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Approximation by translates of refinable functions

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Summary. The functions $f_1(x), \ldots, f_r(x)$ are *refinable* if they are combinations of the rescaled and translated functions $f_i(2x-k)$. This is very common in scientific computing on a regular mesh. The space V_0 of approximating functions with meshwidth $h = 1$ is a subspace of V_1 with meshwidth $h = 1/2$. These refinable spaces have refinable basis functions. The accuracy of the computations depends on p, the *order of approximation*, which is determined by the degree of polynomials $1, x, \ldots, x^{p-1}$ that lie in V_0.

Most refinable functions (such as scaling functions in the theory of wavelets) have no simple formulas. The functions $f_i(x)$ are known only through the coefficients c_k in the refinement equation—scalars in the traditional case, $r \times r$ matrices for multiwavelets. The scalar "sum rules" that determine p are well known. We find the conditions on the matrices c_k that yield approximation of order p from V_0. These are equivalent to the Strang–Fix conditions on the Fourier transforms $\hat{f}_i(\omega)$, but for refinable functions they can be explicitly verified from the c_k.

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1. Introduction.

A function $f(x)$ is *refinable* if it satisfies a two-scale equation

$$f(x) \;=\; \sum_{k=0}^{N} c_k\, f(2x-k). \tag{1}$$

This is a "refinement equation" or "dilation equation." It is satisfied by splines $f(x)$, and by finite elements. It is the starting point for the construction of wavelets (in that theory, $f(x)$ is the scaling function). Equation (1) is also basic

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to the theory of recursive subdivision, where $f(x)$ is the fixed point—the function that stays invariant at each subdivision step. The choice of the coefficients $c_0, \ldots, c_N$ controls the properties of $f(x)$. Thus the c_k totally govern the effectiveness of the wavelets or the subdivision scheme. We assume a finite number of coefficients (N finite, f supported on $[0, N]$), and we study one property in particular.

That property is *approximation by translates*. For wavelets and finite elements and splines, a known or unknown function is approximated by combinations $\sum b_k\, f(\frac{x}{h} - k)$. The asymptotic accuracy of this approximation is decided by the number of polynomials $1, x, \ldots, x^{p-1}$ that can be exactly reproduced from combinations of the translates $f(x-k)$. The approximation error will decrease like h^p, as $h \to 0$. When the functions are piecewise polynomials, the approximation order p is clear. But in general this number must be determined by the coefficients c_k that produce $f(x)$. *We want to compute p from those coefficients.*

In the scalar case, when the c_k are real or complex numbers, the requirement is a set of p "sum rules":

$$\sum_{k=0}^{N} (-1)^k\, k^j\, c_k \;=\; 0, \qquad j = 0, \ldots, p-1. \tag{2}$$

The example $c_0 = c_1 = 1$ satisfies only the first sum rule ($p = 1$). For those coefficients, $f(x)$ is the unit box function—the characteristic function of $[0, 1]$. Equation (1) says that this box is the sum of two half-boxes. The sum of all translates is clearly a constant function on the whole line. The error in approximation by translates of $f(x)$ (in other words, by piecewise constant functions), is $O(h)$ when the meshwidth is h. This confirms the equality between the order of accuracy (the power of h, which is $p = 1$) and the number of sum rules. The precise relation between approximation order and polynomial reproduction has a large literature.

For cubic splines, the coefficients c_k are $\frac{1}{8}$, $\frac{4}{8}$, $\frac{6}{8}$, $\frac{4}{8}$, $\frac{1}{8}$. Those coefficients satisfy $p = 4$ sum rules. The refinable function $f(x)$ is a cubic B-spline, and the error in approximation is $O(h^4)$. Splines of any degree $p-1$ satisfy p sum rules and yield pth order accuracy—the approximation error is $O(h^p)$. Note that the sum of coefficients is always $c_0 + \cdots + c_N = 2$, from integrating both sides of equation (1) and changing variables:

$$\int_{-\infty}^{\infty} f(x)\, dx \;=\; \sum_{k=0}^{N} c_k \int_{-\infty}^{\infty} f(2x-k)\, dx \;=\; \frac{1}{2} \sum_{k=0}^{N} c_k \int_{-\infty}^{\infty} f(x)\, dx.$$

If this integral of $f(x)$ is nonzero—as we want and need, to produce at least the constant function 1 in a stable way from the translates—then

$$\frac{1}{2} \sum_{k=0}^{N} c_k \;=\; 1 \qquad \text{or} \qquad c_0 + \cdots + c_N = 2. \tag{3}$$

The contribution of this paper is to study the *matrix case*, when the coefficients c_k are r by r matrices. Then $f(x)$ is a vector-valued function with components

$f_1(x), \ldots, f_r(x)$. That set of functions is still called *refinable*. If the translates $f_i(x-k)$ can reproduce the polynomials $1, x, \ldots, x^{p-1}$, the approximation accuracy is still p. Our question is: *What are the sum rules in the matrix case*? We are looking for the conditions on the matrices $c_0, \ldots, c_N$ that determine p. These conditions are much weaker than a literal extension of (2) from scalars to matrices.

A previous note [SS1] indicated the correct matrix condition and sketched the reasoning. Here we give a precise formulation and proof. There is a finite and constructive test on the c_k, and it yields the combinations of $f_i(x-n)$ that produce each power x^j for $j < p$. We will connect those matrix conditions to the so-called Strang–Fix conditions [SF1], [SF2] on the functions $f_1(x), \ldots, f_r(x)$. An essential point is that *in the refinable case*, when we start with matrix coefficients c_k rather than functions $f_i(x)$ and their translates, *the test for accuracy p becomes constructive and convenient.*

We discuss examples, and applications to finite elements. Those are refinable! Piecewise polynomial spaces on regular grids are an important source of examples. The approximation conditions extend to the multidimensional case $x \in \mathbf{R}^d$, where the outstanding reference is [CDM]. The conditions also appear in the Daubechies–Lagarias analysis of the convergence of the cascade algorithm [DL], [D], which is used to compute $f(x)$.

A new source of examples is the theory of "multiwavelets" developed by Donovan, Geronimo, Hardin, and Massopust [GHM], [DGHM]. By fractal interpolation they created two functions f_1 and f_2 that satisfy a matrix refinement equation of the form (1). Those scaling functions lead to two orthogonal wavelets [DGHM], [SS2], with extra properties that could not be achieved with $p = 2$ in the scalar case: especially symmetry and short support. (Orthogonality and symmetry come from further conditions [D], [SN] imposed on the c_k.) More multiwavelets are appearing [SS3], [GL], [P1]. Some of them are piecewise polynomial "finite elements with orthogonality." It is rather amazing that orthogonal polynomials (now piecewise) should be undiscovered for so long.

2. Fourier transform and the approximation conditions.

In the scalar case, the Strang–Fix conditions for approximation of order p are usually applied to the Fourier transform of $f(x)$:

(4) $\hat{f}(\omega)$ must have zeros of order p at all $\omega = 2\pi n, \;\; n \neq 0.$

This connects directly to the sum rules, when $f(x)$ is refinable. The Fourier transform of equation (1) is found to be

$$\hat{f}(\omega) \;=\; \left(\frac{1}{2}\sum_{k=0}^{N} c_k\, e^{-ik\omega/2}\right)\hat{f}(\omega/2) \;=\; M(\omega/2)\,\hat{f}(\omega/2). \tag{5}$$

Replacing ω by $\omega/2$, the right side reduces further to $M(\omega/2)\,M(\omega/4)\,\hat{f}(\omega/4)$. In the limit of this recursion we find $\hat{f}(\omega)$ as an infinite product:

$$(6) \qquad \hat{f}(\omega) \;=\; \left(\prod_{j=1}^{\infty} M(\omega/2^j)\right)\hat{f}(0) \quad \text{with} \quad M(\omega) \;=\; \frac{1}{2}\sum_{k=0}^{N} c_k\, e^{-ik\omega}.$$

Equation (3) ensures that $M(0) = 1$. The p sum rules translate directly to a condition on this trigonometric polynomial $M(\omega)$, which has the coefficients $\frac{1}{2}c_k$:

$$(7) \qquad M(\omega) \text{ has a zero of order } p \text{ at } \omega = \pi.$$

It is straightforward to see how this zero of $M(\omega)$ produces the required zeros of $\hat{f}(\omega)$. At $\omega = 2\pi$, the first factor in the infinite product (6) is $M(\pi)$ itself. At every $\omega = 2\pi n$ we write $n = 2^j q$ with q odd, and find that the $(j+1)$st factor in the product is $M(q\pi)$. By periodicity this is $M(\pi)$. Then the pth order zero of M produces the pth order zeros of $\hat{f}$ required in (4).

The matrix case is not so straightforward. Equation (5) still leads to the infinite product in (6), but convergence for all ω is not automatic [HC]. (Equation (3) does not become $M(0) =$ identity matrix; we can only expect that an eigenvalue of $M(0)$ equals one.) Condition (4) is changed and therefore condition (7) must change. With r functions $f_1, \ldots, f_r$, the approximating order is p if and only if there exists a *superfunction* $S(x)$ which achieves that order by itself. That superfunction is a combination of translates:

$$(8) \qquad S(x) \;=\; \sum_{n=0}^{s}\sum_{i=1}^{r} a_{in}\, f_i(x-n) \;\text{ yields an } \hat{S}(\omega) \text{ that satisfies (4).}$$

The problem has always been: *How to find* $S(x)$? In concrete examples, the answer is often clear (and the order p is clear). For piecewise polynomials, it frequently happens that $S(x)$ is a spline—created from the r functions $f_i(x)$. Shorter support is the attraction of finite elements and the reason for preferring r functions. (A single spline is supported on p intervals, and a single scaling function and orthogonal wavelet are supported on at least $2p-1$ intervals—both inconveniently long in the presence of boundaries.) For nonpolynomial functions $f_1, \ldots, f_r$ it can be very difficult to implement the superfunction test. We refer to the deep analysis in [BDR1], [BDR2], [J]. For the refinable case this paper does not need to work with the superfunction.

Note added in proof: We have just received an excellent paper by Plonka [P2] that computes and uses the (refinable) superfunction. She also gives a new formulation of the accuracy condition, through a factorization of the matrix polynomial $M(\omega)$. This is the extension to matrices of the factor $(1 + e^{-i\omega})^p$ that produces p zeros at $\omega = \pi$ in the scalar case.

It was spline theory and especially finite element theory that led to conditions (4) and (8). The question of "refinability" did not arise when these conditions were established. But it is easy to see that finite element spaces (on a regular mesh) are indeed refinable. The space V_0 spanned by the finite element basis functions with meshsize $h = 1$ is invariably a subspace of V_1 with meshsize $h = 1/2$. Every C^1 cubic on unit intervals (to pick a specific example with

$r=2$) is a fortiori a C^1 cubic on half-unit intervals. The basis functions f_1 and f_2 interpolate $y(0)=1$ and $y'(0)=1$, respectively, with all other nodal values set to zero. A suitable combination of translates will produce the cubic B-spline $S(x)$. These functions are refinable, and the 2 by 2 matrix coefficients c_k must satisfy $p=4$ sum rules—but they are not the sum rules in (2).

The property $V_0 \subset V_1$, refinability of spaces and of basis functions, is surely important to numerical analysis. It underlies the multigrid method and the "hierarchical bases" for finite elements and other trial spaces. It allows local mesh refinement; the mesh can be adapted to the problem. The shorter the support, the more local and convenient this adaptation will be. We need a way to determine the order p, when the basis functions arise as solutions of (1). The solution formula (6) is very rarely useful, and condition (8) becomes virtually impossible to verify. But refinability overcomes the difficulty. We will find an equivalent and convenient test on the matrix polynomial $M(\omega)$.

3. Eigenvalues and eigenvectors of $M(\pi)$.

The case $p=1$ of this test will be no surprise. Where the scalar requirement was $M(\pi)=0$, the matrix requirement becomes $\det M(\pi)=0$. Thus $M(\pi)$ *is singular*. This is the "zero at π" expected in wavelet theory. It leads to constant functions in V_0 and a vanishing moment for wavelets. The left nullvector u (which is also a left eigenvector of $M(0)$) plays a critical role:

$$(9)\quad u\,M(0)=u \text{ and } u\,M(\pi)=0 \text{ if and only if } \sum_{k=-\infty}^{\infty} u\,f(x-k) \equiv \text{constant.}$$

Here $u\,f(x-k)$ is the dot product $u_1\,f_1(x-k)+\cdots+u_r\,f_r(x-k)$ from row times column. By exhibiting how to construct a (nonzero) constant function from the translates of $f_1,\ldots,f_r$, the order of accuracy is confirmed to be at least $p=1$.

Notice that the link from information on $M(0)$ and $M(\pi)$ to the "first sum rule" is still correct:

$$(10)\quad u\,M(0)=u \text{ and } u\,M(\pi)=0 \text{ if and only if } u\sum c_{2k}=u\sum c_{2k+1}=u.$$

This comes from adding and subtracting the equations

$$\begin{aligned} u\,M(0) &= \frac{1}{2}u\,(c_0+c_1+c_2+\ldots) = u,\\ u\,M(\pi) &= \frac{1}{2}u\,(c_0-c_1+c_2-\ldots) = 0. \end{aligned}$$

We now indicate the two steps behind statement (9), which is the case $p=1$ of our main theorem. First, use the refinement equation to replace $f(x)$ in the crucial function $G_0(x)$:

$$G_0(x) \;\equiv\; \sum_k u\, f(x-k) \;=\; \sum_k u \sum_\ell c_\ell\, f(2x-2k-\ell). \tag{11}$$

Now separate (11) into "even and odd sums" and apply the assumption (10). The result of that step is that $G_0(x) = G_0(2x)$. From this we deduce (not instantly, and needing hypotheses) that $G_0(x)$ is constant and nonzero. This is statement (9)—the translates of $f_1, \ldots, f_r$ produce a constant function. The converse in(9) comes (also not instantly) by reversing the argument.

The steps that lead to $G_0(x) = G_0(2x)$ are simple, and typical of more complicated arguments to follow later. In fact, this is the case $j = 0$ in equation (20), which is the kernel of the whole reasoning. Here we rewrite $G_0(x)$ in (11) as

$$\begin{aligned}
G_0(x) \;&=\; \sum_\ell u\, c_{2\ell} \sum_k f(2x-2k-2\ell) \;+\; \sum_\ell u\, c_{2\ell+1} \sum_k f(2x-2k-2\ell-1) \\
&=\; \sum_\ell u\, c_{2\ell} \sum_k f(2x-2k) \;+\; \sum_\ell u\, c_{2\ell+1} \sum_k f(2x-2k-1) \\
&=\; u \sum_k f(2x-2k) \;+\; u \sum_k f(2x-2k-1) \\
&=\; G_0(2x).
\end{aligned}$$

The higher sum rules naturally involve the derivatives $M'(\pi), M''(\pi), \ldots, M^{(p-1)}(\pi)$. But in the matrix case they are *not* a statement that these matrices are singular. Instead, a sequence of vectors $y_0^{(j)}$ will enter equations like $u\, M(0) = u$ and $u\, M(\pi) = 0$—in fact the first vector $y_0^{(0)}$ is u. Here are the (recursive) requirements on $M(0)$ and $M(\pi)$:

Theorem 1. *Assume f is an integrable solution of the matrix refinement equation* (1) *such that the integer translates of $f_1, \ldots, f_r$ are independent. Then f has accuracy p if and only if there are vectors $y_0^{(0)}, \ldots, y_0^{(p-1)} \in \mathbf{C}^r$ satisfying* (12) *for $j = 0, \ldots, p-1$:*

$$\begin{aligned}
&\sum_{m=0}^{j} \binom{j}{m} 2^m\, i^{j-m}\, y_0^{(m)}\, M^{(j-m)}(\pi) \;=\; 0, \\
&\sum_{m=0}^{j} \binom{j}{m} 2^m\, i^{j-m}\, y_0^{(m)}\, M^{(j-m)}(0) \;=\; y_0^{(j)}.
\end{aligned} \tag{12}$$

The proof of Theorem 1 occupies us through Sections 4 and 5. The only real inputs are equation (12) and the refinement equation (1). The approach is to regard (1) as an operator equation $F(x) = L\, F(2x)$, for a *downsampled Toeplitz operator* L. Then (12) implies that L has eigenvalues $1, \frac{1}{2}, \ldots, (\frac{1}{2})^{p-1}$ with eigenvectors of a special form. Those two steps in (13) and (15) lead to $G_j(x) = 2^{-j}\, G_j(2x)$ and eventually to $G_j(x) = C\, x^j$. The combination $G_j(x)$, using the row vectors $y_0^{(j)}$ in (12), demonstrates how to construct x^j from the translates of $f(x)$.

Following the proof of Theorem 1, Section 6 illustrates the implementation of these *matrix sum rules*. Section 7 shows how the matrix sum rules (12) simplify in the scalar case to the usual sum rules (2). Our final section is motivated by the following fact: Theorem 1 requires us to verify that the integer translates of $f_1, \ldots, f_r$ are independent. In Section 8 we derive a condition directly on the matrices c_k that gives one direction of Theorem 1 without the need to verify independence. This condition applies even when independence fails. Specifically, we show that if $M(0) = \frac{1}{2} \sum c_k$ has 1 as a simple eigenvalue and all other eigenvalues are strictly less than 1 in absolute value, then the matrix sum rules in (12) imply accuracy p for f. This leads to further observations of the link between accuracy and the form of the eigenvectors $y^{(j)}$, and to some open questions.

4. Implications of accuracy.

Our insight into the proof of Theorem 1 follows from transforming the refinement equation (1) into a two-scale matrix equation $L\,F(2x) = F(x)$ with infinite matrices and vectors. L is the doubly infinite matrix given in block form as $L_{ij} = c_{2i-j}$:

$$L = \begin{bmatrix} \ddots & & & & & & & & & \\ \cdots & c_3 & c_2 & c_1 & c_0 & & & & & \\ & & \cdots & c_3 & c_2 & c_1 & c_0 & & & \\ & & & & \cdots & c_3 & c_2 & c_1 & c_0 & \\ & & & & & & & & & \ddots \end{bmatrix}.$$

Note the double shift between rows. F is a vectorized form of f:

$$F(x) = \begin{bmatrix} \vdots \\ f(x-1) \\ f(x) \\ f(x+1) \\ \vdots \end{bmatrix}.$$

Recall that $f(x)$ is itself a column vector: $f(x) = (f_1(x), \ldots, f_r(x))^{\mathrm{t}}$. Multiplying L times $F(2x)$ expresses the refinement equation (1) in the equivalent form

$$L\,F(2x) = F(x). \tag{13}$$

Here and throughout, equalities are interpreted as holding almost everywhere.

Assume now that an integrable scaling function f is given which has accuracy p. (Existence and uniqueness of solutions to matrix refinement equations is considered in [HC].) Integrability implies compact support; in fact, $\mathrm{supp}(f) \subset [0, N]$. Accuracy p says there are row vectors $y_k^{(j)}$ (each of length r) such that

$$\sum_k y_k^{(j)}\, f(x+k) = x^j \tag{14}$$

for each $j = 0, \ldots, p-1$. Because f has compact support, the summation over k in (14) is finite for each x. Let $y^{(j)} = \begin{bmatrix} \cdots & y_0^{(j)} & y_1^{(j)} & y_2^{(j)} & \cdots \end{bmatrix}$. Then (14) is

$$y^{(j)}\, F(x) \;=\; x^j.$$

So from (13),

$$y^{(j)}\, L\, F(x) \;=\; y^{(j)}\, F(x/2) \;=\; (x/2)^j \;=\; 2^{-j}\, y^{(j)}\, F(x). \tag{15}$$

Here we insert a crucial assumption—reasonable in applications, required for wavelets, satisfied by "almost all" refinement equations—that the translates of $f_1, \ldots, f_r$ are independent. That is,

$$\sum_{i=1}^{r} \sum_{k} a_{ik}\, f_i(x+k) \;\equiv\; 0 \qquad \Longleftrightarrow \qquad \text{every } a_{ik} = 0.$$

In vector form a is an arbitrary infinite row vector:

$$a\, F(x) \;\equiv\; 0 \qquad \Longleftrightarrow \qquad a = 0.$$

In this case, (15) implies that $y^{(j)}$ is a *left eigenvector* of L:

$$y^{(j)}\, L \;=\; 2^{-j}\, y^{(j)}.$$

Thus, accuracy p implies that L has eigenvalues $1, \frac{1}{2}, \ldots, (\frac{1}{2})^{p-1}$.

Accuracy implies more: the equation $y^{(j)} F(x-\ell) = (x-\ell)^j$ imparts structure to the row vector $y^{(j)}$. We have

$$\begin{aligned} \sum_k y_{k+\ell}^{(j)}\, f(x+k) \;=\; \sum_k y_k^{(j)}\, f(x-\ell+k) \;&=\; (x-\ell)^j \\ &=\; \sum_{m=0}^{j} \binom{j}{m} x^m\, (-\ell)^{j-m} \\ &=\; \sum_{m=0}^{j} \binom{j}{m} (-\ell)^{j-m} \sum_k y_k^{(m)}\, f(x+k). \end{aligned} \tag{16}$$

Our independence assumption implies that the coefficients of $f(x+k)$ on both sides of (16) must be equal:

$$y_{k+\ell}^{(j)} \;=\; \sum_{m=0}^{j} \binom{j}{m} (-\ell)^{j-m}\, y_k^{(m)}. \tag{17}$$

In particular, the case $k = 0$ yields

$$y_{\ell}^{(j)} \;=\; \sum_{m=0}^{j} \binom{j}{m} (-\ell)^{j-m}\, y_0^{(m)}. \tag{18}$$

Thus the finite vectors $y_0^{(0)}, \ldots, y_0^{(j)}$, each with r components, determine the entire eigenvector $y^{(j)}$. For $j = 0, 1, 2$ we call these finite vectors u, v, w:

$$\begin{aligned} y^{(0)} &= [\cdots \quad u \quad u \quad \cdots \quad u \quad \cdots], \\ y^{(1)} &= [\cdots \quad v \quad v-u \quad \cdots \quad v-ku \quad \cdots], \\ y^{(2)} &= [\cdots \quad w \quad w-2v+u \quad \cdots \quad w-2kv+k^2u \quad \cdots]. \end{aligned}$$

We return below to the vectors $y_0^{(j)}$. Now we prove the converse to the above: Eigenvalues of L combined with prescribed structure for the eigenvectors implies accuracy. Specifically, assume that f is an integrable solution of the matrix refinement equation, that the translates of $f_1, \ldots, f_r$ are independent, and that L has eigenvalues $1, \frac{1}{2}, \ldots, (\frac{1}{2})^{p-1}$ with corresponding left eigenvectors of the form (18). Write G_j for the combination of the translates of $f_1, \ldots, f_r$ determined by the eigenvector $y^{(j)}$:

$$G_j(x) \;=\; \sum_k y_k^{(j)}\, f(x+k) \;=\; y^{(j)}\, F(x). \tag{19}$$

Then from the refinement equation $L\,F(2x) = F(x)$ we find

$$G_j(x) \;=\; y^{(j)}\, L\, F(2x) \;=\; 2^{-j}\, y^{(j)}\, F(2x) \;=\; 2^{-j}\, G_j(2x). \tag{20}$$

When G_j is continuous, this implies immediately that $G_j(x)$ is a multiple of x^j. But this is true even in the discontinuous case, as we now show. We use the independence of translates hypothesis to ensure that G_j is a *nonzero* multiple of x^j. In Section 8 we give an alternative hypothesis, directly on the matrices c_k, which also ensures that the multiple is nonzero.

We proceed by induction. For $j = 0$, (19) and (20) give $G_0(x+1) = G_0(x)$ and $G_0(x) = G_0(2x)$. The mapping $\tau x = 2x \bmod 1$ from $[0, 1)$ onto itself is ergodic. Therefore, the Birkhoff Ergodic Theorem [Wal, Theorem 1.14] implies that there is a constant C such that

$$G_0(x) \;=\; \frac{1}{n} \sum_{k=0}^{n-1} G_0(\tau^k x) \;\to\; C \quad \text{a.e.} \qquad \text{as } n \to \infty. \tag{21}$$

Hence $G_0(x) = C$ a.e. Our independence assumption implies that $C \neq 0$. By scaling the eigenvector $u = y_0^{(0)}$, we can set $C = 1$, or $G_0(x) = 1$ a.e.

Suppose now that $G_m(x) = x^m$ for $m = 0, \ldots, j-1$. Although G_j is not 1-periodic, there is a functional relation between $G_j(x)$ and $G_j(x+1)$. To derive this, note from (17) that

$$y_{k-1}^{(j)} \;=\; \sum_{m=0}^{j} \binom{j}{m} y_k^{(m)}.$$

Therefore,

$$G_j(x+1) = \sum_k y_k^{(j)} f(x+1+k) = \sum_k y_{k-1}^{(j)} f(x+k)$$

$$= \sum_k \sum_{m=0}^{j} \binom{j}{m} y_k^{(m)} f(x+k) = \sum_{m=0}^{j} \binom{j}{m} G_m(x)$$

$$= G_j(x) + \sum_{m=0}^{j-1} \binom{j}{m} x^m = G_j(x) + (x+1)^j - x^j. \tag{22}$$

Set $H_j(x) = G_j(x) - x^j$. Then, by (20) and (22), $H_j(x+1) = H_j(x)$ and $H_j(x) = 2^{-j} H_j(2x)$. Therefore, the Birkhoff Ergodic Theorem implies that there is a constant C_j such that

$$\frac{1}{n} \sum_{k=0}^{n-1} H_j(\tau^k x) \to C_j \quad \text{a.e.} \qquad \text{as } n \to \infty.$$

The value of C_j is unimportant here: we simply observe that

$$\frac{1}{n} \sum_{k=0}^{n-1} H_j(\tau^k x) = \frac{2^{jn} - 1}{n\,(2^j - 1)} H_j(x)$$

and that $(2^{jn} - 1)/(n\,(2^j - 1)) \to \infty$ as $n \to \infty$. This means we must have $H_j(x) = 0$ a.e., completing the induction.

We have proved:

Theorem 2. *Assume f is an integrable solution of the matrix refinement equation (3) such that the integer translates of $f_1, \dots, f_r$ are independent. Then f has accuracy p if and only if L has eigenvalues $1, \frac{1}{2}, \dots, (\frac{1}{2})^{p-1}$ with corresponding left eigenvectors $y^{(0)}, \dots, y^{(p-1)}$ satisfying (18) for some $y_0^{(0)}, \dots, y_0^{(p-1)} \in \mathbf{C}^r$. In this case, there exists a nonzero constant C such that $y^{(j)} F(x) = C\, x^j$ a.e. for $j = 0, \dots, p-1$.*

We still must show how the eigenvalue structure equation (18) relates to the matrix sum rules (12).

5. Form of the eigenvectors.

The vectors $y_0^{(0)}, \dots, y_0^{(p-1)}$ determine the accuracy. They cannot be arbitrary.

Theorem 3. *Given vectors $y_0^{(0)}, \dots, y_0^{(p-1)} \in \mathbf{C}^r$, let (18) define the vectors $y_k^{(j)}$ and therefore $y^{(j)}$ for $j = 0, \dots, p-1$ and all k. Then $y^{(j)} L = 2^{-j} y^{(j)}$ for $j = 0, \dots, p-1$ if and only if the following two finite equations are satisfied for $j = 0, \dots, p-1$:*

$$
(23)\qquad
\begin{aligned}
\sum_k y_k^{(j)}\, c_{2k+1} &= 2^{-j}\, y_{-1}^{(j)} = 2^{-j} \sum_{m=0}^{j} \binom{j}{m} y_0^{(m)},\\
\sum_k y_k^{(j)}\, c_{2k} &= 2^{-j}\, y_0^{(j)}.
\end{aligned}
$$

Proof. Looking at the block structure of L, we rewrite $y^{(j)}\, L = 2^{-j}\, y^{(j)}$ as the infinite set of equations

$$
(24)\qquad
\begin{aligned}
\sum_k y_{k+\ell}^{(j)}\, c_{2k+1} &= 2^{-j}\, y_{2\ell-1}^{(j)},\\
\sum_k y_{k+\ell}^{(j)}\, c_{2k} &= 2^{-j}\, y_{2\ell}^{(j)},
\end{aligned}
\qquad \ell \in \mathbf{Z}.
$$

The equations in (23) are the particular case of (24) where $\ell = 0$. So, we need only prove that if (24) is true when $\ell = 0$ then it is true for all ℓ.

We proceed by induction on j. For $j = 0$, (24) simplifies dramatically since $y_k^{(0)} = y_0^{(0)}$ for all k. In fact, (24) becomes

$$
\begin{aligned}
\sum_k y_0^{(0)}\, c_{2k+1} &= y_0^{(0)},\\
\sum_k y_0^{(0)}\, c_{2k} &= y_0^{(0)},
\end{aligned}
$$

which has no dependence on ℓ, and equals (23) with $j = 0$.

Now suppose that (23) is valid for $0 \le j \le n-1$. We show that the first equation in (24) is valid for $j = n$ and all $\ell \in \mathbf{Z}$ as follows. Using the induction hypothesis, we compute:

$$
\begin{aligned}
\sum_k y_{k+\ell}^{(n)}\, c_{2k+1} &= \sum_k \sum_{m=0}^{n} \binom{n}{m} (-\ell)^{n-m}\, y_k^{(m)}\, c_{2k+1} && \text{by (17)}\\
&= \sum_{m=0}^{n} \binom{n}{m} (-\ell)^{n-m} \sum_k y_k^{(m)}\, c_{2k+1}\\
&= \sum_{m=0}^{n} \binom{n}{m} (-\ell)^{n-m}\, 2^{-m}\, y_{-1}^{(m)} && \text{by (23)}\\
&= 2^{-n} \sum_{m=0}^{n} \binom{n}{m} (-2\ell)^{n-m}\, y_{-1}^{(m)}\\
&= 2^{-n}\, y_{2\ell-1}^{(n)}. && \text{by (17)}
\end{aligned}
$$

An analogous computation shows that the second equation in (24) holds for $j = n$ and all $\ell \in \mathbf{Z}$, and finishes the proof. □

For $j = 0, 1, 2$, the equations in (23) have the form:

$$\sum_k u\,c_{2k+1} \;=\; u, \qquad \sum_k u\,c_{2k} \;=\; u,$$

$$\sum_k (v-ku)\,c_{2k+1} \;=\; \frac{1}{2}\,(v+u), \qquad \sum_k (v-ku)\,c_{2k} \;=\; \frac{1}{2}\,v,$$

$$\sum_k (w-2kv+k^2u)\,c_{2k+1} \;=\; \frac{1}{4}\,(w+2v+u), \quad \sum_k (w-2kv+k^2u)\,c_{2k} \;=\; \frac{1}{4}\,w.$$

The summations over k are all finite.

Now that we have reduced accuracy to a finite system of finite equations, we can reformulate it to resemble the scalar sum rules (2). For simplicity, introduce the following alternating and non-alternating sums of the matrices c_k:

$$A_j \;=\; \sum_{k=0}^{N} (-1)^k\,k^j\,c_k \qquad \text{and} \qquad S_j \;=\; \sum_{k=0}^{N} k^j\,c_k.$$

In terms of the symbol $M(\omega) = \frac{1}{2}\sum c_k\,e^{-ik\omega}$, these are:

$$A_j \;=\; \frac{2}{(-i)^j}\,M^{(j)}(\pi) \qquad \text{and} \qquad S_j \;=\; \frac{2}{(-i)^j}\,M^{(j)}(0).$$

Then we obtain the following result, which, when combined with Theorem 2, proves Theorem 1. Note that the matrix sum rules given by (25) are the same as those given by (12), except that they are stated in terms of A_j and S_j instead of $M^{(j)}(\pi)$ and $M^{(j)}(0)$.

Theorem 4. *Given the vectors $y_0^{(0)}, \dots, y_0^{(p-1)}$, let (18) define the vectors $y_k^{(j)}$ and therefore $y^{(j)}$ for $j = 0, \dots, p-1$ and all k. Then $y^{(j)}\,L = 2^{-j}\,y^{(j)}$ for $j = 0, \dots, p-1$ if and only if the following two finite equations are satisfied for $j = 0, \dots, p-1$:*

$$\begin{aligned} &\sum_{m=0}^{j} \binom{j}{m}\,2^m\,(-1)^{j-m}\,y_0^{(m)}\,A_{j-m} \;=\; 0, \\ &\sum_{m=0}^{j} \binom{j}{m}\,2^m\,(-1)^{j-m}\,y_0^{(m)}\,S_{j-m} \;=\; 2\,y_0^{(j)}. \end{aligned} \tag{25}$$

Proof. We must show that (23) and (25) are equivalent. For convenience of notation, define the sums

$$\begin{aligned} E_j \;&=\; \sum_k \sum_{m=0}^{j} \binom{j}{m}\,2^m\,(-2k)^{j-m}\,y_0^{(m)}\,c_{2k}, \\ O_j \;&=\; \sum_k \sum_{m=0}^{j} \binom{j}{m}\,2^m\,(-2k-1)^{j-m}\,y_0^{(m)}\,c_{2k+1}, \end{aligned} \tag{26}$$

so that the left-hand side of the first equation in (25) is $E_j - O_j$, and the left-hand side of the second equation in (25) is $E_j + O_j$.

Assume now that (23) holds for each $j = 0, \ldots, p-1$. To show that (25) holds, we only have to show that $E_j = O_j = y_0^{(j)}$. First, note that if $0 \le m \le j$ and $0 \le \ell \le j - m$ then

$$\binom{j}{\ell+m}\binom{\ell+m}{m} = \binom{j}{m}\binom{j-m}{\ell}. \tag{27}$$

Using this, the binomial theorem, and equations (18) and (23), we then compute:

$$\begin{aligned}
E_j &= \sum_k \sum_{m=0}^{j} \binom{j}{m} 2^j \, (-k)^{j-m} \, y_0^{(m)} \, c_{2k} && \text{by (26)} \\
&= 2^j \sum_k y_k^{(j)} \, c_{2k} && \text{by (18)} \\
&= y_0^{(j)} && \text{by (23)} \\
&= \sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^{j-\ell} \, y_{-1}^{(\ell)} && \text{by (17)} \\
&= \sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^{j-\ell} \, 2^\ell \sum_k y_k^{(\ell)} \, c_{2k+1} && \text{by (23)} \\
&= \sum_k \sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^{j-\ell} \, 2^\ell \sum_{m=0}^{\ell} \binom{\ell}{m} (-k)^{\ell-m} \, y_0^{(m)} \, c_{2k+1} && \text{by (18)} \\
&= \sum_k \sum_{m=0}^{j} \sum_{\ell=m}^{j} \binom{j}{\ell} (-1)^{j-\ell} \, 2^\ell \binom{\ell}{m} (-k)^{\ell-m} \, y_0^{(m)} \, c_{2k+1} \\
&= \sum_k \sum_{m=0}^{j} \sum_{\ell=0}^{j-m} \binom{j}{\ell+m} (-1)^{j-\ell-m} \, 2^{\ell+m} \binom{\ell+m}{m} (-k)^\ell \, y_0^{(m)} \, c_{2k+1} \\
&= \sum_k \sum_{m=0}^{j} \binom{j}{m} (-1)^{j-m} \, 2^m \sum_{\ell=0}^{j-m} \binom{j-m}{\ell} (2k)^\ell \, y_0^{(m)} \, c_{2k+1} && \text{by (27)} \\
&= \sum_k \sum_{m=0}^{j} \binom{j}{m} (-1)^{j-m} \, 2^m \, (2k+1)^{j-m} \, y_0^{(m)} \, c_{2k+1} \\
&= O_j && \text{by (26).}
\end{aligned}$$

Now we show the converse implication. Assume that (25) holds for each $j = 0, \dots, p-1$. By adding and subtracting the two equations in (25) we see that $E_j = O_j = y_0^{(j)}$. Applying (18) and (26), we therefore have

$$\begin{aligned} 2^j \sum_k y_k^{(j)} c_{2k} &= \sum_k \sum_{m=0}^{j} \binom{j}{m} 2^j (-k)^{j-m} y_0^{(m)} c_{2k} \\ &= \sum_k \sum_{m=0}^{j} \binom{j}{m} 2^m (-2k)^{j-m} y_0^{(m)} c_{2k} \\ &= E_j \\ &= y_0^{(j)}, \end{aligned}$$

giving the second equation in (23).

Before proving the first equation in (23), note that

$$\sum_{\ell=m}^{j} \binom{j}{\ell} \binom{\ell}{m} (-1)^{\ell-m} = \begin{cases} 0, & m = 0, \dots, j-1, \\ 1, & m = j. \end{cases} \tag{28}$$

Therefore,

$$\begin{aligned} 2^j \sum_k y_k^{(j)} c_{2k+1} &= \sum_{m=0}^{j} 2^m \sum_{\ell=m}^{j} \binom{j}{\ell} \binom{\ell}{m} (-1)^{\ell-m} \sum_k y_k^{(m)} c_{2k+1} && \text{by (28)} \\ &= \sum_{\ell=0}^{j} \binom{j}{\ell} \sum_{m=0}^{\ell} \binom{\ell}{m} (-1)^{\ell-m} 2^m \sum_k y_k^{(m)} c_{2k+1} \\ &= \sum_{\ell=0}^{j} \binom{j}{\ell} O_\ell && \text{by (26)} \\ &= \sum_{\ell=0}^{j} \binom{j}{\ell} y_0^{(\ell)} \\ &= y_{-1}^{(j)}. && \text{by (18)} \end{aligned}$$

This gives the first equation in (23). $\square$

The coefficients in the equations on the left-hand side of (25) are easy to remember: they are the coefficients in the binomial expansion of $(2a-b)^j$. For $j = 0, 1, 2$, the equations in (25) have the form:

$$
(29)\qquad
\begin{aligned}
u\,A_0 &= 0, & u\,S_0 &= 2\,u,\\
2\,v\,A_0 - u\,A_1 &= 0, & 2\,v\,S_0 - u\,S_1 &= 2\,v,\\
4\,w\,A_0 - 4\,v\,A_1 + u\,A_2 &= 0, & 4\,w\,S_0 - 4\,v\,S_1 + u\,S_2 &= 2\,w.
\end{aligned}
$$

6. Examples.

To illustrate how the "matrix sum rules" given by (12) or (25) apply, we check a case where we know beforehand what the result must be: a cubic finite element with $p = 4$ from [SS3]. This matrix refinement equation is specified by

$$
c_0 = \begin{bmatrix} 1/2 & 3/4 \\ -1/8 & -1/8 \end{bmatrix}, \qquad c_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad c_2 = \begin{bmatrix} 1/2 & -3/4 \\ 1/8 & -1/8 \end{bmatrix}.
$$

The corresponding scaling functions f_1 and f_2 are shown in Figure 1.

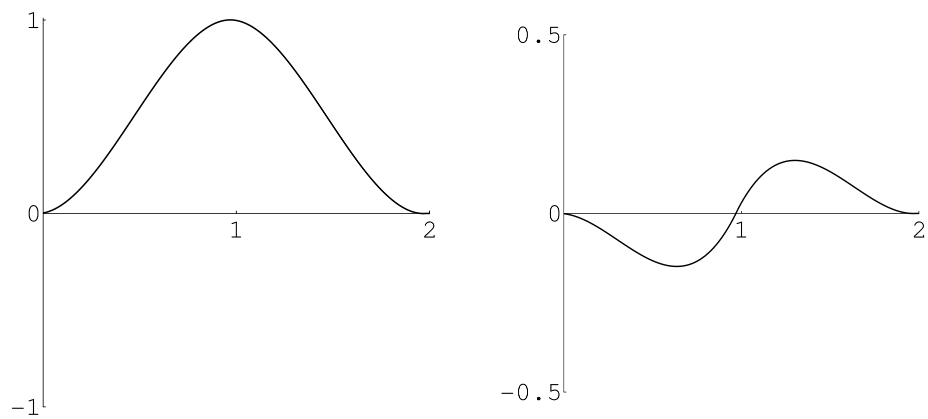

Fig. 1. Cubic finite element scaling functions f_1 and f_2

We begin by computing the matrices A_j and S_j:

$$
\begin{aligned}
A_0 &= \begin{bmatrix} 0 & 0 \\ 0 & -3/4 \end{bmatrix}, & S_0 &= \begin{bmatrix} 2 & 0 \\ 0 & 1/4 \end{bmatrix},\\
A_1 &= \begin{bmatrix} 0 & -3/2 \\ 1/4 & -3/4 \end{bmatrix}, & S_1 &= \begin{bmatrix} 2 & -3/2 \\ 1/4 & 1/4 \end{bmatrix},\\
A_2 &= \begin{bmatrix} 1 & -3 \\ 1/2 & -1 \end{bmatrix}, & S_2 &= \begin{bmatrix} 3 & -3 \\ 1/2 & 0 \end{bmatrix},\\
A_3 &= \begin{bmatrix} 3 & -6 \\ 1 & -3/2 \end{bmatrix}, & S_3 &= \begin{bmatrix} 5 & -6 \\ 1 & -1/2 \end{bmatrix}.
\end{aligned}
$$

Now we simply proceed through the equations in (25) in turn. In terms of u, v, w, and a fourth vector x, these can be rewritten as four simultaneous matrix equations, each dependent on the solution of the preceding one:

$$\begin{cases} u\,A_0 \;=\; 0, \\ u\,(S_0 - 2\,I) \;=\; 0, \end{cases} \tag{30}$$

$$\begin{cases} v\,(2\,A_0) \;=\; u\,A_1, \\ v\,(2\,S_0 - 2\,I) \;=\; u\,S_1, \end{cases} \tag{31}$$

$$\begin{cases} w\,(4\,A_0) \;=\; v\,(4\,A_1) - u\,A_2, \\ w\,(4\,S_0 - 2\,I) \;=\; v\,(4\,S_1) - u\,S_2, \end{cases} \tag{32}$$

$$\begin{cases} x\,(8\,A_0) \;=\; w\,(12\,A_1) - v\,(6\,A_2) + u\,A_3, \\ x\,(8\,S_0 - 2\,I) \;=\; w\,(12\,S_1) - v\,(6\,S_2) + u\,S_3. \end{cases} \tag{33}$$

Equation (30) has the unique solution, up to normalization, of $u = [\,1 \quad 0\,]$. With this value for u, equation (31) has the unique solution $v = [\,1 \quad 1\,]$. Note that there is now no choice for the normalization for v—it is fixed by the normalization of u. Proceeding to the next level, the system (32) has the unique solution $w = [\,1 \quad 2\,]$. Applying these values for u, v, w to (33) gives the final solution $x = [\,1 \quad 3\,]$. The fact that (30)–(33) are solvable implies that the accuracy is at least $p = 4$. The next system is not solvable, so in fact the accuracy is exactly $p = 4$. Moreover, the values of u, v, w, x tell us exactly how to construct the eigenvectors $y^{(0)}$, $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, and hence give the coefficients needed to reconstruct the polynomials 1, x, x^2, and x^3 from translates of f_1 and f_2.

To be precise, to conclude accuracy from Theorems 2 and 4 we must know that the translates of f_1 and f_2 are independent. However, in Section 8 we show that the accuracy conclusion follows without needing to prove independence, simply because the matrix

$$\Delta \;=\; M(0) \;=\; \frac{1}{2}(c_0 + c_1 + c_2) \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1/8 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1/8 < 1$.

7. Reduction in the scalar case.

When the matrices c_k are simply scalars, the matrix sum rule equations in (25) simplify to the usual scalar sum rules (2), which are themselves simply the conditions $A_0 = \cdots = A_{p-1} = 0$. For simplicity, we show this just for $j = 0, 1, 2$ with the u, v, w notation.

Begin with (29). We must have $u \neq 0$ since $y^{(0)} = [\cdots \quad u \quad u \quad u \quad \cdots]$ must be nontrivial. Then (29) implies

$$
\begin{aligned}
A_0 &= 0, & S_0 &= 2, \\
A_1 &= 0, & v &= \frac{u\,S_1}{2}, \\
A_2 &= 0, & w &= \frac{4\,v\,S_1 - u\,S_2}{6}.
\end{aligned} \tag{34}
$$

Thus accuracy implies $S_0 = \sum c_k = 2$ and the sum rules $A_0 = \cdots = A_{p-1} = 0$. Moreover, the numbers v, w are completely determined by u. Thus, up to normalization, the eigenvector $y^{(j)}$ for the eigenvalue $(\frac{1}{2})^j$ for L is completely determined.

Conversely, start with $S_0 = \sum c_k = 2$ and sum rules $A_0 = \cdots = A_{p-1} = 0$. The extra equations in (34) are uniquely solvable for v, w. This in turn gives (29) and hence accuracy. So for the scalar case, accuracy reduces merely to a question of sum rules, or, equivalently, eigenvalues of L. The eigenvector structure is necessarily determined. In summary:

Theorem 5. *Assume f is an integrable solution of the scalar refinement equation $(r = 1)$, such that the integer translates of f are independent. Then f has accuracy p if and only if $\sum c_k = 2$ and the scalar sum rules (2) hold.*

In the following section, we see that the independence hypothesis in Theorem 5 can be dropped in one direction: the assumption $\sum c_k = 2$ together with the scalar sum rules (2) is sufficient to imply accuracy.

8. Independence of translates.

In this section we consider the hypothesis in Theorems 1–5 that the translates of $f_1, \ldots, f_r$ be independent. We find that, in one direction at least, this hypothesis can be replaced by a direct condition on the matrices c_k.

Consider again the discussion in Section 4 which led to Theorem 2. Begin with the assumption that eigenvectors $y^{(0)}, \ldots, y^{(p-1)}$ exist with internal structure prescribed by (18). From this, we proved that $f_1, \ldots, f_r$ had accuracy p if the integer translates of these functions were independent. We used this independence hypothesis at only one point: to prove that the constant C appearing in (21) was nonzero. Now we give alternative hypotheses that also imply that C is nonzero. These hypotheses hold for any matrix refinement equation of current practical interest. Specifically, we require that the matrix $\Delta = M(0) = \frac{1}{2}\sum c_k$ satisfy:

(H1) $\Delta^\infty = \lim_{n\to\infty} \Delta^n$ exists,
(H2) 1 is an eigenvalue of Δ, with multiplicity 1.

Hypothesis (H1) is the natural extension to the matrix case of the scalar requirement (3) that $\sum c_k = 2$, i.e., $\Delta = M(0) = 1$. (H1) implies that the infinite matrix product $\prod_{j=1}^{\infty} M(\omega/2^j)$ converges uniformly on compact sets to a continuous function [HC]. This means that $\hat{f}$ is determined, just as in the scalar case (6), by this infinite product:

$$\hat{f}(\omega) \;=\; \left(\prod_{j=1}^{\infty} M(\omega/2^j)\right) \hat{f}(0).$$

Note then that $\hat{f}(0) = \Delta^\infty \hat{f}(0)$. Therefore $\Delta \hat{f}(0) = \Delta \Delta^\infty \hat{f}(0) = \hat{f}(0)$, i.e., $\hat{f}(0)$ is a right 1-eigenvector for Δ.

This assumption that Δ^∞ exists means that the Jordan canonical form of Δ looks in block form like

$$\Delta \;\sim\; \begin{bmatrix} I_s & 0 \\ 0 & J \end{bmatrix}, \tag{35}$$

where I_s is an $s \times s$ identity matrix with $0 \le s \le r$ and J has all eigenvalues strictly less than 1 in absolute value. We know that $\hat{f}(0)$ must be a right 1-eigenvalue for Δ, so $s \ge 1$, and hypothesis (H2) simply states that $s = 1$. This ensures that at most one integrable solution to the matrix refinement equation can exist up to normalization by scalar multiples—in other words, f is unique [HC]. Thus (H1) and (H2) together simply state that 1 is a simple eigenvalue for Δ and all other eigenvalues are less than 1 in absolute value.

Hypotheses (H1) and (H2) are tied to the constant C as follows. We know by the argument in Section 4 that

$$C \;=\; G_0(x) \;=\; \sum_k y_k^{(0)} f(x+k) \;=\; \sum_k u\, f(x+k) \text{ a.e.}$$

Hence,

$$C \;=\; \int_0^1 G_0(x)\, dx \;=\; u \sum_k \int_0^1 f(x+k)\, dx \;=\; u \int_{-\infty}^{\infty} f(x)\, dx \;=\; u\, \hat{f}(0),$$

where integrals $\int_a^b f(x)\, dx$ are performed componentwise. Hence, we need only show that (H1) and (H2) imply that $u\, \hat{f}(0) \ne 0$. We have seen already that $\hat{f}(0)$ is a right 1-eigenvector for Δ. The *left* 1-eigenvector is u! For, we have assumed that $y^{(0)}$ is a left 1-eigenvector for the infinite matrix L with structure given by (18), so by Theorem 4—which was proven without any independence hypotheses—we know that equation (25) must hold for $j = 0, \ldots, p-1$. Included in this is the statement that $uS_0 = 2u$; by definition, $S_0 = \sum c_k = 2\Delta$.

Thus u and $\hat{f}(0)$ are left and right 1-eigenvectors, respectively, for Δ. But the fact that Δ has the block form (35) with $s = 1$ ensures that the inner product of these left and right eigenvectors must be nonzero.

We have proved the following.

Theorem 6. *Assume f is an integrable solution of the matrix refinement equation* (1) *and suppose that the matrix $\Delta = \frac{1}{2} \sum c_k$ satisfies* (H1) *and* (H2). *If there are vectors $y_0^{(0)}, \ldots, y_0^{(p-1)} \in \mathbf{C}^r$ satisfying the matrix sum rules* (12) *or* (25) *for $j = 0, \ldots, p-1$, then f has accuracy p.*

Note that hypotheses (H1) and (H2) are *always* satisfied in the scalar case ($r = 1$) because Δ is then just the number 1. Therefore, the scalar result (Theorem 5) can be simplified in one direction: If $\sum c_k = 2$ and the scalar sum rules (2) hold for $j = 0, \dots, p-1$ then the accuracy will be p, without the need for the independence hypothesis. Moreover, because of the simplifications that occur in (25) for the scalar case, this says that, in the scalar case, the mere existence of eigenvalues $1, \frac{1}{2}, \dots, (\frac{1}{2})^{p-1}$ for L is sufficient to ensure accuracy p, without the need to assume that the eigenvectors $y^{(0)}, \dots, y^{(p-1)}$ have the structure specified by (18).

This fact does not carry over to the matrix case. Set

$$c_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad c_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Then the infinite matrix L has both 1 and 1/2 as eigenvalues. However, the solution to the corresponding matrix refinement equation is

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \chi_{[0,1]} \\ 0 \end{bmatrix},$$

which only has accuracy $p = 1$.

Because the matrix Δ for this example does satisfy (H1) and (H2), the trouble must be that the eigenvector $y^{(1)}$ does not satisfy (18). In fact,

$$\begin{aligned} y^{(0)} &= [\cdots \quad u \quad u \quad u \quad \cdots], \\ y^{(1)} &= [\cdots \quad t \quad t \quad t \quad \cdots], \end{aligned}$$

with $u = [1 \quad 0]$ and $t = [0 \quad 1]$, while (18) requires $y_k^{(1)} = v - ku$ for some v. Equivalently, the matrix sum rules (25) hold for $j = 0$, but not for $j = 1$.

The translates of f_1 and f_2 in this example are not independent. This leads to the following question.

Problem. *Assume f is an integrable solution of the matrix refinement equation* (1) *such that the integer translates of $f_1, \dots, f_r$ are independent. If L has eigenvalues $1, \frac{1}{2}, \dots, (\frac{1}{2})^{p-1}$, must f have accuracy p? In other words, must the matrix sum rules* (25) *be satisfied for each $j = 0, \dots, p-1$?*

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