the cosets of the sublattice $A(\Gamma)$ in the quotient group $\Gamma/A(\Gamma)$.

We see then that the eigenvalue/eigenvector condition of the one-dimensional case is a consequence of the very special form of $A_{[s]}$ when $d = 1$, namely that $A_{[s]} = A^s$ with A a scalar. In the special case $A = cI_d$, which plays a role in the construction of "separable" wavelets in higher dimensions, we have $A_{[s]} = c^s I_{d_s}$, and then (1.7) is again, as in the one-dimensional case, an eigenvector equation. In the general case it is instead a kind of "generalized eigenvector equation."

In the course of our analysis we prove some results that apply to the shift-invariant space generated by arbitrary functions $f_1, \dots, f_r$. We show that even for arbitrary $f = (f_1, \dots, f_r)^{\mathrm{T}}$ with independent translates, accuracy p implies that the coefficients $y_{\alpha,i}(k)$ such that

$$x^\alpha \;=\; \sum_{k\in\Gamma}\sum_{i=1}^{r} y_{\alpha,i}(k)\, f_i(x+k)$$

are evaluations of polynomials at lattice points. Hence for each polynomial $q \in S(f)$ with $\deg(q) < p$, there exist polynomials $u_{q,1}, \dots, u_{q,r}$ such that

$$q(x) \;=\; \sum_{i=1}^{r}\sum_{k\in\Gamma} u_{q,i}(k)\, f_i(x+k). \tag{1.8}$$

This result can also be viewed as a restatement of the Strang–Fix conditions for multiple functions in higher dimensions.

In addition, we prove the following related result, although we make no actual use of it in this paper. We show that if any polynomial q lying in $S(f)$ can be written as in (1.8) with coefficients that are evaluations of polynomials at lattice points, then

$$\frac{\partial q}{\partial x_j}(x) \;=\; \sum_{k\in\Gamma}\sum_{i=1}^{r} \frac{\partial u_{q,i}}{\partial x_j}(k)\, f_i(x+k), \tag{1.9}$$

and hence any derivative of q also lies in $S(f)$. This result can also be obtained by using Appell polynomials [deB90].

The outline of our paper is as follows. Following the presentation of our notation in Section 2, we give the precise statement of our results in Section 3. The proofs of these results are given in Section 4. Section 5 contains some applications of these results to the specific case of the "quincunx" or "twin dragon" dilation matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. This matrix is one of the most popular for the construction of nonseparable wavelet bases for $L^2(\mathbf{R}^2)$. Finally, we provide in the Appendix a discussion of the convergence of the infinite matrix product $\prod_{j=1}^{\infty} M(B^j\omega)$, where $B = (A^{-1})^{\mathrm{T}}$ and $M(\omega) = (1/m)\sum_{k\in\Lambda} c_k\, e^{-2\pi i k\cdot\omega}$. This product arises when considering the Fourier transform of the refinement equation, and plays a role in the proof of Theorem 3.9.

Note added in proof. Following completion of this paper we became aware of some related results obtained independently. In [Jia98], Jia considered the accuracy of a single function in higher-dimensions. In [Jng96], Jiang obtained some results on accuracy of multiple functions in higher dimensions, but only for a restricted class of dilation matrices.

2. NOTATION

2.1. *General Notation and Remarks*

The space $\mathbf{C}^r = \mathbf{C}^{r\times 1}$ is the set of ordinary $r \times 1$ column vectors with complex entries, and $\mathbf{C}^{1\times r}$ is the set of $1 \times r$ row vectors with complex entries. In particular, $f(x) \in \mathbf{C}^r$ is an $r \times 1$ column vector for each $x \in \mathbf{R}^d$.

We use the standard multi-index notation $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, where $x = (x_1, \dots, x_d)^{\mathrm{T}} \in \mathbf{R}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d)$ with each α_i a nonnegative integer. The degree of α is $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The number of multi-indices α of degree s is $d_s = \binom{s+d-1}{d-1}$. We write $\beta \le \alpha$ if $\beta_i \le \alpha_i$ for $i = 1, \dots, d$.

Recall that $m = |\det(A)|$ is an integer. Therefore, the quotient group $\Gamma/A(\Gamma)$ has order m. A *full set of digits* $d_1, \dots, d_m \in \Gamma$ is a complete set of representatives of $\Gamma/A(\Gamma)$. In this case, Γ is partitioned into the disjoint cosets

$$\Gamma_i \;=\; A(\Gamma) - d_i \;=\; \{Ak - d_i : k \in \Gamma\}.$$

For example, if $d = 1$, $\Gamma = \mathbf{Z}$, and $A = m$, then $0, \dots, m-1$ is a full set of digits.

Recall that the lattice Γ is the set of integer linear combinations of the vectors $u_1, \dots, u_d \in \mathbf{R}^d$. Therefore, the rectangular parallelepiped

$$P \;=\; \{x_1 u_1 + \cdots + x_d u_d : 0 \le x_i < 1\}$$

is a *fundamental domain* for the group $\mathbf{R}^d/\Gamma$, i.e., it is a full set of representatives of $\mathbf{R}^d/\Gamma$. $\mathbf{R}^d$ is therefore partitioned into the disjoint sets $\{P + k\}_{k\in\Gamma}$. Note that $\mathbf{R}^d/\Gamma$ is isomorphic to the d-dimensional torus $\mathbf{R}^d/\mathbf{Z}^d$ via a simple change of variables. In fact, $\{W^{-1}x : x \in P\} = [0,1)^d$ if W is the matrix with $u_1, \dots, u_d$ as columns.

Integrals of the vector-valued function $f = (f_1, \dots, f_r)^{\mathrm{T}}$ are computed componentwise. If f is integrable then we define its Fourier transform by

$$\begin{aligned}\hat f(\omega) \;&=\; \int_{\mathbf{R}^d} f(x)\, e^{-2\pi i x\cdot\omega}\, dx \\ &=\; \left(\int_{\mathbf{R}^d} f_1(x)\, e^{-2\pi i x\cdot\omega}\, dx,\; \dots,\; \int_{\mathbf{R}^d} f_r(x)\, e^{-2\pi i x\cdot\omega}\, dx\right)^{\mathrm{T}}.\end{aligned}$$

In particular,

$$\hat{f}(0) = (\hat{f}_1(0), \dots, \hat{f}_r(0))^{\mathrm{T}} = \left(\int_{\mathbf{R}^d} f_1(x)\, dx, \dots, \int_{\mathbf{R}^d} f_r(x)\, dx \right)^{\mathrm{T}}.$$

Suppose that $f\colon \mathbf{R}^d \to \mathbf{C}^r$ is integrable and refinable. If we define $B = (A^{-1})^{\mathrm{T}}$, then $\hat{f}$ will satisfy the equation

$$\hat{f}(\omega) = M(B\omega)\, \hat{f}(B\omega),$$

where $M(\omega) = (1/m) \sum_{k \in \Lambda} c_k\, e^{-2\pi i k \cdot \omega}$ is the matrix-valued symbol of the refinement equation. In particular, if we set

$$\Delta = M(0) = \frac{1}{m} \sum_{k \in \Lambda} c_k,$$

then

$$\hat{f}(0) = M(0)\, \hat{f}(0) = \Delta\, \hat{f}(0).$$

Therefore, $\hat{f}(0)$ is a right 1-eigenvector of Δ if $\hat{f}(0) \neq 0$.

In the one-dimensional case with $d = 1$, $A = 2$, $\Gamma = \mathbf{Z}$, and $\Lambda = \{0, \dots, N\}$, it is known that if f is an integrable solution of the refinement equation and $\hat{f}(0) = 0$, then there is a positive integer n such that the refinement equation with coefficients $2^{-n}c_0, \dots, 2^{-n}c_N$ has an integrable solution g satisfying $\hat{g}(0) \neq 0$. Moreover, f is the nth distributional derivative of g in this case. The accuracy of f is clearly determined from the accuracy of g. These facts were established for the PSI case in [DL91] and for the FSI case in [HC96]. We believe that analogues of these facts should hold in higher dimensions as well, although we are not aware of any papers addressing this issue. As a consequence of these remarks, we concentrate in this paper on those refinement equations whose solutions f satisfy $\hat{f}(0) \neq 0$.

2.2. *Generalized Matrix Notation*

The notation of this paper is complicated by the multitude of indices involved. These are of three basic types: one related to the dimension of $\mathbf{R}^d$, a second due to the multiplicity of functions $f_1, \dots, f_r$, and a third related to elements of the lattice Γ. We therefore introduce the following generalized matrix/vector notation, which greatly simplifies the abstract formulation of our results.

Let J and K be finite or countable index sets. If $m_{j,k} \in \mathbf{C}$ for $j \in J$ and $k \in K$, then we say that $M = [m_{j,k}]_{j \in J, k \in K} \in \mathbf{C}^{J \times K}$ is a "$J \times K$ matrix." We also allow block entries. For example, if each $m_{j,k}$ is an $r \times s$ matrix with complex entries then we refer to $M = [m_{j,k}]_{j \in J, k \in K} \in (\mathbf{C}^{r \times s})^{J \times K}$ as a $J \times K$ matrix with $r \times s$ block entries. We say that the block $m_{j,k}$ lies in "row j, column k." Analogues of all the usual matrix definitions and operations apply. For example, the transpose of M is $M^{\mathrm{T}} = [m^{\mathrm{T}}_{j,k}]_{k\in K, j\in J}$. The $J \times J$ identity matrix is $I = [\delta_{i,j}]_{i,j\in J}$, where $\delta_{i,j} = 1$ if $i = j$ and 0 if $i \neq j$.

If $M = [m_{j,k}]_{j\in J, k\in K}$ is a $J \times K$ matrix and $N = [n_{k,\ell}]_{k\in K, \ell\in L}$ is a $K \times L$ matrix such that the product of the block $m_{j,k}$ with the block $n_{k,\ell}$ makes sense, then we define the product of M with N to be the $J \times L$ matrix

$$MN = \left[\sum_{k\in K} m_{j,k}\, n_{k,\ell} \right]_{j\in J, \ell\in L}.$$

All summations encountered in this paper will contain only finitely many nonzero terms, and therefore are always well-defined.

A column vector is a $J \times 1$ matrix, with scalar or block entries. We denote a column vector by $v = [v_j]_{j\in J}$. A row vector is a $1 \times J$ matrix. We use the notation $u = (u_j)_{j\in J}$ to denote a row vector. A row vector is the transpose of a column vector.

2.3. *The Refinement Equation and the Operator L*

Using our generalized matrix notation, we can recast the refinement equation (1.1) as an infinite matrix-vector equation.

A fundamental operator associated with the refinement equation is the $\Gamma \times \Gamma$ matrix L with $r \times r$ block entries c_{Ai-j}, i.e.,

$$L = [\, c_{Ai-j} \,]_{i,j\in\Gamma},$$

where we assume that $c_k = 0$ if $k \notin \Lambda$. Note that only finitely many entries of any given row or column of L are nonzero.

For each $x \in \mathbf{R}^d$, let $F(x)$ be the infinite column vector with $r \times 1$ block entries $f(x+k)$, i.e.,

$$F(x) = [\, f(x+k) \,]_{k\in\Gamma}.$$

Note that for a given x, only finitely many entries $f(x+k)$ of $F(x)$ are nonzero since f has compact support.

If f satisfies the refinement equation (1.1), then

$$\begin{aligned} L\,F(Ax) &= [\, c_{Ai-j} \,]_{i,j\in\Gamma}\, [\, f(Ax+j) \,]_{j\in\Gamma} \\ &= \left[\sum_{j\in\Gamma} c_{Ai-j}\, f(Ax+j) \right]_{i\in\Gamma} \\ &= \left[\sum_{k\in\Gamma} c_k\, f(Ax+Ai-k) \right]_{i\in\Gamma} \\ &= [\, f(x+i) \,]_{i\in\Gamma} \\ &= F(x). \end{aligned}$$

The converse is also true, so the refinement equation (1.1) can be rewritten as

$$L\,F(Ax) \;=\; F(x).$$

We will say that translates of f along Γ are *independent* if for every choice of row vectors $b_k \in \mathbf{C}^{1\times r}$,

$$\sum_{k\in\Gamma} b_k\, f(x+k) = 0 \qquad \Longleftrightarrow \qquad b_k = 0 \text{ for every } k.$$

Equivalently, for every choice of infinite row vector $b = (b_k)_{k\in\Gamma}$ with block entries $b_k \in \mathbf{C}^{1\times r}$,

$$b\,F(x) \;=\; 0 \qquad \Longleftrightarrow \qquad b \;=\; 0.$$

2.4. *Translation and Dilation of Multidimensional Polynomials*

Recall that the number of monomials x^α of degree s is $d_s = \binom{s+d-1}{d-1}$. For each integer $s \geq 0$ we define the vector-valued function $X_{[s]} \colon \mathbf{R}^d \to \mathbf{C}^{d_s}$ by

$$X_{[s]}(x) \;=\; [\, x^\alpha \,]_{|\alpha|=s}\,, \qquad x \in \mathbf{R}^d.$$

In this section we shall define the matrix $A_{[s]}$ and matrix of polynomials $Q_{[s,t]}$ which naturally arise when considering the dilation and translation of the vector of monomials $X_{[s]}$. We will see that these matrices satisfy the fundamental equations

$$\begin{aligned} X_{[s]}(Ax) \;&=\; A_{[s]}\, X_{[s]}(x), \\ X_{[s]}(x-y) \;&=\; \sum_{t=0}^{s} Q_{[s,t]}(y)\, X_{[t]}(x). \end{aligned}$$

Consider the behavior of $X_{[s]}(x)$ under translation by an element $y \in \mathbf{R}^d$. If x^α has degree s, then $(x-y)^\alpha$ as a polynomial in x has degree s and can involve terms x^β for $0 \leq |\beta| \leq s$. Let $q_{\alpha,\beta}(y)$ be the coefficients of this polynomial, i.e.,

$$\begin{aligned} &\sum_{0\leq|\beta|\leq s} q_{\alpha,\beta}(y)\, x^\beta \\ &\qquad = (x-y)^\alpha \\ &\qquad = (x_1-y_1)^{\alpha_1} \cdots (x_d-y_d)^{\alpha_d} \\ &\qquad = \prod_{i=1}^{d} \sum_{\beta_i=0}^{\alpha_i} \binom{\alpha_i}{\beta_i} (-y_i)^{\alpha_i-\beta_i}\, x_i^{\beta_i} \\ &\qquad = \sum_{\beta_1=0}^{\alpha_1} \cdots \sum_{\beta_d=0}^{\alpha_d} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} (-y_1)^{\alpha_1-\beta_1} \cdots (-y_d)^{\alpha_d-\beta_d}\, x_1^{\beta_1} \cdots x_d^{\beta_d}. \end{aligned}$$

In particular, the nonzero terms of $(x-y)^\alpha$ occur only when $\beta \le \alpha$. Therefore, if we set

$$\binom{\alpha}{\beta} = \begin{cases} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}, & \text{if } \beta_i \le \alpha_i \text{ for every } i, \\ 0, & \text{if } \beta_i > \alpha_i \text{ for some } i, \end{cases}$$

then $q_{\alpha,\beta}(y) = (-1)^{|\alpha|-|\beta|} \binom{\alpha}{\beta} y^{\alpha-\beta}$. Thus, each $q_{\alpha,\beta}$ is itself a polynomial, which is either identically zero or is a monomial of degree $|\alpha| - |\beta|$.

For each integer $0 \le t \le s$, define the matrix of polynomials $Q_{[s,t]} \colon \mathbf{R}^d \to \mathbf{C}^{d_s \times d_t}$ by

$$Q_{[s,t]}(y) = [\, q_{\alpha,\beta}(y) \,]_{|\alpha|=s,|\beta|=t} = (-1)^{s-t} \left[\binom{\alpha}{\beta} y^{\alpha-\beta} \right]_{|\alpha|=s,|\beta|=t}. \quad (2.1)$$

Note that each entry of $Q_{[s,t]}$ is either 0 or is a monomial of degree $s-t$. By definition,

$$\begin{aligned} X_{[s]}(x-y) &= [\, (x-y)^\alpha \,]_{|\alpha|=s} \\ &= \left[\sum_{t=0}^{s} \sum_{|\beta|=t} (-1)^{s-t} \binom{\alpha}{\beta} y^{\alpha-\beta} x^\beta \right]_{|\alpha|=s} \\ &= \sum_{t=0}^{s} Q_{[s,t]}(y)\, X_{[t]}(x). \end{aligned}$$

Consider next the behavior of $X_{[s]}(x)$ under dilation by an arbitrary $d \times d$ matrix A. Let $a_{i,j}$ denote the entries of A. If $|\alpha| = s$, then $(Ax)^\alpha$ is still a homogeneous polynomial of degree s, but possibly involving all terms x^β with $|\beta| = s$. Let $a^s_{\alpha,\beta}$ be the coefficients of the polynomial $(Ax)^\alpha$, i.e.,

$$\begin{aligned} \sum_{|\beta|=s} a^s_{\alpha,\beta}\, x^\beta = (Ax)^\alpha &= (Ax)_1^{\alpha_1} \cdots (Ax)_d^{\alpha_d} \\ &= \prod_{i=1}^{d} (a_{i,1}x_1 + \cdots + a_{i,d}x_d)^{\alpha_i}. \end{aligned}$$

Let $A_{[s]}$ be the $d_s \times d_s$ matrix

$$A_{[s]} = [\, a^s_{\alpha,\beta} \,]_{|\alpha|=s,|\beta|=s}.$$

Then we have

$$X_{[s]}(Ax) = [\, (Ax)^\alpha \,]_{|\alpha|=s} = [\, a^s_{\alpha,\beta} \,]_{|\alpha|=s,|\beta|=s} \, [\, x^\alpha \,]_{|\alpha|=s} = A_{[s]}\, X_{[s]}(x).$$

We emphasize that the definition of $A_{[s]}$ is valid for any $d \times d$ matrix A, although we shall apply it most often to the dilation matrix A appearing in the refinement equation (1.1).

As an example, consider $A_{[1]}$. There is a natural ordering of the degree-1 polynomials, namely $x_1, \dots, x_d$. With this ordering, we have $X_{[1]}(x) = (x_1, \dots, x_d)^{\mathrm{T}} = x$. Therefore,

$$A_{[1]}\, X_{[1]}(x) \;=\; X_{[1]}(Ax) \;=\; \begin{bmatrix} (Ax)_1 \\ \vdots \\ (Ax)_d \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1}x_1 + \cdots + a_{1,d}x_d \\ \vdots \\ a_{d,1}x_1 + \cdots + a_{d,d}x_d \end{bmatrix} \;=\; A\, X_{[1]}(x).$$

Thus $A_{[1]} = A$ with this ordering.

2.5. *Some Special Matrices and Polynomial Functions*

Given a collection

$$\{v_\alpha = (v_{\alpha,1}, \dots, v_{\alpha,r}) \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$$

of row vectors of length r, we shall associate a number of special matrices and functions. These play an important role in our analysis of accuracy. We use the notation of this section extensively throughout the paper.

We group the v_α by degree to form $d_s \times 1$ column vectors $v_{[s]}$ with block entries that are the $1 \times r$ row vectors v_α. Specifically, we set

$$v_{[s]} \;=\; [v_\alpha]_{|\alpha|=s}, \qquad 0 \le s < p.$$

Thus $v_{[s]} \in (\mathbf{C}^{1\times r})^{d_s \times 1}$. Alternatively, we could view $v_{[s]}$ as a $d_s \times r$ matrix

$$v_{[s]} \;=\; \begin{bmatrix} v_{\alpha_1,1} & \cdots & v_{\alpha_1,r} \\ \vdots & \ddots & \vdots \\ v_{\alpha_{d_s},1} & \cdots & v_{\alpha_{d_s},r} \end{bmatrix}.$$

However, the block viewpoint for $v_{[s]}$, and for the other vectors defined below, is especially convenient for our analysis. Note that $v_{[0]} = [v_0] = v_0$, since $\alpha = 0$ is the only multi-index of degree 0.

We shall now define several vector-valued or matrix-valued functions, each of whose entries is a polynomial. In general, if $u = [u_{j,k}]_{j\in J, k\in K}\colon \mathbf{R}^d \to \mathbf{C}^{J\times K}$ and each $u_{j,k}\colon \mathbf{R}^d \to \mathbf{C}$ is a polynomial, then we will say that u is a *matrix of polynomials*. The *degree* of u is the maximum degree of the $u_{j,k}$, i.e., $\deg(u) = \max\{\deg(u_{j,k})\}_{j\in J, k\in K}$.

For each α, we define a row vector of polynomials $y_\alpha : \mathbf{R}^d \to \mathbf{C}^{1\times r}$ by

$$\begin{aligned} y_\alpha(x) &= \sum_{0\le\beta\le\alpha} (-1)^{|\alpha|-|\beta|} \binom{\alpha}{\beta} v_\beta \, x^{\alpha-\beta} \\ &= \sum_{0\le\beta\le\alpha} \left((-1)^{|\beta|} \binom{\alpha}{\alpha-\beta} v_{\alpha-\beta} \right) x^\beta. \end{aligned} \tag{2.2}$$

Note that if we write $y_\alpha(x) = (y_{\alpha,1}(x), \dots, y_{\alpha,r}(x))$, then the coefficients of the polynomial $y_{\alpha,i}$ are completely determined by the scalars $v_{\beta,i}$ for those β with $0 \le \beta \le \alpha$. Further, $\deg(y_\alpha) = \max\{\deg(y_{\alpha,1}), \dots, \deg(y_{\alpha,r})\} \le |\alpha|$, and

$$\deg(y_\alpha) = |\alpha| \quad \iff \quad v_0 \ne 0.$$

Note that $y_\alpha(0) = v_\alpha$, and that $y_0(x) = v_0$ for every x.

As with the vectors v_α, we collect the vectors of polynomials y_α by degree and arrange them as block entries in a column vector to form the matrix of polynomials $y_{[s]} : \mathbf{R}^d \to (\mathbf{C}^{1\times r})^{d_s \times 1}$. Specifically,

$$\begin{aligned} y_{[s]}(x) &= [\, y_\alpha(x) \,]_{|\alpha|=s} \\ &= \left[\sum_{t=0}^{s} \sum_{|\beta|=t} (-1)^{s-t} \binom{\alpha}{\beta} x^{\alpha-\beta} \, v_\beta \right]_{|\alpha|=s} \\ &= \sum_{t=0}^{s} Q_{[s,t]}(x) \, v_{[t]}. \end{aligned} \tag{2.3}$$

Thus, for a given x, $y_{[s]}(x)$ is a $d_s \times 1$ column vector with block entries that are the $1 \times r$ row vectors $y_\alpha(x)$. Note that the coefficients of the matrix of polynomials $y_{[s]}$ are entirely determined by the matrices $v_{[t]}$ for $0 \le t \le s$. In addition, $y_{[s]}(0) = v_{[s]}$, and $y_{[0]}(x) = y_0(x) = v_0$ for every x. Moreover, $\deg(y_{[s]}) \le s$, and $\deg(y_{[s]}) = s$ if and only if $v_0 \ne 0$.

Finally, for each x we collect the blocks $y_{[s]}(x+k)$ into an infinite row vector to form a function $Y_{[s]} : \mathbf{R}^d \to \big((\mathbf{C}^{1\times r})^{d_s\times 1}\big)^{1\times\Gamma}$. Specifically,

$$Y_{[s]}(x) = \big(y_{[s]}(x+k)\big)_{k\in\Gamma}. \tag{2.4}$$

We adopt the convention that

$$Y_{[s]} = Y_{[s]}(0) = \big(y_{[s]}(k)\big)_{k\in\Gamma}.$$

Thus $Y_{[s]}$ is the row vector of evaluations of the matrix of polynomials $y_{[s]}$ at lattice points. Note that since $y_{[0]}(x) = v_0$ for every x, $Y_{[0]}(x)$ is the "constant" vector $Y_{[0]}(x) = (v_0)_{k\in\Gamma}$.

EXAMPLE 2.1. Consider the above definitions in the one-dimensional case. If $d = 1$ then $d_s = 1$ for every s, since there is a single polynomial x^s of degree s. In this case,

$$\begin{aligned}
&v_\alpha = v_s \in \mathbf{C}^{1\times r}, \\
&v_{[s]} = [v_\alpha]_{|\alpha|=s} = v_s \in \mathbf{C}^{1\times r}, \\
&y_s(x) = \sum_{t=0}^{s} (-1)^{s-t} \binom{s}{t} x^{s-t}\, v_t \quad \text{maps } \mathbf{R} \to \mathbf{C}^{1\times r}, \\
&y_{[s]}(x) = [y_\alpha(x)]_{|\alpha|=s} = y_s(x) \qquad \text{maps } \mathbf{R} \to \mathbf{C}^{1\times r}, \\
&Y_{[s]}(x) = \bigl(y_s(x+k)\bigr)_{k\in\Gamma} \qquad \text{maps } \mathbf{R} \to (\mathbf{C}^{1\times r})^{1\times\Gamma}.
\end{aligned}$$

In particular, $Y_{[s]}(x)$ is an infinite row vector whose entries are the $1 \times r$ row vectors $y_s(x + k)$. Moreover, since $\Gamma = b\mathbf{Z}$ for some constant b, $Y_{[s]}(x)$ is simply an "ordinary" infinite row vector of the form

$$Y_{[s]}(x) \;=\; (\,\cdots\,, \quad y_s(x-b), \quad y_s(x), \quad y_s(x+b), \quad \cdots),$$

with blocks $y_s(x + kb)$ that are ordinary $1 \times r$ row vectors. $\square$

3. STATEMENT OF RESULTS

3.1. *Results for Arbitrary Functions*

Our initial result states that for arbitrary (not necessarily refinable) functions f with independent translates, the coefficients that are used to reconstruct the polynomials x^α from translates of f are themselves polynomials evaluated at lattice points. This result can also be viewed as a restatement of the Strang–Fix conditions for multiple functions in higher dimensions.

THEOREM 3.1. *Assume that $f\colon \mathbf{R}^d \to \mathbf{C}^r$ is compactly supported, and that translates of f along Γ are independent.*

If f has accuracy p, then there exists a collection $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$ of row vectors such that

(i) $v_0 \neq 0$, *and*

(ii) $X_{[s]}(x) \;=\; \sum_{k\in\Gamma} y_{[s]}(k)\, f(x+k) \;=\; Y_{[s]}\, F(x)$ *for* $0 \le s < p$,

where $Y_{[s]} = \big(y_{[s]}(k)\big)_{k\in\Gamma}$ *is the row vector of evaluations at lattice points of the matrix of polynomials* $y_{[s]}(x) = \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}$ *defined by* (2.3).

In particular, if q *is any polynomial with* $\deg(q) < p$*, then there exists a unique row vector of polynomials* $u_q \colon \mathbf{R}^d \to \mathbf{C}^{1\times r}$*, with* $\deg(u_q) = \deg(q)$*, such that*

$$q(x) \;=\; \sum_{k\in\Gamma} u_q(k)\, f(x+k).$$

Note that since translates of f are assumed to be independent, the coefficients $y_{[s]}(k)$ in statement (ii) of Theorem 3.1 are unique.

Since $X_{[s]}(x) = [x^\alpha]_{|\alpha|=s}$ and $y_{[s]} = [y_\alpha]_{|\alpha|=s}$, it follows from statement (ii) of Theorem 3.1 that the individual polynomials x^α are obtained from translates of f by the formula $x^\alpha = \sum_{k\in\Gamma} y_\alpha(k)\, f(x+k)$, where $y_\alpha(x) = \sum_{0\le\beta\le\alpha} (-1)^{|\alpha|-|\beta|} \binom{\alpha}{\beta}\, x^{\alpha-\beta}\, v_\beta$ is the row vector of polynomials defined in (2.2). Since $v_0 \ne 0$, $\deg(y_\alpha) = |\alpha|$ and $\deg(y_{[s]}) = s$.

Remark 3.2. Suppose that $f \colon \mathbf{R}^d \to \mathbf{C}^r$ is compactly supported with independent translates, and has accuracy p. Let $\Pi_{p,r}$ be the space of all row vectors of polynomials $u \colon \mathbf{R}^d \to \mathbf{C}^{1\times r}$ with $\deg(u) < p$. Then Theorem 3.1 states that the linear mapping $T \colon \Pi_{p,1} \to \Pi_{p,r}$ defined by $T(q) = u_q$ is injective and preserves degree. The dimensions of $\Pi_{p,1}$ and $\Pi_{p,r} = \Pi_{p,1} \times \cdots \times \Pi_{p,1}$ are equal only when $r = 1$. Therefore T is surjective if and only if $r = 1$. As a consequence, in the PSI case ($r = 1$), for each polynomial $u \in \Pi_{p,1}$ we have that the function $q(x) = \sum_{k\in\Gamma} u(k)\, f(x+k)$ is itself a multivariate polynomial with $\deg(q) = \deg(u)$.

However, T cannot be surjective when $r > 1$. As a consequence, there must exist polynomials $u \in \Pi_{p,r}$ such that $q(x) = \sum_{k\in\Gamma} u(k)\, f(x+k)$ is not a polynomial. To construct a specific example, consider any constant vector of polynomials $u(x) \equiv u_0 \in \mathbf{C}^{1\times r}$. If $q(x) = \sum_{k\in\Gamma} u(k)\, f(x+k)$ is a polynomial, then we must have $\deg(q) = \deg(u) = 0$. Thus q is constant. However, translates of f are independent, so this implies that u_0 is a multiple of v_0. Therefore, if u_0 is not a multiple of v_0 then q cannot be a polynomial. □

The following result states that, regardless of whether f has accuracy p or not, if any monomial x^α can be reproduced from lattice translates of f using coefficients that are themselves polynomials evaluated at lattice points, then the monomial x^β can also be reproduced from translates of f for each $0 \le \beta \le \alpha$. Moreover, the coefficients used to obtain x^β are the evaluations at lattice points of a constant times the $\alpha - \beta$ derivative of the coefficients used to obtain x^α. This result can also be obtained by using Appell polynomials. We make no use of this result in the sequel.

THEOREM 3.3. *Assume that* $f\colon \mathbf{R}^d \to \mathbf{C}^r$ *is compactly supported, and let* α *be any multi-index. If* $u\colon \mathbf{R}^d \to \mathbf{C}^{1\times r}$ *is a row vector of polynomials such that*

$$x^\alpha \;=\; \sum_{k\in\Gamma} u(k)\, f(x+k),$$

then for each $0 \le \beta \le \alpha$,

$$x^\beta \;=\; C_\beta \sum_{k\in\Gamma} (D^{\alpha-\beta}u)(k)\, f(x+k),$$

where

$$D^\gamma u \;=\; \left(\frac{\partial^{|\gamma|}}{\partial x^\gamma}u_1, \dots, \frac{\partial^{|\gamma|}}{\partial x^\gamma}u_r\right)$$

and

$$C_\gamma \;=\; (-1)^{|\alpha-\gamma|}\,\frac{\gamma!}{\alpha!} \;=\; (-1)^{|\alpha-\gamma|}\,\frac{\gamma_1!}{\alpha_1!}\cdots\frac{\gamma_d!}{\alpha_d!}.$$

3.2 *Results for Refinable Functions*

The following result gives necessary and/or sufficient conditions for a refinable function to have accuracy p.

THEOREM 3.4. *Assume that* $f\colon \mathbf{R}^d \to \mathbf{C}^r$ *satisfies the refinement equation* (1.1), *and that* f *is integrable and compactly supported. Consider the following statements.*

(I) f *has accuracy* p.

(II) *There exists a collection of row vectors* $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$ *such that*

  (i) $v_0 \hat{f}(0) \neq 0$, *and*

  (ii) $Y_{[s]} = A_{[s]}\, Y_{[s]}\, L$ *for* $0 \le s < p$,

where $Y_{[s]} = \big(y_{[s]}(k)\big)_{k\in\Gamma}$ *is the row vector of evaluations at lattice points of the matrix of polynomials* $y_{[s]}(x) = \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}$ *defined by* (2.3).

Then we have the following.

(a) *If translates of* f *along* Γ *are independent, then statement* (I) *implies statement* (II).

(b) *Statement* (II) *implies statement* (I). *In this case, if we scale all the vectors* v_α *by the factor* $C = (v_0 \hat{f}(0))^{-1}\,|P|$, *then*

$$X_{[s]}(x) \;=\; \sum_{k\in\Gamma} y_{[s]}(k)\, f(x+k) \;=\; Y_{[s]}\, F(x), \quad 0 \le s < p. \tag{3.1}$$

Recall that $Q_{[s,t]}$ is a matrix of polynomials, and that $\deg(Q_{[s,t]}) = s-t$. Hence $\deg(y_{[s]}) = s$ if and only if $v_0 \neq 0$. In particular, the hypothesis $v_0\hat{f}(0) \neq 0$ in statement (II) of Theorem 3.4 implies that $v_0 \neq 0$, and therefore that $y_{[s]}$ has degree s.

Remark 3.5. Let us comment on the significance of hypothesis (i) in statement (II) of Theorem 3.4, which states that $v_0\hat{f}(0) \neq 0$. In the proof of Theorem 3.4(b), we will see that the $s = 0$ case of hypothesis (ii) in statement (II) implies that $G_{[0]}(x) = \sum_{k\in\Gamma} v_0\, f(x+k) = C$, a constant. Therefore f will have accuracy at least $p = 1$ if $C \neq 0$. By integrating $G_{[0]}$ over the fundamental domain P, we show that the value of C is $C = (v_0\hat{f}(0))\,|P|^{-1}$. Hence the constant polynomial 1 is reproduced from translates of f if $v_0\hat{f}(0) \neq 0$.

It is apparent then that we could replace the hypothesis $v_0\hat{f}(0) \neq 0$ by the hypothesis that $G_{[0]}(x) = \sum_{k\in\Gamma} v_0\, f(x+k)$ does not vanish everywhere. This version of the hypothesis might be advantageous if f is known to be continuous, for then it suffices to show that $G_{[0]}(x) \neq 0$ for a single x. On the other hand, the vectors v_0 and $\hat{f}(0)$ can be computed directly from the matrices c_k in many cases. As discussed in Section 2.1, $\hat{f}(0)$ is a right 1-eigenvector of the matrix $\Delta = (1/m)\sum_{k\in\Lambda} c_k$. As a consequence of Theorem 3.6 below, which gives some equivalent formulations of hypothesis (ii) in statement (II), the vector v_0 is a left 1-eigenvector of the same matrix Δ. If the eigenvalue 1 for Δ is simple, which is the case in most practical examples, then v_0 and $\hat{f}(0)$ are uniquely determined up to scalar multiples. Moreover, in this case we automatically have $v_0\hat{f}(0) \neq 0$ since the product of the left and right 1-eigenvectors of a matrix is nonzero when the eigenvalue 1 is simple. These facts are made explicit in Theorem 3.9. □

The following result gives several equivalent formulations of requirement (ii) in statement (II) of Theorem 3.4.

THEOREM 3.6. *Let $m = |\det(A)|$, and let $d_1, \dots, d_m \in \Gamma$ be a full set of digits. Set $\Gamma_i = A(\Gamma) - d_i$.*

Given a collection $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \leq |\alpha| < p\}$ of row vectors, let $y_{[s]}(x) = \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}$ be the matrix of polynomials defined by (2.3) and let $Y_{[s]} = \bigl(y_{[s]}(k)\bigr)_{k\in\Gamma}$ be the row vector of evaluations of this polynomial at lattice points.

If $v_0 \neq 0$, then the following statements are equivalent.

(a) $Y_{[p-1]} = A_{[p-1]}\, Y_{[p-1]}\, L$.

(b) $Y_{[s]} = A_{[s]}\, Y_{[s]}\, L$ *for* $0 \leq s < p$.

(c) $v_{[s]} \;=\; \sum_{k\in\Gamma_i} \sum_{t=0}^{s} Q_{[s,t]}(k)\, A_{[t]}\, v_{[t]}\, c_k$ *for* $0 \leq s < p$ *and* $i = 1, \dots, m$.

Since only finitely many c_k are nonzero, the summations in statement (c) of Theorem 3.6 are all finite.

Note that if $s = 0$ then statement (c) in Theorem 3.6 reduces to the requirement that $v_0 = v_0 \sum_{k\in\Gamma_i} c_k$ for $i = 1, \dots, m$. Since Γ is the disjoint union of the Γ_i, this implies that $v_0 = v_0\Delta$, where $\Delta = (1/m) \sum_{k\in\Lambda} c_k$. Hence v_0 is a left 1-eigenvector of Δ.

An important implication of statement (c) in Theorem 3.6 is that the vectors v_α are determined directly by the matrices c_k and can be computed without explicit knowledge of f. These vectors determine the coefficients $y_{[s]}(k)$ needed to reproduce the vector of monomials $X_{[s]}(x)$ from translates of f. Hence these coefficients can be derived directly from the matrices c_k.

Statement (c) of Theorem 3.6 is a finite system of linear equations, stated in terms of the collections $v_{[s]} = [v_\alpha]_{|\alpha|=s}$. If desired, it is possible to rewrite this system in terms of the v_α themselves, by simply writing out the entries of both sides of the equations. If we do this, then we have the following form of statement (c):

$$v_\alpha \;=\; \sum_{k\in\Gamma_i} \sum_{t=0}^{s} \sum_{|\beta|=t} \sum_{|\gamma|=t} (-1)^{s-t} \binom{\alpha}{\beta} k^{\alpha-\beta}\, a^t_{\beta,\gamma}\, v_\gamma\, c_k, \qquad \begin{cases} 0 \le s < p, \\ |\alpha| = s, \\ i = 1, \dots, m. \end{cases}$$

We shall refer to either form of this system of equations as the *sum rules.*

Note that in either form, this system has a block triangular structure since the equation for $v_{[s]}$ involves only $v_{[0]}, \dots, v_{[s]}$. Hence the system can be checked recursively, i.e., $v_{[s+1]}$ is solved for after $v_{[0]}, \dots, v_{[s]}$ have been found.

In the case of a single refinable function ($r = 1$), the coefficients c_k in the refinement equation are simply scalars. Hence, they commute with any matrix or vector. This leads to the following dramatic simplification of the sum rules.

THEOREM 3.7. *Assume that* $r = 1$. *Let* $m = |\det(A)|$, *and let* $d_1, \dots, d_m \in \Gamma$ *be a full set of digits. Set* $\Gamma_i = A(\Gamma) - d_i$. *Then the following statements are equivalent.*

(a) *There exists a collection of scalars* $\{v_\alpha \in \mathbf{C} : 0 \le |\alpha| < p\}$ *so that* $v_0 \ne 0$ *and the equivalent statements* (a)–(c) *in Theorem 3.6 hold.*

(b) $\displaystyle\sum_{k\in\Gamma} c_k = m$ *and* $\displaystyle\sum_{k\in\Gamma_1} k^\alpha c_k = \cdots = \sum_{k\in\Gamma_m} k^\alpha c_k$ *for* $0 \le |\alpha| < p$.

Note that statement (b) in Theorem 3.7 for the case $d = 1$, $r = 1$, $A = 2$, $\Gamma = \mathbf{Z}$, and $\Lambda \subset \{0, \dots, N\}$ precisely yields the sum rules in (1.3).

EXAMPLE 3.8. The following example shows that the hypothesis $r = 1$ in Theorem 3.7 is necessary. Set $d = 1$, $r = 2$, $A = 2$, $\Gamma = \mathbf{Z}$, $\Lambda = \{0, 1, 2\}$, and

$$c_0 \;=\; \begin{bmatrix} 1/2 & 3/4 \\ -1/8 & -1/8 \end{bmatrix}, \qquad c_1 \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad c_2 \;=\; \begin{bmatrix} 1/2 & -3/4 \\ 1/8 & -1/8 \end{bmatrix}.$$

The solution to this refinement equation is a cubic finite element pair with accuracy $p = 4$ [HSS96]. We do have the $s = 0$ requirement $c_0 + c_2 = c_1$, but the $s = 1$ requirement fails since $0c_0 + 2c_2 \neq 1c_1$. Thus statement (b) in Theorem 3.7 is not valid for $p = 2$, hence is certainly not valid for $p = 4$. However, the vectors $v_0 = (1, 0)$, $v_1 = (1, 1)$, $v_2 = (1, 2)$, $v_3 = (1, 3)$ satisfy statement (c) in Theorem 3.6 with $p = 4$. $\square$

Statement (II) in Theorem 3.4 includes the requirement that $v_0 \hat{f}(0) \neq 0$. The following theorem gives some sufficient conditions on f or directly on the matrices c_k so that $v_0 \neq 0$ implies $v_0 \hat{f}(0) \neq 0$.

THEOREM 3.9. *Assume that $f\colon \mathbf{R}^d \to \mathbf{C}^r$ satisfies the refinement equation* (1.1), *and that f is integrable and compactly supported. Let $m = |\det(A)|$, and let $d_1, \dots, d_m \in \Gamma$ be a full set of digits. Assume that $v_0 \in \mathbf{C}^{1\times r}$ satisfies statement* (c) *in Theorem 3.6 for the case $s = 0$, i.e.,*

$$v_0 \;=\; v_0 \sum_{k \in \Gamma_i} c_k, \qquad i = 1, \dots, m. \tag{3.2}$$

If $v_0 \neq 0$, then either of the following two conditions is sufficient to imply that $v_0 \hat{f}(0) \neq 0$, and therefore that f has accuracy at least $p = 1$:

(a) *translates of f along Γ are independent, or*
(b) *the matrix $\Delta = (1/m) \sum_{k \in \Lambda} c_k$ has eigenvalues $\lambda_1 = 1$ and $|\lambda_2|, \dots, |\lambda_r| < 1$.*

Remark 3.10. Note that (3.2) states that v_0 is a left 1-eigenvector for each of the matrices $\Delta_i = \sum_{k \in \Gamma_i} c_k$. In the PSI case, v_0 is a nonzero scalar, so this requirement reduces to $\Delta_i = 1$ for $i = 1, \dots, m$. Hence $\Delta = (1/m) \sum_{k \in \Lambda} c_k = (1/m) \sum_{i=1}^{m} \Delta_i = 1$, so condition (b) in Theorem 3.9 is an immediate consequence of (3.2) when $r = 1$. $\square$

By combining parts of Theorems 3.4, 3.6, and 3.9, we immediately conclude the following simple statement of necessary and sufficient conditions for accuracy in the case where f has independent translates.

THEOREM 3.11. *Assume that $f\colon \mathbf{R}^d \to \mathbf{C}^r$ satisfies the refinement equation* (1.1), *that f is integrable and compactly supported, and that translates of f along Γ are independent. Then the following two statements are equivalent.*

(a) *f has accuracy p.*

(b) *There exists a collection of row vectors $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$ so that $v_0 \ne 0$ and the equivalent statements* (a)–(c) *in Theorem 3.6 hold.*

If $r = 1$ and $d_1, \dots, d_m \in \Gamma$ is a full set of digits, then these two statements are further equivalent to the following statement.

(c) $\displaystyle\sum_{k\in\Gamma} c_k = m$ *and* $\displaystyle\sum_{k\in\Gamma_1} k^\alpha c_k = \cdots = \sum_{k\in\Gamma_m} k^\alpha c_k$ *for* $0 \le |\alpha| < p$.

Remark 3.12. Refinable functions whose translates along Γ are orthonormal play a key role in the construction of orthonormal wavelet bases for $L^2(\mathbf{R}^d)$. In this case there is an interesting relationship between the vectors v_α and the moments $m_{\alpha,i} = \int x^\alpha f_i(x)\,dx$. In particular, if f has accuracy p and orthonormal translates, then for each $|\alpha| < p$ we have from Theorem 3.11 that $x^\alpha = \sum_{k\in\Gamma} y_\alpha(k)\, f(x+k)$, where $y_\alpha = (y_{\alpha,1}, \dots, y_{\alpha,d})$ is the row vector of polynomials defined in (2.2). Since x^α is real-valued, we therefore have

$$\begin{aligned}
m_{\alpha,i} &= \int_{\mathbf{R}^d} \overline{x^\alpha}\, f_i(x)\,dx \\
&= \int_{\mathbf{R}^d} \sum_{k\in\Gamma} \overline{y_\alpha(k)\, f(x+k)}\, f_i(x)\,dx \\
&= \sum_{k\in\Gamma} \sum_{j=1}^{r} \overline{y_{\alpha,j}(k)} \int_{\mathbf{R}^d} \overline{f_j(x+k)}\, f_i(x)\,dx \\
&= \overline{y_{\alpha,i}(0)} \;=\; \overline{v_{\alpha,i}}.
\end{aligned}$$

Thus, if $m_\alpha = (m_{\alpha,1}, \dots, m_{\alpha,d})$ is the row vector of the αth moments of $f_1, \dots, f_r$, then $v_\alpha = \overline{m}_\alpha$. □

4. PROOFS

4.1. *Preliminary Lemmas*

We will prove a number of useful lemmas in this section.

First, we derive some properties of the matrix of polynomials $Q_{[s,t]}$.

LEMMA 4.1. (a) *$Q_{[s,s]}(y) = I$.*

(b) *$Q_{[s,0]}(y) = (-1)^s\, X_{[s]}(y) = (-1)^s\, [y^\alpha]_{|\alpha|=s}$.*

(c) *If $0 \le t \le s$, then*

$$Q_{[s,t]}(x+y) \;=\; \sum_{u=t}^{s} Q_{[s,u]}(y)\, Q_{[u,t]}(x). \tag{4.1}$$

Proof. Parts (a) and (b) follow from definition of $Q_{[s,t]}(y)$ in (2.1).

(c) For each x, y, $z \in \mathbf{R}^d$ we have

$$\begin{aligned} \sum_{t=0}^{s} Q_{[s,t]}(x+y)\, X_{[t]}(z) &= X_{[s]}(z-(x+y)) \\ &= \sum_{t=0}^{s} Q_{[s,t]}(y)\, X_{[t]}(z-x) \\ &= \sum_{t=0}^{s} Q_{[s,t]}(y) \sum_{u=0}^{t} Q_{[t,u]}(x)\, X_{[u]}(z) \\ &= \sum_{u=0}^{s} \sum_{t=u}^{s} Q_{[s,t]}(y)\, Q_{[t,u]}(x)\, X_{[u]}(z) \\ &= \sum_{t=0}^{s} \sum_{u=t}^{s} Q_{[s,u]}(y)\, Q_{[u,t]}(x)\, X_{[t]}(z), \end{aligned} \tag{4.2}$$

where we have interchanged the order of summation and then relabeled the summation indices. For each fixed x and y, the only way that the first vector of polynomials in z in (4.2) can equal the last vector of polynomials in (4.2) is for the coefficients of these polynomials to be identical, which happens if (4.1) holds. □

Next, we derive some properties of the matrices $A_{[s]}$.

LEMMA 4.2. *Let A, B be arbitrary $d \times d$ matrices. Then the following statements hold.*

(a) *If $d = 1$ (so A is scalar), then $A_{[s]} = A^s$.*
(b) *$A_{[0]} = 1$ (scalar), and $A_{[1]} = A$.*
(c) *$(AB)_{[s]} = A_{[s]}\, B_{[s]}$. Hence, if A is invertible then so is $A_{[s]}$, and $(A_{[s]})^{-1} = (A^{-1})_{[s]}$.*
(d) *Let $\lambda = (\lambda_1, \dots, \lambda_d)^{\mathrm{T}}$ be the vector of eigenvalues of A. Then the eigenvalues of $A_{[s]}$ are $[\lambda^\alpha]_{|\alpha|=s}$.*
(e) *If A is expansive and $s > 0$, then $A_{[s]}$ is expansive.*

Proof. (a), (b) Trivial.

(c) This ollows from

$$(AB)_{[s]}\, X_{[s]}(x) \;=\; X_{[s]}(ABx) \;=\; A_{[s]}\, X_{[s]}(Bx) \;=\; A_{[s]}\, B_{[s]}\, X_{[s]}(x).$$

(d) Although we usually regard the real matrix A as operating on real Euclidean space $\mathbf{R}^d$ in this paper, to compute its eigenvalues we regard it as operating on complex Euclidean space $\mathbf{C}^d$.

Recall that the number of multi-indices of degree s is $d_s = \binom{s+d-1}{d-1}$. Thus $A_{[s]}$ is a $d_s \times d_s$ matrix. Since any given ordering of the multi-indices of degree s corresponds to a particular choice of basis for $\mathbf{C}^{d_s}$, it is clear that the eigenvalues of $A_{[s]}$ do not depend on the choice of ordering used. In addition, the eigenvalues of $A_{[s]}$ are independent of the basis for $\mathbf{C}^d$ in which the matrix A is expressed, for if B is invertible, then $(BAB^{-1})_{[s]} = B_{[s]}\, A_{[s]}\, B_{[s]}^{-1}$.

Therefore, choose a basis for $\mathbf{C}^d$ in which A is lower-triangular. We will show that $A_{[s]}$ is also lower-triangular when we impose the following linear ordering of the multi-indices of degree s:

$$\alpha \preceq \beta \iff \begin{cases} \alpha = \beta \quad \text{or} \\ \exists\, k \text{ such that } \alpha_1 = \beta_1, \dots, \alpha_k = \beta_k, \alpha_{k+1} > \beta_{k+1}. \end{cases} \tag{4.3}$$

Denote the entries of A by $a_{i,j}$. Since A is lower-triangular, the entries $a^s_{\alpha,\beta}$ of $A_{[s]}$ satisfy

$$\sum_{|\beta|=s} a^s_{\alpha,\beta}\, x^\beta \;=\; (Ax)^\alpha \;=\; (a_{1,1}x_1)^{\alpha_1} (a_{2,1}x_1 + a_{2,2}x_2)^{\alpha_2} \cdots$$
$$\times\, (a_{d,1}x_1 + \cdots + a_{d,d}x_d)^{\alpha_d}. \tag{4.4}$$

Let $|\beta| = s$. If $\beta = \alpha$ then we must have

$$a^s_{\alpha,\alpha}\, x^\alpha \;=\; (a_{1,1}x_1)^{\alpha_1} \cdots (a_{d,d}x_d)^{\alpha_d} \;=\; \lambda^\alpha x^\alpha,$$

where $\lambda = (\lambda_1, \dots, \lambda_d)^{\mathrm{T}} = (a_{1,1}, \dots, a_{d,d})^{\mathrm{T}}$ is the vector of diagonal entries of A. In particular, the diagonal entries of $A_{[s]}$ are λ^α for $|\alpha| = s$. On the other hand, if $\beta \neq \alpha$, then $k = \max\{i : \alpha_i = \beta_i\} < d-1$. To obtain the term $a^s_{\alpha,\beta}\, x^\beta$ in the left-hand side of (4.4), we are forced to choose in the right-hand side of (4.4) the terms $(a_{1,1}x_1)^{\alpha_1}, \dots, (a_{k,k}x_k)^{\alpha_k}$ out of the first k factors, and to choose no terms involving $x_1, \dots, x_k$ out of the remaining $d-k$ factors. Hence, if $a^s_{\alpha,\beta}\, x^\beta$ is nonzero, then it must be one term resulting from the expansion of the product

$$(a_{1,1}x_1)^{\alpha_1} \cdots (a_{k,k}x_k)^{\alpha_k} (a_{k+1,k+1}x_{k+1})^{\alpha_{k+1}} \cdots$$
$$\times\, (a_{d,k+1}x_{k+1} + \cdots + a_{d,d}x_d)^{\alpha_d}.$$

As a consequence, $\beta_{k+1} \geq \alpha_{k+1}$, and therefore, by definition of k, $\beta_{k+1} > \alpha_{k+1}$.

Thus, $a^s_{\alpha,\beta}$ can only be nonzero when $\beta \preceq \alpha$. Therefore $A_{[s]}$ is lower-triangular in this ordering, and the values λ^α appear on the diagonal.

(e) This follows immediately from (d). $\square$

If B is an invertible matrix then we have the following connection between $B_{[s]}$ and $Q_{[s,t]}$.

LEMMA 4.3. *Let B be an invertible matrix. If $0 \le t \le s$ and $y \in \mathbf{R}^d$, then*

$$Q_{[s,t]}(By) \;=\; B_{[s]}\, Q_{[s,t]}(y)\, B_{[t]}^{-1}.$$

Proof. For each x, $y \in \mathbf{R}^d$ we have

$$\begin{aligned}
\sum_{t=0}^{s} Q_{[s,t]}(By)\, X_{[t]}(x) &= X_{[s]}(x - By) \\
&= B_{[s]}\, X_{[s]}(B^{-1}x - y) \\
&= B_{[s]} \sum_{t=0}^{s} Q_{[s,t]}(y)\, X_{[t]}(B^{-1}x) \\
&= \sum_{t=0}^{s} B_{[s]}\, Q_{[s,t]}(y)\, B_{[t]}^{-1}\, X_{[t]}(x). \quad \square
\end{aligned}$$

Next, we consider the behavior under translation of the matrix of polynomials $y_{[s]}$.

LEMMA 4.4. *Given a collection $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$ of row vectors, let $y_{[s]}(x) = \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}$ be the matrix of polynomials defined by* (2.3). *Then*

$$y_{[s]}(x+y) \;=\; \sum_{t=0}^{s} Q_{[s,t]}(y)\, y_{[t]}(x).$$

As a consequence, if $Y_{[s]}(x) = \bigl(y_{[s]}(x+k)\bigr)_{k\in\Gamma}$, then

$$Y_{[s]}(x+y) \;=\; \sum_{t=0}^{s} Q_{[s,t]}(y)\, Y_{[t]}(x).$$

Proof. We compute

$$\begin{aligned}
y_{[s]}(x+y) &= \sum_{t=0}^{s} Q_{[s,t]}(x+y)\, v_{[t]} && \text{by (2.3)} \\
&= \sum_{t=0}^{s} \sum_{u=t}^{s} Q_{[s,u]}(y)\, Q_{[u,t]}(x)\, v_{[t]} && \text{by Lemma 4.1(c)} \\
&= \sum_{u=0}^{s} \sum_{t=0}^{u} Q_{[s,u]}(y)\, Q_{[u,t]}(x)\, v_{[t]} && \text{interchanging summations} \\
&= \sum_{u=0}^{s} Q_{[s,u]}(y)\, y_{[u]}(x) && \text{by (2.3).} \quad \square
\end{aligned}$$

4.2. *Proof of Theorem* 3.1

Proof of Theorem 3.1. Since f has accuracy p, there exist row vectors $w_{\alpha,k} \in C^{1\times r}$ so that each polynomial x^α with degree $0 \le |\alpha| < p$ can be written

$$x^\alpha \;=\; \sum_{k\in\Gamma} w_{\alpha,k}\, f(x+k) \text{ a.e.}$$

For each $k \in \Gamma$, group the vectors $w_{\alpha,k}$ by degree to form the column vectors

$$w_{[s]}(k) \;=\; [w_{\alpha,k}]_{|\alpha|=s}.$$

Then, for each $\ell \in \Gamma$ define the infinite row vector

$$W_{[s]}(\ell) \;=\; \bigl(w_{[s]}(k+\ell)\bigr)_{k\in\Gamma}.$$

Next, set $v_\alpha = w_{\alpha,0}$, and, following the notation of Section 2.5, define the vectors $v_{[s]}$ and matrix of polynomials $y_{[s]}$ by

$$v_{[s]} \;=\; [v_\alpha]_{|\alpha|=s} \qquad \text{and} \qquad y_{[s]}(x) \;=\; \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}.$$

Then, considering the polynomials x^α by degree, we have for $0 \le s < p$ that

$$\begin{aligned} X_{[s]}(x) \;&=\; [\,x^\alpha\,]_{|\alpha|=s} \\ &=\; \Biggl[\sum_{k\in\Gamma} w_{\alpha,k}\, f(x+k)\Biggr]_{|\alpha|=s} \\ &=\; \sum_{k\in\Gamma} w_{[s]}(k)\, f(x+k) \\ &=\; W_{[s]}(0)\, F(x). \end{aligned}$$

Therefore, for each $\ell \in \Gamma$,

$$\begin{aligned} W_{[s]}(\ell)\, F(x) \;&=\; W_{[s]}(0)\, F(x-\ell) \\ &=\; X_{[s]}(x-\ell) \\ &=\; \sum_{t=0}^{s} Q_{[s,t]}(\ell)\, X_{[t]}(x) \\ &=\; \Biggl(\sum_{t=0}^{s} Q_{[s,t]}(\ell)\, W_{[t]}(0)\Biggr) F(x). \end{aligned}$$

Considering our assumption that translates of f are independent, this implies that

$$\begin{aligned}\bigl(w_{[s]}(k+\ell)\bigr)_{k\in\Gamma} &= W_{[s]}(\ell)\\ &= \sum_{t=0}^{s} Q_{[s,t]}(\ell)\,W_{[t]}(0)\\ &= \biggl(\sum_{t=0}^{s} Q_{[s,t]}(\ell)\,w_{[t]}(k)\biggr)_{k\in\Gamma}.\end{aligned}$$

In particular, taking $k=0$ we obtain

$$w_{[s]}(\ell) \;=\; \sum_{t=0}^{s} Q_{[s,t]}(\ell)\,w_{[t]}(0) \;=\; \sum_{t=0}^{s} Q_{[s,t]}(\ell)\,v_{[t]} \;=\; y_{[s]}(\ell).$$

Consider now the case $s=0$. Since $y_{[0]}(k)=v_0$ for every k, we have

$$1 \;=\; x^0 \;=\; X_{[0]}(x) \;=\; \sum_{k\in\Gamma} y_{[0]}(k)\,f(x+k) \;=\; v_0\sum_{k\in\Gamma} f(x+k).$$

Therefore we must have $v_0\neq 0$.

Finally, suppose that $q(x)=\sum_{0\le|\alpha|\le s} t_\alpha\,x^\alpha$ is any polynomial with $\deg(q)=s<p$. Since $y_{[s]}=[y_\alpha]_{|\alpha|=s}$, we have that $x^\alpha=\sum_{k\in\Gamma} y_\alpha(k)\,f(x+k)$. Therefore,

$$q(x) \;=\; \sum_{k\in\Gamma}\biggl(\sum_{0\le|\alpha|\le s} t_\alpha\,y_\alpha(k)\biggr) f(x+k) \;=\; \sum_{k\in\Gamma} u_q(k)\,f(x+k). \tag{4.5}$$

Since translates of f are independent, the coefficients $u_q(k)$ in (4.5) are unique. However, $u_q(k)$ is the evaluation at lattice points of the row vector of polynomials $u_q(x)=\sum_{0\le|\alpha|\le s} t_\alpha\,y_\alpha(x)$. Since such evaluations uniquely determine a polynomial, we conclude that u_q is unique.

It therefore remains only to show that $\deg(u_q)=s$. For this, recall that $y_\alpha(x)=\sum_{0\le\beta\le a}(-1)^{|\alpha|-|\beta|}\binom{\alpha}{\beta}x^{\alpha-\beta}\,v_\beta$. Since $v_0\neq 0$, we have $\deg(y_\alpha)=|\alpha|$. Moreover, y_α contains only a single term of degree $|\alpha|$, namely, $(-1)^{|\alpha|}\,x^\alpha\,v_0$. Therefore, $\deg(u_q)=\max\{|\alpha| : t_\alpha\neq 0\}=s$. □

4.3. *Proof of Theorem* 3.3

Proof of Theorem 3.3. By hypothesis, there is a row vector of polynomials $u\colon \mathbf{R}^d\to\mathbf{C}^r$ such that

$$x^\alpha \;=\; \sum_{k\in\Gamma} u(k)\,f(x+k). \tag{4.6}$$

Note that $(x+\ell)^\alpha = \sum_{k\in\Gamma} u(k-\ell)\, f(x+k)$ for $\ell \in \Gamma$. For each fixed x, define

$$g_x(y) \;=\; (x+y)^\alpha \qquad \text{and} \qquad h_x(y) \;=\; \sum_{k\in\Gamma} u(k-y)\, f(x+k).$$

Then g_x and h_x are both polynomials in the variable $y \in \mathbf{R}^d$. Moreover, $g_x(\ell) = h_x(\ell)$ for every lattice point $\ell \in \Gamma$. Therefore $g_x(y) = h_x(y)$ for every $y \in \mathbf{R}^d$, i.e.,

$$(x+y)^\alpha \;=\; \sum_{k\in\Gamma} u(k-y)\, f(x+k). \tag{4.7}$$

Let e_j be the multi-index of degree 1 with a 1 in the jth coordinate and 0's elsewhere. Then, by taking the derivative $\partial/\partial y_j$ of both sides of (4.7) and setting $y=0$, we find that

$$\alpha_j\, x^{\alpha-e_j} \;=\; (-1) \sum_{k\in\Gamma} (D^{e_j}u)(k)\, f(x+k). \tag{4.8}$$

Since (4.6) holds for almost every x, (4.8) holds a.e. as well. The proof then follows by repetition of this argument. □

4.4. *Proof of Theorem* 3.4(a)

In this section we will prove part (a) of Theorem 3.4, which we restate in the following form.

THEOREM 4.5. *Assume that $f\colon \mathbf{R}^d \to \mathbf{C}^r$ satisfies the refinement equation* (1.1) *and is compactly supported. Assume also that translates of f along Γ are independent.*

If f has accuracy p then there exists a collection of row vectors $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$ so that if $y_{[s]}(x) = \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}$ is the matrix of polynomials defined by (2.3) *and $Y_{[s]} = \big(y_{[s]}(k)\big)_{k\in\Gamma}$ is the row vector of evaluations of this polynomial at lattice points, then*

(i) *$v_0 \hat{f}(0) \ne 0$, and*

(ii) *$Y_{[s]} = A_{[s]}\, Y_{[s]}\, L$ for $0 \le s < p$.*

Proof. Since f has accuracy p and translates of f along Γ are independent, Theorem 3.1 implies that there exist row vectors $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$ such that $v_0 \ne 0$ and

$$X_{[s]}(x) \;=\; \sum_{k\in\Gamma} y_{[s]}(k)\, f(x+k) \;=\; Y_{[s]}\, F(x), \qquad 0 \le s < p.$$

If We combine this with the refinement equation $F(x) = L\, F(Ax)$ and with the definition of $A_{[s]}$, then we see that

$$\begin{aligned} Y_{[s]}\, F(Ax) &= X_{[s]}(Ax) \\ &= A_{[s]}\, X_{[s]}(x) \\ &= A_{[s]}\, Y_{[s]}\, F(x) \\ &= A_{[s]}\, Y_{[s]}\, L\, F(Ax). \end{aligned}$$

Considering our assumption that translates of f are independent, this implies that $Y_{[s]}^{\mathrm{T}} = A_{[s]}\, Y_{[s]}^{\mathrm{T}}\, L$ for $0 \le s < p$.

Consider now the case $s = 0$. Since $y_{[0]}(k) = v_0$ for all k, we have

$$1 \;=\; x^0 \;=\; X_{[0]}(x) \;=\; \sum_{k \in \Gamma} v_0\, f(x+k) \text{ a.e.}$$

Recall that the rectangular parallelepiped $P = \{x_1 u_1 + \cdots + x_d u_d : 0 \le x_i < 1\}$ is a fundamental domain for Γ. Therefore, computing integrals on f componentwise,

$$v_0 \hat{f}(0) \;=\; v_0 \int_{\mathbf{R}^d} f(x)\, dx \;=\; v_0 \sum_{k \in \Gamma} \int_P f(x+k)\, dx \;=\; \int_P 1\, dx \;=\; |P| \;\ne\; 0,$$

which completes the proof. □

4.5. *Proof of Theorem* 3.4(*b*)

In this section we will prove part (b) of Theorem 3.4, which we restate in the following form.

THEOREM 4.6. *Assume that* $f \colon \mathbf{R}^d \to \mathbf{C}^r$ *satisfies the refinement equation* (1.1), *and that* f *is integrable and compactly supported.*

Assume that there exists a collection of row vectors $\{v_\alpha \in \mathbf{C}^{1 \times r} : 0 \le |\alpha| < p\}$ *such that*

(i) $v_0 \hat{f}(0) \ne 0$, *and*

(ii) $Y_{[s]} = A_{[s]}\, Y_{[s]}\, L$ *for* $0 \le s < p$,

where $Y_{[s]} = \big(y_{[s]}(k)\big)_{k \in \Gamma}$ *is the row vector of evaluations at lattice points of the matrix of polynomials* $y_{[s]}(x) = \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}$ *defined by* (2.3).

Then f *has accuracy* p, *and*

$$Y_{[s]}\, F(x) \;=\; \sum_{k \in \Gamma} y_{[s]}(k)\, f(x+k) \;=\; C\, X_{[s]}(x), \qquad 0 \le s < p,$$

where $C = (v_0 \hat{f}(0))\, |P|^{-1}$.

Proof. For each $0 \le s < p$, define the vector-valued function $G_{[s]}: \mathbf{R}^d \to \mathbf{C}^{d_s}$ by

$$G_{[s]}(x) \;=\; \sum_{k\in\Gamma} y_{[s]}(k)\, f(x+k) \;=\; Y_{[s]}\, F(x). \tag{4.9}$$

Note that for each fixed x, only finitely many terms in the summation defining $G_{[s]}(x)$ are nonzero.

Using the equation $Y_{[s]} = A_{[s]}\, Y_{[s]}\, L$ and the refinement equation $L\, F(Ax) = F(x)$, we have

$$\begin{aligned} G_{[s]}(Ax) \;&=\; Y_{[s]}\, F(Ax) \\ &=\; A_{[s]}\, Y_{[s]}\, L\, F(Ax) \\ &=\; A_{[s]}\, Y_{[s]}\, F(x) \\ &=\; A_{[s]}\, G_{[s]}(x). \end{aligned} \tag{4.10}$$

Since $X_{[s]}(Ax) = A_{[s]}\, X_{[s]}(x)$, we see that $G_{[s]}(x)$ and $X_{[s]}(x)$ behave identically under dilation by A. We will show that there is a constant C independent of s so that $G_{[s]}(x) = C\, X_{[s]}(x)$ for $0 \le s < p$, and we will show that the explicit value of C is $C = (v_0 \hat{f}(0))\, |P|^{-1}$.

First, we need some basic facts concerning the mapping $x \mapsto Ax$. Let W be the matrix with the lattice generators $u_1, \dots, u_d$ as columns. Then $W^{-1}AW$ maps $\mathbf{Z}^d$ into itself, hence induces a homomorphism σ of the d-dimensional torus $\mathbf{R}^d/\mathbf{Z}^d$ into itself, defined by $\sigma(x+\mathbf{Z}^d) = W^{-1}AWx+\mathbf{Z}^d$. Since $W^{-1}AW$ is expansive, it follows from [Wal82, Corollary 1.10.1] that σ is ergodic. Since $x+\mathbf{Z}^d \mapsto W^{-1}x+\Gamma$ is a topological group isomorphism of $\mathbf{R}^d/\mathbf{Z}^d$ onto $\mathbf{R}^d/\Gamma$, the mapping $\tau(x+\Gamma) = Ax+\Gamma$ is therefore an ergodic mapping of $\mathbf{R}^d/\Gamma$ onto itself. Recall that the rectangular parallelepiped $P = \{x_1u_1 + \cdots + x_du_d : 0 \le x_i < 1\}$ is a fundamental domain for $\mathbf{R}^d/\Gamma$. Therefore we can view τ as a mapping of P onto itself, defined by $\tau(x) = Ax+j$ where j is the unique element of Γ such that $Ax+j \in P$. Since Haar measure on $\mathbf{R}^d/\Gamma$ corresponds to Lebesgue measure on P, τ is an ergodic mapping of P onto itself.

We now proceed by induction to show that $G_{[s]}(x) = C\, X_{[s]}(x)$ for $0 \le s < p$ with C independent of s.

Consider the case $s = 0$. Here $G_{[0]}(x)$ is scalar-valued. Since $A_{[0]}$ is the constant 1, Eq. (4.10) states that $G_{[0]}(Ax) = G_{[0]}(x)$. Further, $y_{[0]}(k) = v_0$ for every k, so $G_{[0]}(x) = \sum_{k\in\Gamma} v_0\, f(x+k)$. Therefore, for each $\ell \in \Gamma$ we have

$$G_{[0]}(x-\ell) \;=\; \sum_{k\in\Gamma} v_0\, f(x-\ell+k) \;=\; \sum_{k\in\Gamma} v_0\, f(x+k) \;=\; G_{[0]}(x).$$

Thus $G_{[0]}(x)$ satisfies

$$G_{[0]}(Ax) \;=\; G_{[0]}(x) \qquad \text{and} \qquad G_{[0]}(x-\ell) \;=\; G_{[0]}(x), \quad \ell \in \Gamma.$$

Hence $G_{[0]}(\tau(x)) = G_{[0]}(x)$ for each $x \in P$. Since τ is ergodic, it follows that $G_{[0]}$ is constant a.e. on P [Wal82, Theorem 1.6]. By periodicity, we therefore have $G_{[0]}(x) = C$ a.e. on $\mathbf{R}^d$. We can evaluate this constant explicitly, since

$$\begin{aligned} C\,|P| &= \int_P G_{[0]}(x)\,dx \\ &= v_0 \sum_{k\in\Gamma} \int_P f(x+k)\,dx \\ &= v_0 \int_{\mathbf{R}^d} f(x)\,dx \\ &= v_0 \hat{f}(0) \neq 0. \end{aligned}$$

In particular, $C = (v_0 \hat{f}(0))\,|P|^{-1} \neq 0$.

Suppose now, inductively, that $G_{[t]}(x) = C\,X_{[t]}(x)$ a.e. for $0 \le t < s$. Using the notation $Y_{[s]}(x) = \bigl(y_{[s]}(x+k)\bigr)_{k\in\Gamma}$ as in (2.4), we then have

$$\begin{aligned} G_{[s]}(x-\ell) &= Y_{[s]}\,F(x-\ell) \\ &= Y_{[s]}(\ell)\,F(x) \\ &= \sum_{t=0}^{s} Q_{[s,t]}(\ell)\,Y_{[t]}\,F(x) && \text{by Lemma 4.4} \\ &= \sum_{t=0}^{s} Q_{[s,t]}(\ell)\,G_{[t]}(x) \\ &= Q_{[s,s]}(\ell)\,G_{[s]}(x) + \sum_{t=0}^{s-1} Q_{[s,t]}(\ell)\,G_{[t]}(x) \\ &= Q_{[s,s]}(\ell)\,G_{[s]}(x) + C \sum_{t=0}^{s-1} Q_{[s,t]}(\ell)\,X_{[t]}(x) && \text{by induction} \\ &= Q_{[s,s]}(\ell)\,G_{[s]}(x) + C \sum_{t=0}^{s} Q_{[s,t]}(\ell)\,X_{[t]}(x) - C\,Q_{[s,s]}(\ell)\,X_{[s]}(x) \\ &= G_{[s]}(x) + C\,X_{[s]}(x-\ell) - C\,X_{[s]}(x) && \text{by definition of } Q_{[s,t]}. \end{aligned}$$

Therefore, if we define $H_{[s]}(x) = G_{[s]}(x) - C\,X_{[s]}(x)$, then

$$H_{[s]}(Ax) = A_{[s]}\,H_{[s]}(x) \qquad \text{and} \qquad H_{[s]}(x-\ell) = H_{[s]}(x), \quad \ell \in \Gamma.$$

This implies that

$$H_{[s]}(\tau(x)) = A_{[s]}\,H_{[s]}(x).$$

Let $E \subset P$ be a set of positive measure on which $H_{[s]}$ is bounded, say $\|H_{[s]}(x)\| \le M$ for $x \in E$, where $\|\cdot\|$ is any fixed norm on $\mathbf{C}^{d_s}$. Since τ is ergodic, we know from the Ergodic Theorem [Wal82, p. 35] that for almost every $x \in P$,

$$\lim_{n\to\infty} \frac{\#\{0 < k \le n : \tau^k(x) \in E\}}{n} \;=\; |E| \;>\; 0. \tag{4.11}$$

Fix any $x \in P$ such that (4.11) holds. Then there exists an increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers such that $\tau^{n_j}(x) \in E$ for each j. Hence

$$M \;\ge\; \|H_{[s]}(\tau^{n_j}(x))\| \;=\; \|(A_{[s]})^{n_j}\, H_{[s]}(x)\|.$$

However, Lemma 4.2(e) implies that $A_{[s]}$ is expansive since A is expansive and $s > 0$. Therefore $\|(A_{[s]})^{n_j}\, H_{[s]}(x)\|$ diverges to infinity if $H_{[s]}(x) \ne 0$. Therefore we must have $H_{[s]}(x) = 0$ a.e. on P. Since $H_{[s]}$ is Γ-periodic, it must therefore vanish a.e. on $\mathbf{R}^d$. Hence $G_{[s]}(x) = C\, X_{[s]}(x)$ a.e., which completes the proof. □

4.6. *Proof of Theorem* 3.6

We will prove Theorem 3.6 in this section. First we require the following lemma.

LEMMA 4.7. *Let matrices $B_t \in \mathbf{C}^{d_t \times r}$ be given for $0 \le t \le s$. If $\sum_{t=0}^{s} Q_{[s,t]}(A\ell)\, B_t = 0$ for each $\ell \in \Gamma$, then $B_t = 0$ for $0 \le t \le s$.*

Proof. Denote the elements of B_t by $B_t = [b^t_{\beta,i}]_{|\beta|=t, i=1,\dots,r}$. Consider a single element $b^t_{\beta_0,i_0}$ of B_t. Since $|\beta_0| = t$, we can find a multi-index $\alpha_0 \ge \beta_0$ with $|\alpha_0| = s$. Consider the polynomial

$$u(x) \;=\; \sum_{|\beta|=t} (-1)^{s-t} \binom{\alpha_0}{\beta}\, x^{\alpha_0-\beta}\, b^t_{\beta,i_0}.$$

Then $u(x)$ is element (α_0, i_0) of the matrix $\sum_{t=0}^{s} Q_{[s,t]}(x)\, B_t$. Hence $u(A\ell) = 0$ for every $\ell \in \Gamma$, or, equivalently, $u(\ell) = 0$ for every $\ell \in A(\Gamma)$. Since $A(\Gamma)$ is itself a full-rank lattice in $\mathbf{R}^d$, this implies that u is the zero polynomial, and therefore we must have $\binom{\alpha_0}{\beta}\, b^t_{\beta,i_0} = 0$ for every β of degree t. In particular, $\binom{\alpha_0}{\beta_0}\, b^t_{\beta_0,i_0} = 0$. Since $\beta_0 \le \alpha_0$ we know that $\binom{\alpha_0}{\beta_0} \ne 0$, so we must have $b^t_{\beta_0,i_0} = 0$. Thus every entry of B_t is zero. □

The following result is an expanded version of Theorem 3.6. It is helpful to observe that the row vector $A_{[s]}\, Y_{[s]}\, L$ can be written

$$A_{[s]}\, Y_{[s]}\, L \;=\; A_{[s]}\, \bigl(y_{[s]}(k)\bigr)_{k\in\Gamma}\, [\,c_{Ak-\ell}\,]_{k,\ell\in\Gamma} \;=\; \Bigl(A_{[s]} \sum_{k\in\Gamma} y_{[s]}(k)\, c_{Ak-\ell}\Bigr)_{\ell\in\Gamma}.$$

THEOREM 4.8. *Let $m = |\det(A)|$, and let $d_1, \dots, d_m \in \Gamma$ be a full set of digits. Set $\Gamma_i = A(\Gamma) - d_i$.*

Given a collection $\{v_\alpha \in \mathbf{C}^{1\times r} : 0 \le |\alpha| < p\}$ of row vectors, let $y_{[s]}(x) = \sum_{t=0}^{s} Q_{[s,t]}(x)\, v_{[t]}$ be the matrix of polynomials defined by (2.3) and let $Y_{[s]} = \bigl(y_{[s]}(k)\bigr)_{k\in\Gamma}$ be the row vector of evaluations of this polynomial at lattice points.

If $v_0 \ne 0$, then the following statements are equivalent.

(a) $Y_{[p-1]} = A_{[p-1]}\, Y_{[p-1]}\, L$. *Equivalently,*

$$y_{[p-1]}(\ell) \;=\; A_{[p-1]} \sum_{k\in\Gamma} y_{[p-1]}(k)\, c_{Ak-\ell} \qquad \text{for } \ell \in \Gamma.$$

(b) $Y_{[s]} = A_{[s]}\, Y_{[s]}\, L$ *for* $0 \le s < p$. *Equivalently,*

$$y_{[s]}(\ell) \;=\; A_{[s]} \sum_{k\in\Gamma} y_{[s]}(k)\, c_{Ak-\ell} \qquad \text{for } 0 \le s < p \text{ and } \ell \in \Gamma.$$

(c) $$y_{[s]}(d_i) \;=\; A_{[s]} \sum_{k\in\Gamma} y_{[s]}(k)\, c_{Ak-d_i} \qquad \text{for } 0 \le s < p \text{ and } i = 1, \dots, m.$$

(d) $$v_{[s]} \;=\; \sum_{k\in\Gamma_i} \sum_{t=0}^{s} Q_{[s,t]}(k)\, A_{[t]}\, v_{[t]}\, c_k \qquad \text{for } 0 \le s < p \text{ and } i = 1, \dots, m.$$

Proof. (a) $\Rightarrow$ (b). Assume that (a) holds. Then

$$\begin{aligned}
&\sum_{s=0}^{p-1} Q_{[p-1,s]}(Aj) \left(A_{[s]} \sum_{k\in\Gamma} y_{[s]}(k)\, c_{Ak-\ell} \right) \\
&\quad= \sum_{s=0}^{p-1} A_{[p-1]}\, Q_{[p-1,s]}(j) \sum_{k\in\Gamma} y_{[s]}(k)\, c_{Ak-\ell} && \text{by Lemma 4.3} \\
&\quad= A_{[p-1]} \sum_{k\in\Gamma} \left(\sum_{s=0}^{p-1} Q_{[p-1,s]}(j)\, y_{[s]}(k) \right) c_{Ak-\ell} \\
&\quad= A_{[p-1]} \sum_{k\in\Gamma} y_{[p-1]}(j+k)\, c_{Ak-\ell} && \text{by Lemma 4.4} \\
&\quad= A_{[p-1]} \sum_{k\in\Gamma} y_{[p-1]}(k)\, c_{Ak-(Aj+\ell)} \\
&\quad= y_{[p-1]}(Aj+\ell) && \text{by hypothesis (a)} \\
&\quad= \sum_{s=0}^{p-1} Q_{[p-1,s]}(Aj)\, y_{[s]}(\ell) && \text{by Lemma 4.4.}
\end{aligned}$$

Lemma 4.7 therefore implies that $A_{[s]} \sum_{k\in\Gamma} y_{[s]}(k)\, c_{Ak-\ell} = y_{[s]}(\ell)$ for $0 \le s < p$ and $\ell \in \Gamma$, so statement (b) holds.

(b) $\Rightarrow$ (a), (c). Trivial.

(c) $\Rightarrow$ (b). Assume that (c) holds, and fix any $\ell \in \Gamma$. Then $\ell = Aj + d_i$ for a unique choice of $j \in \Gamma$ and $i = 1, \dots, m$. Therefore,

$$\begin{aligned}
y_{[s]}(\ell) &= y_{[s]}(Aj + d_i) \\
&= \sum_{t=0}^{s} Q_{[s,t]}(Aj)\, y_{[t]}(d_i) && \text{by Lemma 4.4} \\
&= \sum_{t=0}^{s} \sum_{k\in\Gamma} Q_{[s,t]}(Aj)\, A_{[t]}\, y_{[t]}(k)\, c_{Ak-d_i} && \text{by hypothesis (c)} \\
&= \sum_{k\in\Gamma} \sum_{t=0}^{s} A_{[s]}\, Q_{[s,t]}(j)\, y_{[t]}(k)\, c_{Ak-d_i} && \text{by Lemma 4.3} \\
&= A_{[s]} \sum_{k\in\Gamma} y_{[s]}(k+j)\, c_{Ak-d_i} && \text{by Lemma 4.4} \\
&= A_{[s]} \sum_{k\in\Gamma} y_{[s]}(k)\, c_{Ak-(Aj+d_i)} \\
&= A_{[s]} \sum_{k\in\Gamma} y_{[s]}(k)\, c_{Ak-\ell}.
\end{aligned}$$

(c) $\Rightarrow$ (d). Assume that (c) holds. Then

$$\begin{aligned}
v_{[s]} &= y_{[s]}(-d_i + d_i) \\
&= \sum_{t=0}^{s} Q_{[s,t]}(-d_i)\, y_{[t]}(d_i) && \text{by Lemma 4.4} \\
&= \sum_{t=0}^{s} \sum_{k\in\Gamma} Q_{[s,t]}(-d_i)\, A_{[t]}\, y_{[t]}(k)\, c_{Ak-d_i} && \text{by hypothesis (c)} \\
&= \sum_{k\in\Gamma} \sum_{t=0}^{s} \sum_{u=0}^{t} Q_{[s,t]}(-d_i) A_{[t]}\, Q_{[t,u]}(k)\, v_{[u]}\, c_{Ak-d_i} && \text{by (2.3)} \\
&= \sum_{k\in\Gamma} \sum_{t=0}^{s} \sum_{u=0}^{t} Q_{[s,t]}(-d_i)\, Q_{[t,u]}(Ak)\, A_{[u]}\, v_{[u]}\, c_{Ak-d_i} && \text{by Lemma 4.3} \\
&= \sum_{k\in\Gamma} \sum_{u=0}^{s} \sum_{t=u}^{s} Q_{[s,t]}(-d_i)\, Q_{[t,u]}(Ak)\, A_{[u]}\, v_{[u]}\, c_{Ak-d_i} && \text{interchanging summations} \\
&= \sum_{k\in\Gamma} \sum_{u=0}^{s} Q_{[s,u]}(Ak - d_i)\, A_{[u]}\, v_{[u]}\, c_{Ak-d_i} && \text{by Lemma 4.1(c)} \\
&= \sum_{k\in\Gamma_i} \sum_{u=0}^{s} Q_{[s,u]}(k)\, A_{[u]}\, v_{[u]}\, c_k && \text{by definition of } \Gamma_i.
\end{aligned}$$

(d) $\Rightarrow$ (c). Assume that (d) holds. Then

$$
\begin{aligned}
& A_{[s]} \sum_{k\in\Gamma} y_{[t]}(k)\, c_{Ak-d_i} \\
&= \sum_{k\in\Gamma} \sum_{t=0}^{s} A_{[s]}\, Q_{[s,t]}(k)\, v_{[t]}\, c_{Ak-d_i} && \text{by (2.3)} \\
&= \sum_{k\in\Gamma} \sum_{t=0}^{s} Q_{[s,t]}(Ak)\, A_{[t]}\, v_{[t]}\, c_{Ak-d_i} && \text{by Lemma 4.3} \\
&= \sum_{k\in\Gamma} \sum_{t=0}^{s} Q_{[s,t]}(d_i + Ak - d_i)\, A_{[t]}\, v_{[t]}\, c_{Ak-d_i} \\
&= \sum_{k\in\Gamma} \sum_{t=0}^{s} \sum_{u=t}^{s} Q_{[s,u]}(d_i)\, Q_{[u,t]}(Ak-d_i)\, A_{[t]}\, v_{[t]}\, c_{Ak-d_i} && \text{by Lemma 4.1(c)} \\
&= \sum_{k\in\Gamma} \sum_{u=0}^{s} \sum_{t=0}^{u} Q_{[s,u]}(d_i)\, Q_{[u,t]}(Ak-d_i)\, A_{[t]}\, v_{[t]}\, c_{Ak-d_i} && \text{interchanging summations} \\
&= \sum_{u=0}^{s} Q_{[s,u]}(d_i)\, v_{[u]} && \text{by hypothesis (d)} \\
&= y_{[s]}(d_i) && \text{by (2.3).} \quad \square
\end{aligned}
$$

4.7. *Proof of Theorem* 3.7

Proof of Theorem 3.7. Note first that since $r = 1$, the coefficients c_k are scalars, hence commute with any matrix or vector.

(a) $\Rightarrow$ (b). Assume that there exist scalars $\{v_\alpha \in \mathbf{C} : 0 \le |\alpha| < p\}$ so that $v_0 \neq 0$ and statement (c) of Theorem 3.6 holds, i.e., for each $0 \le s < p$ we have

$$
v_{[s]} \;=\; \sum_{k\in\Gamma_i} \sum_{t=0}^{s} Q_{[s,t]}(k)\, A_{[t]}\, v_{[t]}\, c_k, \qquad i = 1, \dots, m. \tag{4.12}
$$

We proceed by induction on the degree of α to show that $\sum_{k\in\Gamma_i} k^\alpha c_k$ is independent of i.

For $s = 0$, we have by (4.12) that

$$
v_0 \;=\; v_{[0]} \;=\; \sum_{k\in\Gamma_i} Q_{[0,0]}(k)\, A_{[0]}\, v_{[0]}\, c_k \;=\; v_0 \sum_{k\in\Gamma_i} c_k.
$$

Since v_0 is a nonzero scalar, this implies that $\sum_{k\in\Gamma_i} c_k = 1$ is independent of i, and that $\sum_{k\in\Gamma} c_k = m$.

Assume now, inductively, that $\sum_{k\in\Gamma_i} k^\alpha c_k$ is independent of i for all multi-indices α with degrees $0 \le |\alpha| < s$. Recall the definition $Q_{[s,t]}(k) = (-1)^{s-t}[\binom{\alpha}{\beta}\, k^{\alpha-\beta}]_{|\alpha|=s,|\beta|=t}$. If $0 < t \le s$ then $0 \le s-t < s$, so for these t the matrix

$$M_{[s,t]} \;=\; \sum_{k\in\Gamma_i} Q_{[s,t]}(k)\, c_k \;=\; (-1)^{s-t}\left[\binom{\alpha}{\beta} \sum_{k\in\Gamma_i} k^{\alpha-\beta}\, c_k\right]_{|\alpha|=s,|\beta|=t} \tag{4.13}$$

is independent of i. Since $Q_{[s,0]}(k) = (-1)^s\, X_{[s]}(k)$ and $A_{[0]} = 1$, we therefore have by (4.12) that

$$\begin{aligned} v_{[s]} \;&=\; \sum_{k\in\Gamma_i}\sum_{t=0}^{s} Q_{[s,t]}(k)\, A_{[t]}\, v_{[t]}\, c_k \\ &=\; \left(\sum_{k\in\Gamma_i} Q_{[s,0]}(k)\, c_k\right) A_{[0]}\, v_{[0]} \;+\; \sum_{t=1}^{s}\left(\sum_{k\in\Gamma_i} Q_{[s,t]}(k)\, c_k\right) A_{[t]}\, v_{[t]} \\ &=\; (-1)^s\left(\sum_{k\in\Gamma_i} X_{[s]}(k)\, c_k\right) v_0 \;+\; \sum_{t=1}^{s} M_{[s,t]}\, A_{[t]}\, v_{[t]}. \end{aligned}$$

Since v_0 is a nonzero scalar, this implies that $\sum_{k\in\Gamma_i} X_{[s]}(k)\, c_k$ does not depend on i, and hence that $\sum_{k\in\Gamma_i} k^\alpha c_k$ does not depend on i for any $|\alpha| = s$. This completes the induction.

(b) $\Rightarrow$ (a). Assume that $\sum_{k\in\Gamma} c_k = m$ and that $\sum_{k\in\Gamma_i} k^\alpha c_k$ is independent of i for each $0 \le |\alpha| < p$. Then the matrices $M_{[s,t]}$ defined in (4.13) do not depend on i for any $0 \le t \le s < p$. We shall inductively define scalars $v_\alpha \in \mathbf{C}^r$ so that $v_{[s]} = [v_\alpha]_{|\alpha|=s}$ satisfies (4.12) for $0 \le s < p$. As a consequence, statement (c) of Theorem 3.6 will be fulfilled.

Define $v_0 = 1$. By hypothesis, $\sum_{k\in\Gamma_i} c_k = 1$ for $i = 1, \dots, m$, so $v_{[0]} = [v_0] = 1$ satisfies (4.12) for $s = 0$. Assume now, inductively, that $v_{[s]} = [v_\alpha]_{|\alpha|=s}$ has been defined so that (4.12) is satisfied for $0 \le s < u$. Since $A_{[u]}$ is expansive, we know that $I - A_{[u]}$ is invertible. Therefore, we can define

$$v_{[u]} \;=\; (I - A_{[u]})^{-1} \sum_{t=0}^{u-1} M_{[u,t]}\, A_{[t]}\, v_{[t]}. \tag{4.14}$$

We must show that this $v_{[u]}$ satisfies (4.12) for $s = u$. First, rewrite (4.14) as

$$v_{[u]} \;=\; A_{[u]}\, v_{[u]} \;+\; \sum_{t=0}^{u-1} M_{[u,t]}\, A_{[t]}\, v_{[t]}.$$

Then, since $Q_{[u,u]}(k) = I$ and $\sum_{k\in\Gamma_i} c_k = 1$, we have

$$\begin{aligned}
\sum_{k\in\Gamma_i} \sum_{t=0}^{u} & Q_{[u,t]}(k)\, A_{[t]}\, v_{[t]}\, c_k \\
&= \sum_{k\in\Gamma_i} Q_{[u,u]}(k)\, A_{[u]}\, v_{[u]}\, c_k \;+\; \sum_{t=0}^{u-1}\biggl(\sum_{k\in\Gamma_i} Q_{[u,t]}(k)\, c_k\biggr) A_{[t]}\, v_{[t]} \\
&= A_{[u]}\, v_{[u]} \;+\; \sum_{t=0}^{u-1} M_{[u,t]}\, A_{[t]}\, v_{[t]} \\
&= v_{[u]}.
\end{aligned}$$

Thus (4.12) holds for $s = u$. $\square$

4.8. *Proof of Theorem* 3.9

Proof of Theorem 3.9. (a) Define $G_{[0]}(x)$ as in (4.9), i.e.,

$$G_{[0]}(x) \;=\; v_0 \sum_{k\in\Gamma} f(x+k).$$

Then the argument of the proof of Theorem 4.6 shows that $G_{[0]}(x) = C$ a.e., with $C = (v_0 \hat{f}(0))|P|^{-1}$. Hence $v_0 \hat{f}(0) \neq 0$ if and only if $C \neq 0$. However, if translates of f along Γ are independent, then we must have $C \neq 0$ since $C = \sum_{k\in\Gamma} v_0\, f(x+k)$ and $v_0 \neq 0$.

(b) Assume that the matrix $\Delta = (1/m)\sum_{k\in\Lambda} c_k$ has eigenvalues $\lambda_1 = 1$ and $|\lambda_2|, \dots, |\lambda_r| < 1$. Define $B = (A^{-1})^{\mathrm{T}}$, and let $M(\omega) = (1/m)\sum_{k\in\Lambda} c_k\, e^{-2\pi i k\cdot\omega}$ be the matrix-valued symbol of the refinement equation. Note that $\Delta = M(0)$. The refinement equation implies that the Fourier transform $\hat{f}$ of f satisfies $\hat{f}(\omega) = M(B\omega)\, \hat{f}(B\omega)$. In particular,

$$\hat{f}(0) \;=\; M(0)\, \hat{f}(0) \;=\; \Delta\, \hat{f}(0). \tag{4.15}$$

We prove in the Appendix that since Δ^∞ converges, the infinite matrix product $P(\omega) = \prod_{j=1}^{\infty} M(B^j\omega)$ converges uniformly on compact sets, and that as a consequence $\hat{f}(\omega) = P(\omega)\, \hat{f}(0)$. Hence we must have $\hat{f}(0) \neq 0$, so (4.15) implies that $\hat{f}(0)$ is the right 1-eigenvector for Δ.

On the other hand, since $v_0 = v_0 \sum_{k\in\Gamma_i} c_k$ and Γ is the disjoint union of the Γ_i, we have

$$v_0 \;=\; v_0\, \frac{1}{m} \sum_{i=1}^{m} \sum_{k\in\Gamma_i} c_k \;=\; v_0 \Delta.$$

Since v_0 is nonzero, it therefore is a left 1-eigenvector for Δ. However, 1 is a simple eigenvalue for Δ, and the dot product of the left and right 1-eigenvectors must be nonzero when 1 is simple, so $v_0\hat{f}(0) \neq 0$. $\square$

5. THE QUINCUNX MATRIX

The quincunx matrix is $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. It maps the lattice $\Gamma = \mathbf{Z}^2$ into itself via an expansion by $\sqrt{2}$ and a rotation by $\pi/4$. The sublattice $A(\Gamma)$ is the quincunx lattice. The determinant of A is $m = 2$. If we choose digits

$$d_1 \;=\; 0 \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad d_2 \;=\; \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

then the attractor

$$K \;=\; \left\{ \sum_{n=1}^{\infty} A^{-n}\varepsilon_n : \varepsilon_n \in \{d_1, d_2\} \right\} \;=\; \bigcup_{i=1}^{2} A^{-1}(K + d_i)$$

of the iterated function system (IFS) determined by A and d_1, d_2 has Lebesgue measure 1, and therefore tiles $\mathbf{R}^2$ by translates along $\mathbf{Z}^2$ [GM92]. The set K is commonly known as the "twin dragon" fractal. The characteristic function of K is the solution to the refinement equation with $r = 1$, $\Lambda = \{d_1, d_2\}$, and $c_{d_1} = c_{d_2} = 1$.

The quincunx matrix is a popular choice for the construction of nonseparable two-dimensional wavelets [GM92], [KV92], [CD93], [Vil94]. We shall write out explicitly the three lowest-order sum rule requirements from Theorem 3.6(c) for the quincunx matrix for the general FSI case. We shall then examine the much simpler PSI case, and apply our results to the family of refinable functions constructed in [KV92].

5.1. *Sum Rules for Multiple Refinable Functions* (*Quincunx Case*)

We shall write out the "sum rules" of Theorem 3.6(c) for $p \leq 3$. We let the number r of refinable functions $f_1, \dots, f_r$ be arbitrary.

First we must specialize our notation to the quincunx case. The lattice $\Gamma = \mathbf{Z}^2$ has two cosets under dilation by A, namely

$$\Gamma_1 \;=\; A(\mathbf{Z}^2) - d_1 \;=\; \{Ak : k \in \mathbf{Z}^2\} \;=\; \left\{ \begin{bmatrix} k_1 - k_2 \\ k_1 + k_2 \end{bmatrix} : k \in \mathbf{Z}^2 \right\}$$

$$\Gamma_2 \;=\; A(\mathbf{Z}^2) - d_2 \;=\; \left\{ Ak - \begin{bmatrix} 1 \\ 0 \end{bmatrix} : k \in \mathbf{Z}^2 \right\} \;=\; \left\{ \begin{bmatrix} k_1 - k_2 - 1 \\ k_1 + k_2 \end{bmatrix} : k \in \mathbf{Z}^2 \right\}.$$

In order to write out the matrix $A_{[s]}$ and matrix of polynomials $Q_{[s,t]}$, we must choose an ordering of the multi-indices α of degree $|\alpha| = s$. We use the ordering defined by (4.3). That is, for $s = 0$, 1, 2 the multi-indices of degree s are ordered as follows:

$$\begin{aligned} s = 0: &\quad (0,0), \\ s = 1: &\quad (1,0) \prec (0,1), \\ s = 2: &\quad (2,0) \prec (1,1) \prec (0,2). \end{aligned}$$

With this ordering, we have

$$X_{[0]}(x) = 1, \qquad X_{[1]}(x) = \begin{bmatrix} x^{(1,0)} \\ x^{(0,1)} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$X_{[2]}(x) = \begin{bmatrix} x^{(2,0)} \\ x^{(1,1)} \\ x^{(0,2)} \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}.$$

Using the definition $X_{[s]}(Ax) = A_{[s]}\, X_{[s]}(x)$, we therefore have

$$A_{[0]} = 1, \qquad A_{[1]} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad A_{[2]} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Using the definition $X_{[s]}(x-y) = \sum_{t=0}^{s} Q_{[s,t]}(y)\, X_{[s]}(x)$, we have

$$Q_{[0,0]}(y) = I_1 = 1,$$

$$Q_{[1,0]}(y) = \begin{bmatrix} -y_1 \\ -y_2 \end{bmatrix}, \qquad Q_{[1,1]}(y) = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$Q_{[2,0]}(y) = \begin{bmatrix} y_1^2 \\ y_1 y_2 \\ y_2^2 \end{bmatrix}, \qquad Q_{[2,1]}(y) = \begin{bmatrix} -2y_1 & 0 \\ -y_2 & -y_1 \\ 0 & -2y_2 \end{bmatrix},$$

$$Q_{[2,2]}(y) = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrices $v_{[s]}$ have the form

$$v_{[0]} \;=\; [\, v_{(0,0)} \,], \qquad v_{[1]} \;=\; \begin{bmatrix} v_{(1,0)} \\ v_{(0,1)} \end{bmatrix}, \qquad v_{[2]} \;=\; \begin{bmatrix} v_{(2,0)} \\ v_{(1,1)} \\ v_{(0,2)} \end{bmatrix},$$

with entries that are row vectors $v_\alpha \in \mathbf{C}^{1\times r}$.

Since $m = 2$ and $\Gamma = \mathbf{Z}^2$, the sum rules in statement (c) of Theorem 3.6 can be rewritten

$$v_{[s]} \;=\; \sum_{k\in\Gamma_i} \sum_{t=0}^{s} Q_{[s,t]}(k)\, A_{[t]}\, v_{[t]}\, c_k, \qquad i = 1, 2. \tag{5.1}$$

Using the fact that $A_{[0]}$ and $Q_{[s,s]}(k)$ are identity matrices of the appropriate sizes, we can therefore write (5.1) for $s = 0,\ 1,\ 2$ as

$$s = 0: \qquad v_{[0]} \;=\; v_{[0]} \sum_{k\in\Gamma_i} c_k, \tag{5.2}$$

$$s = 1: \qquad v_{[1]} \;=\; \sum_{k\in\Gamma_i} \bigl(Q_{[1,0]}(k)\, v_{[0]} + A_{[1]}\, v_{[1]}\bigr)\, c_k, \tag{5.3}$$

$$s = 2: \qquad v_{[2]} \;=\; \sum_{k\in\Gamma_i} \bigl(Q_{[2,0]}(k)\, v_{[0]} + Q_{[2,1]}(k)\, A_{[1]}\, v_{[1]} + A_{[2]}\, v_{[2]}\bigr)\, c_k, \tag{5.4}$$

all for $i = 1,\ 2$. By Theorems 3.4 and 3.6, if there exists $v_{[0]} = v_{(0,0)}$ such that (5.2) holds and $v_{(0,0)}\hat{f}(0) \neq 0$, then f has accuracy at least $p = 1$. If in addition there exists $v_{[1]} = \begin{bmatrix} v_{(1,0)} \\ v_{(0,1)} \end{bmatrix}$ such that (5.3) holds, then f has accuracy $p = 2$. If there is no such $v_{[1]}$, then f is limited to accuracy $p = 1$. And so forth, each higher value of p requiring the existence of additional matrices $v_{[s]}$.

We can further expand (5.2)–(5.4) in terms of the vectors v_α that make up the matrices $v_{[s]}$. Expanding these equations using the values for $A_{[s]}$ and $Q_{[s,t]}(y)$ found earlier, we find that they are equivalent to the following equations, each of which must hold for $i = 1, 2$.

$$s = 0: \qquad v_{(0,0)} \;=\; v_{(0,0)} \sum_{k\in\Gamma_i} c_k,$$

$$s = 1: \qquad \begin{bmatrix} v_{(1,0)} \\[2ex] v_{(0,1)} \end{bmatrix} \;=\; \begin{bmatrix} \displaystyle\sum_{k\in\Gamma_i} \bigl(-k_1\, v_{(0,0)} + v_{(1,0)} - v_{(0,1)}\bigr)\, c_k \\ \displaystyle\sum_{k\in\Gamma_i} \bigl(-k_2\, v_{(0,0)} + v_{(1,0)} + v_{(0,1)}\bigr)\, c_k \end{bmatrix},$$

$$s = 2: \qquad \begin{bmatrix} v_{(2,0)} \\ v_{(1,1)} \\ v_{(0,2)} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k \in \Gamma_i} \bigl(k_1^2\, v_{(0,0)} - 2k_1\, v_{(1,0)} + 2k_1\, v_{(0,1)} + v_{(2,0)} - 2v_{(1,1)} + v_{(0,2)}\bigr)\, c_k \\ \sum_{k \in \Gamma_i} \bigl(k_1 k_2\, v_{(0,0)} - (k_1 + k_2)\, v_{(1,0)} + (-k_1 + k_2)\, v_{(0,1)} + v_{(2,0)} - v_{(0,2)}\bigr)\, c_k \\ \sum_{k \in \Gamma_i} \bigl(k_2^2\, v_{(0,0)} - 2k_2\, v_{(1,0)} - 2k_2\, v_{(0,1)} + v_{(2,0)} + 2v_{(1,1)} + v_{(0,2)}\bigr)\, c_k \end{bmatrix}.$$

5.2. *Sum Rules for a Single Refinable Function* (*Quincunx Case*)

The sum rules for the PSI case ($r = 1$) are much simpler than for the general case. By Theorem 3.7, in the PSI case for the quincunx matrix, the sum rules for accuracy p are

$$\sum_{k \in \mathbf{Z}^2} c_k = 2 \qquad \text{and} \qquad \sum_{k \in \Gamma_1} k^\alpha c_k = \sum_{k \in \Gamma_2} k^\alpha c_k, \quad 0 \le |\alpha| < p. \tag{5.5}$$

If we let $M(\omega) = \frac{1}{2} \sum_{k \in \Lambda} c_k\, e^{-2\pi i k \cdot \omega}$ be the symbol of the refinement equation, then these sum rules are precisely equivalent to the following "zeros at $(1/2, 1/2)$" condition that plays a role in the results of [KV92], [CD93]:

$$M(0,0) = 1 \qquad \text{and} \qquad (D^\alpha M)(1/2, 1/2) = 0, \quad 0 \le |\alpha| < p.$$

As an example, we apply the sum rules to the parameterized family of refinable functions proposed by Kovačević and Vetterli in [KV92]. For each $a = (a_0, a_1, a_2) \in \mathbf{R}^3$, consider the coefficients $c = [c_k]_{k \in \mathbf{Z}^2}$ defined by

$$c \;=\; \nu(a) \begin{bmatrix} & -a_1 & -a_0 a_1 & \\ -a_2 & -a_0 a_2 & -a_0 & 1 \\ & a_0 a_1 a_2 & -a_1 a_2 & \end{bmatrix},$$

where $\nu(a)$ is a normalization factor chosen so that $\sum c_k = 2$. We place the 0 index at the coefficient $-a_0 a_2$. For this example, the sum rules in (5.5) hold for $p = 1$ if and only if

$$-a_0 - a_1 - a_2 + a_0 a_1 a_2 \;=\; -a_0 a_1 - a_0 a_2 - a_1 a_2 + 1. \tag{5.6}$$

For accuracy $p = 2$ we require in addition that

$$-a_0a_1 - a_1a_2 + 2 \;=\; a_2 - a_0 \quad \text{and} \quad -a_0a_1 + a_1a_2 \;=\; -a_1 - a_0a_1a_2. \tag{5.7}$$

The solutions to the system of nonlinear equations in (5.6) and (5.7) are

$$a_0 = \pm\sqrt{3}, \qquad a_1 = \pm\sqrt{3}, \qquad a_2 = 2 \mp \sqrt{3}, \tag{5.8}$$

$$a_0 = \pm\sqrt{3}, \qquad a_1 = 0, \qquad a_2 = 2 \pm \sqrt{3}. \tag{5.9}$$

In any of these cases there is an integrable solution f to the refinement equation, and we conclude that the accuracy of this solution is at least $p = 2$. It is easy to check that none of the choices in (5.8) or (5.9) satisfy the sum rules in (5.5) for $p = 3$, and therefore the accuracy of these f is exactly $p = 2$, i.e., both constant and linear polynomials can be exactly reproduced from translates of f. Kovačević and Vetterli [KV92] conjectured that the solutions f resulting from the choices in (5.8) are continuous. Cohen and Daubechies [CD93] conjectured the same for the solutions resulting from the choice $a_2 = 0$ (which allows only $p = 1$). Villemoes [Vil94] proved that the solutions from (5.8) are continuous, while those from the choice $a_2 = 0$ are discontinuous. In any case, these f have orthogonal translates, and therefore can be used to construct multiresolution analyses and orthonormal wavelet bases for $L^2(\mathbf{R}^2)$.

APPENDIX

Convergence of the Infinite Matrix Product

Suppose that $f\colon \mathbf{R}^d \to \mathbf{C}^r$ satisfies the refinement equation (1.1). Define $m = |\det(A)|$ and $B = (A^{-1})^{\mathrm{T}}$. If f is integrable, then its Fourier transform will satisfy the equation

$$\hat{f}(\omega) \;=\; M(B\omega)\,\hat{f}(B\omega), \tag{A.1}$$

where $M(\omega)$ is the matrix-valued *symbol* of the refinement equation, defined by $M(\omega) = (1/m)\sum_{k\in\Lambda} c_k\, e^{-2\pi i k\cdot\omega}$. Iterating (A.1), we have

$$\hat{f}(\omega) \;=\; \left(\prod_{j=1}^{n} M(B^j\omega)\right)\hat{f}(B^n\omega) \;=\; P_n(\omega)\,\hat{f}(B^n\omega). \tag{A.2}$$

Since A is expansive, the spectral radius of B satisfies $\rho(B) < 1$. Therefore $B^n \to 0$ as $n \to \infty$. Since $\hat{f}$ is continuous, it follows that $\hat{f}(B^n\omega) \to \hat{f}(0)$ as

$n \to \infty$. If it were the case that $P_n(\omega)$ converged as $n \to \infty$, then it would follow from (A.2) that

$$\hat{f}(\omega) \;=\; \left(\prod_{j=1}^{\infty} M(B^j\omega)\right)\hat{f}(0) \;=\; P(\omega)\,\hat{f}(0).$$

These remarks remain valid even if f is merely a compactly supported tempered distribution, since, by the Paley–Wiener theorem for distributions, $\hat{f}$ will still be a continuous function in this case.

In this Appendix, we will provide a condition on the matrix

$$\Delta \;=\; M(0) \;=\; \frac{1}{m}\sum_{k\in\Lambda} c_k$$

that is both necessary and sufficient for the convergence of the infinite matrix product $P(\omega) = \prod_{j=1}^{\infty} M(B^j\omega)$ for each $\omega \in \mathbf{R}^d$. This generalizes a one-dimensional result of [HC96] to the higher-dimensional setting.

The necessary condition is immediate, since $P(0) = \prod_{j=1}^{\infty} M(0) = \Delta^{\infty}$ converges if and only if the matrix Δ has eigenvalues $\lambda_1 = \cdots = \lambda_s = 1$ and $|\lambda_{s+1}|, \dots, |\lambda_r| < 1$, with the eigenvalue 1 nondegenerate. We will show that this condition is also sufficient for the convergence of $P(\omega)$.

The following two lemmas are elementary.

Lemma A.1. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex-valued functions on a set K such that $\lim_{n\to\infty} a_n(\omega) = 0$ uniformly on K. Given $\lambda \in \mathbf{C}$, define*

$$s_n(\omega) \;=\; a_n(\omega) + \lambda a_{n-1}(\omega) + \cdots + \lambda^n a_0(\omega).$$

If $|\lambda| < 1$, then $\lim_{n\to\infty} s_n(\omega) = 0$ uniformly on K.

Lemma A.2. *Assume that S is an $r \times r$ matrix with eigenvalues $\lambda_1 = \cdots = \lambda_s = 1$ and $|\lambda_{s+1}|, \dots, |\lambda_r| < 1$, with the eigenvalue 1 nondegenerate. Then there exists a vector norm $\|\cdot\|$ on $\mathbf{C}^r$ such that the corresponding matrix norm of S is $\|S\| = 1$.*

Theorem A.3. *Assume that $\Delta = M(0) = (1/m)\sum_{k\in\Lambda} c_k$ has eigenvalues $\lambda_1 = \cdots = \lambda_s = 1$ and $|\lambda_{s+1}|, \dots, |\lambda_r| < 1$, with the eigenvalue 1 nondegenerate. Then the infinite matrix product*

$$P(\omega) \;=\; \prod_{j=1}^{\infty} M(B^j\omega)$$

converges uniformly on compact sets to a continuous matrix-valued function.

Proof. We will use a norm $|\cdot|$ on the vector space $\mathbf{R}^d$ and a norm $\|\cdot\|$ on the vector space $\mathbf{C}^r$. We use the same symbols to denote the matrix norms induced by these vector norms. We let $|\omega| = (|\omega_1|^2 + \cdots + |\omega_d|^2)^{1/2}$ denote the Euclidean norm on $\mathbf{R}^d$. By Lemma A.2, there exists a vector norm $\|\cdot\|$ on $\mathbf{C}^r$ so that the corresponding matrix norm of the $r \times r$ matrix Δ is $\|\Delta\| = 1$.

Since A is expansive, the $d \times d$ matrix $B = (A^{-1})^{\mathrm{T}}$ has spectral radius $\rho(B) < 1$. Fix θ with $\rho(B) < \theta < 1$. Then by standard results, there exists a constant R so that the matrix norm of B^n satisfies

$$|B^n| \;\le\; R\,\theta^n, \qquad n \ge 0.$$

Now, if $\{E_j\}_{j=1}^n$ is any collection of $r \times r$ matrices, then it follows from $\|\Delta\| = 1$ that

$$\begin{aligned} \Big\|\prod_{j=1}^n E_j\Big\| \;&\le\; \prod_{j=1}^n \|\Delta + (E_j - \Delta)\| \\ &\le\; \prod_{j=1}^n (1 + \|E_j - \Delta\|) \\ &\le\; \prod_{j=1}^n e^{\|E_j - \Delta\|} \\ &=\; \exp\Big(\sum_{j=1}^n \|E_j - \Delta\|\Big). \end{aligned} \tag{A.3}$$

Note that

$$\begin{aligned} \|M(\omega) - \Delta\| \;&=\; \|M(\omega) - M(0)\| \\ &=\; \Big\|\frac{1}{m}\sum_{k\in\Lambda}(e^{-2\pi i k\cdot\omega} - 1)\,c_k\Big\| \\ &\le\; \frac{1}{m}\Big(\max_{k\in\Lambda}\|c_k\|\Big)\sum_{k\in\Lambda}|e^{-2\pi i k\cdot\omega} - 1| \\ &\le\; \frac{2\pi}{m}\Big(\max_{k\in\Lambda}\|c_k\|\Big)\sum_{k\in\Lambda}|k\cdot\omega| \\ &\le\; \frac{2\pi}{m}\Big(\max_{k\in\Lambda}\|c_k\|\Big)\Big(\sum_{k\in\Lambda}|k|\Big)\,|\omega| \\ &=\; C_1\,|\omega|. \end{aligned}$$

As a consequence,

$$\|M(B^j\omega) - \Delta\| \;\le\; C_1\,|B^j\omega| \;\le\; C_1\,|B^j|\,|\omega| \;\le\; C_1\,R\,\theta^j\,|\omega|, \tag{A.4}$$

and therefore

$$\sum_{j=1}^{n} \|M(B^j\omega) - \Delta\| \;\le\; C_1\, R\, |\omega| \sum_{j=1}^{n} \theta^j \;\le\; C_2\, |\omega|. \tag{A.5}$$

Hence, if we define

$$P_n(\omega) \;=\; \prod_{j=1}^{n} M(B^j\omega),$$

then it follows from (A.3) and (A.5) that $\|P_n(\omega)\| \le e^{C_2|\omega|}$. In particular, if K is compact, then

$$C_K \;=\; \sup_n \sup_{\gamma\in K} \|P_n(\gamma)\| \;<\; \infty. \tag{A.6}$$

Now fix any eigenvalue λ for Δ. Let v be any corresponding λ-eigenvector, normalized so that $\|v\| = 1$. Then for any $\omega \in K$, we have from (A.4) and (A.6) that

$$\begin{aligned} \|P_n(\omega)v - \lambda P_{n-1}(\omega)v\| \;&=\; \|P_{n-1}(\omega)\,(M(B^n\omega)v - \Delta v)\| \\ &\le\; \|P_{n-1}(\omega)\|\, \|M(B^n\omega) - \Delta\|\, \|v\| \\ &\le\; C_K\, C_1\, R\, \theta^n\, |\omega|. \end{aligned} \tag{A.7}$$

Note that (A.7) implies immediately that $P_n(\omega)v$ converges uniformly on K if $\lambda = 1$. Consider then the other eigenvalues, which all satisfy $|\lambda| < 1$. Defining $P_0(\omega) = I$, we have

$$\begin{aligned} \|P_n(\omega)v\| \;&\le\; \|P_n(\omega)v - \lambda P_{n-1}(\omega)v\| \;+\; \|\lambda P_{n-1}(\omega)v - \lambda^2 P_{n-2}(\omega)v\| \\ &\qquad +\; \cdots \;+\; \|\lambda^{n-1}P_1(\omega)v - \lambda^n P_0(\omega)v\| \;+\; \|\lambda^n P_0(\omega)v\| \\ &\le\; C_K\, C_1\, R\, |\omega| \big(\theta^n \;+\; \theta^{n-1}\,|\lambda| \;+\; \cdots \;+\; \theta\,|\lambda|^{n-1} \;+\; |\lambda|^n\big). \end{aligned}$$

By Lemma A.1 or by direct computation, we conclude $\lim_{n\to\infty} P_n(\omega)v = 0$ uniformly on K when $|\lambda| < 1$.

If Δ is diagonalizable, then there is a basis $\{v_1, \dots, v_r\}$ for $\mathbf{C}^r$ consisting of eigenvectors of Δ. We have shown that $P_n(\omega)v_k$ converges uniformly on K for each of these v_k. Therefore, we can conclude that $P_n(\omega)$ itself converges uniformly on compact sets when Δ is diagonalizable.

For nondiagonalizable Δ, we proceed by considering the Jordan decomposition of Δ. Since the eigenvalue 1 for Δ is nondegenerate, we need only consider those eigenvalues λ with $|\lambda| < 1$. Let $U = \{u \in \mathbf{C}^r : (\Delta - \lambda)^k u = 0 \text{ for some } k\}$. There exists a smallest integer $\ell > 0$ such that $(\Delta - \lambda)^\ell u = 0$ for all $u \in U$. By standard Jordan techniques, there exists a basis $\{u_1, \dots, u_\ell\}$ for U such that

$$\Delta u_1 \;=\; \lambda u_1 \qquad \text{and} \qquad \Delta u_k \;=\; \lambda u_k \;+\; u_{k-1}, \quad k = 2, \dots, \ell.$$

We may assume that the u_k are normalized so that $\|u_k\| = 1$.

Since u_1 is a λ-eigenvector for Δ, we know from the above calculations that $\lim_{n\to\infty} P_n(\omega) u_1 = 0$ uniformly on K. Assume, inductively, that $\lim_{n\to\infty} P_n(\omega) u_{k-1} = 0$ uniformly on K for some $k > 1$. Then, by (A.4) and (A.6),

$$\begin{aligned}
&\|P_n(\omega)u_k - \lambda P_{n-1}(\omega)u_k - P_{n-1}(\omega)u_{k-1}\| \\
&\qquad \le\; \|P_{n-1}(\omega)\| \, \|M(B^n\omega)u_k - \lambda u_k - u_{k-1}\| \\
&\qquad \le\; C_K \, \|M(B^n\omega)u_k - \Delta u_k\| \\
&\qquad \le\; C_K \, \|M(B^n\omega) - \Delta\| \, \|u_k\| \\
&\qquad \le\; C_K \, C_1 \, R \, \theta^n \, |\omega|.
\end{aligned}$$

Therefore, adding and subtracting conveniently, we have

$$\begin{aligned}
\|P_n(\omega)u_k\| \;\le\;& \|P_n(\omega)u_k - \lambda P_{n-1}(\omega)u_k - P_{n-1}(\omega)u_{k-1}\| \\
&+\; \|\lambda P_{n-1}(\omega)u_k - \lambda^2 P_{n-2}(\omega)u_k - \lambda P_{n-2}(\omega)u_{k-1}\| \\
&+\; \cdots \;+\; \|\lambda^{n-1}P_1(\omega)u_k - \lambda^n P_0(\omega)u_k - \lambda^{n-1}P_0(\omega)u_{k-1}\| \\
&+\; \|\lambda^n P_0(\omega)u_k\| \;+\; \|P_{n-1}(\omega)u_{k-1}\| \\
&+\; \|\lambda P_{n-2}(\omega)u_{k-1}\| \;+\; \cdots \;+\; \|\lambda^{n-1}P_0(\omega)u_{k-1}\| \\
\le\;& C_K \, C_1 \, R \, |\omega| \bigl(\theta^n \;+\; \theta^{n-1}|\lambda| \;+\; \cdots \;+\; \theta \, |\lambda|^{n-1} \;+\; |\lambda|^n\bigr) \\
&+\; a_{n-1}(\omega) \;+\; |\lambda| \, a_{n-2}(\omega) \;+\; \cdots \;+\; |\lambda|^{n-1} a_0(\omega),
\end{aligned}$$

where $a_n(\omega) = \|P_n(\omega)u_{k-1}\|$. It follows from Lemma A.1 that $P_n(\omega)u_k$ converges to zero uniformly on K, completing the induction.

It remains only to note that since each $P_n(\omega)$ is continuous and $P_n(\omega)$ converges uniformly on compact sets, the limit must be continuous. $\square$

Corollary A.4. *Assume that $\Delta = M(0) = (1/m)\sum_{k\in\Lambda} c_k$ has eigenvalues $\lambda_1 = \cdots = \lambda_s = 1$ and $|\lambda_{s+1}|, \dots, |\lambda_r| < 1$, with the eigenvalue 1 nondegenerate. If there exists a distributional solution f to the refinement equation (1.1) whose Fourier transform $\hat f$ is a continuous function, then $\hat f(\omega) = P(\omega)\,\hat f(0)$. As a consequence, $\hat f(0) \neq 0$, and $\hat f(0)$ is a right 1-eigenvector for Δ.*

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REFERENCES

[deB90] C. de Boor, Quasiinterpolants and approximation power of multivariate splines, *in* "Computation of Curves and Surfaces" (M. Gasca and C. A. Micchelli, eds.), Kluwer, Dordrecht, 1990, pp. 313–345.

[BR92] C. de Boor and A. Ron, The exponentials in the span of the integer translates of a compactly supported function, *J. London Math. Soc.* **45** (1992), 519–535.

[BDR94a] C. de Boor, R. De Vore, and A. Ron, Approximation from shift-invariant subspaces of $L_2(\mathbf{R}^d)$, *Trans. Amer. Math. Soc.* **341** (1994), 787–806.

[BDR94b] C. de Boor, R. De Vore, and A. Ron, The structure of finitely generated shift-invariant subspaces of $L_2(\mathbf{R}^d)$, *J. Funct. Anal.* **119** (1994), 37–78.

[CDM91] A. Cavaretta, W. Dahmen, and C. A. Micchelli, "Stationary Subdivision", Mem. Amer. Math. Soc., vol. 93, no. 453, Amer. Math. Soc., Providence, RI, 1991.

[CD93] A. Cohen and I. Daubechies, Non-separable bidimensional wavelet bases, *Rev. Mat. Iberoamericana* **9** (1993), 51–137.

[CDP97] A. Cohen, I. Daubechies, and G. Plonka, Regularity of refinable function vectors, *J. Fourier Anal. Appl.* **3** (1997), 295–324.

[Dau92] I. Daubechies, "Ten Lectures on Wavelets", SIAM, Philadelphia, 1992.

[DL91] I. Daubechies and J. C. Lagarias, Two-scale difference equations: I. Existence and global regularity of solutions, *SIAM J. Math. Anal.* **22** (1991), 1388–1410.

[DGHM96] G. Donovan, J. S. Geronimo, D. P. Hardin, and P. R. Massopust, Construction of orthogonal wavelets using fractal interpolation functions, *SIAM J. Math. Anal.* **47** (1996), 1158–1192.

[GHM94] J. S. Geronimo, D. P. Hardin, and P. R. Massopust, Fractal functions and wavelet expansions based on several scaling functions, *J. Approx. Th.* **78** (1994), 373–401.

[GLT93] T. N. T. Goodman, S. L. Lee, and W.-S. Tang, Wavelets in wandering subspaces, *Trans. Amer. Math. Soc.* **338** (1993), 639–654.

[GM92] K. Gröchenig and W. R. Madych, Multiresolution analysis, Haar bases, and self-similar tilings of $\mathbf{R}^n$, *IEEE Trans. Inform. Theory* **38** (1992), 556–568.

[HC96] C. Heil and D. Colella, Matrix refinement equations: existence and uniqueness, *J. Fourier Anal. Appl.* **2** (1996), 363–377.

[HSS96] C. Heil, G. Strang, and V. Strela, Approximation by translates of refinable functions, *Numerische Math.* **73** (1996), 75–94.

[Jia95] R. Q. Jia, Refinable shift-invariant spaces: from splines to wavelets, *in* "Approximation Theory VIII," Vol. 2 (C. K. Chui and L. L. Schumaker, eds.), World Scientific, Singapore, 1995, pp. 179–208.

[Jia98] R. Q. Jia, Approximation properties of multivariate wavelets, *Math. Comp.* **67** (1998), 647–665.

[JRZ97] R. Q. Jia, S. D. Riemenschneider, and D. X. Zhou, Approximation by multiple refinable functions, *Canad. J. Math.* **49** (1997), 944–962.

[Jng96] Q. Jiang, Multivariate matrix refinable functions with arbitrary matrix dilation, *preprint* (1996).

[KV92] J. Kovačević and M. Vetterli, Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for $\mathbf{R}^n$, *IEEE Trans. Inform. Theory* **38** (1992), 533–555.

[Mey92] Y. Meyer, "Wavelets and Operators", Cambridge Univ. Press, Cambridge, 1992.

[Plo97] G. Plonka, Approximation order provided by refinable function vectors, *Constr. Approx.* **13** (1997), 221–244.

[PS98] G. Plonka and V. Strela, Construction of multi-scaling functions with approximation and symmetry, *SIAM J. Math. Anal.* **29** (1998), 481–510.

[SF73] G. Strang and G. Fix, A Fourier analysis of the finite-element variational method, *in* "Constructive Aspects of Functional Analysis" (G. Geymonat, ed.), CIME, 1973, pp. 793–840.

[SS94] G. Strang and V. Strela, Orthogonal multiwavelets with vanishing moments, *J. Optical Engrg.* **33** (1994), 2104–2107.

[Vil94] L. F. Villemoes, Continuity of nonseparable quincunx wavelets, *Appl. Comput. Harmon. Anal.* **1** (1994), 180–187.

[Wal82] P. Walters, "An Introduction to Ergodic Theory", Springer–Verlag, New York, 1982.