Author's note: A few typos in the article have been marked in green type; these corrections are not included in the published article.

Absolute Continuity and the Banach–Zaretsky Theorem

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Abstract The Banach–Zaretsky Theorem is a fundamental but often overlooked result that characterizes the functions that are absolutely continuous. This chapter presents basic results on differentiability, absolute continuity, and the Fundamental Theorem of Calculus with an emphasis on the role of the Banach–Zaretsky Theorem.

1 Introduction

I started as a graduate student at the University of Maryland in 1982, and John Benedetto became my Ph.D. advisor around 1986. While working with John, I came across his book “Real Variable and Integration” [1]. Unfortunately, the text was out of print, but I checked the book out of the library and made a “grad student copy” (meaning I xeroxed the text). This became an important reference for me. Much later there was a limited reprinting of the book, and I am proud to own a copy autographed by John himself, dated May 2000. More recently, John and Wojciech Czaja collaborated to create a revised and expanded version of the text [2], published in 2009. An especially appealing aspect of both books is the inclusion of many historical references and discussions of the historical development of the results.

Quoting from the Zentralblatt review (Zbl. 0336.26001) of [1],

> At the heart of the differential and integral calculus is the so-called fundamental theorem of the calculus … In the theory of Lebesgue integration this result takes on the form that a function is the indefinite integral of a Lebesgue integrable function if and only if it is absolutely continuous. – In the introduction to the book under review the author states that the main mathematical reason for having written the book is that none of the other texts

This work was partially supported by a grant from the Simons Foundation.

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M. Hirn et al. (eds.), *Excursions in Harmonic Analysis, Volume 6*, Applied and Numerical Harmonic Analysis, https://doi.org/10.1007/978-3-030-69637-5_2

> in this area of analysis and at this level stress the importance of the notion of absolute continuity as it pertains to such fundamental results .... It is the reviewer's opinion that the author has written a beautiful and extremely useful textbook on real variable and integration theory. ... All in all the author is to be congratulated to have written such a fine addition to the literature. W.A.J. Luxemburg.

Eventually, it became time for me to write my own book on real analysis ([6], which appeared recently). My primary goal was to write a classroom text for students taking their first course on Lebesgue measure and the Lebesgue integral. There are many books on real analysis available, but I wanted to write the book that I wished my instructor had used when I first took that course. One of my principles of writing for this book was that each proof should be both rigorous and *enlightening* (which often, though not always, entails finding a "simple proof"). The two chapters on differentiation and absolute continuity were especially difficult in this regard. In the standard texts, there are a variety of approaches to the proofs of the basic theorems in this area, but most of the standard proofs are very technical, far from "enlightening" for the beginning student. In this chapter (which may be called a *mathematical essay*), I will outline how I dealt with these issues. The highlight is that the *Banach–Zaretsky Theorem*, an elegant but often-overlooked result that I first learned about from John's book (of course), plays a key role in making the presentation "enlightening." Unfortunately, I was still left with one theorem whose proof I was unable to simplify to my satisfaction, and I will discuss this failure as well.

Of course, what is "simple," "enlightening," "elegant," or "obvious" is, almost by definition, in the eye of the beholder. Consequently, the reader may disagree with my opinions of what is important, elegant, or enlightening. I would be happy to engage in further discussion about how best to present absolute continuity, differentiation, and related results to the first-time student.

1.1 The Fundamental Theorem of Calculus

Our fundamental goal is to fully understand the Fundamental Theorem of Calculus (which we sometimes refer to using the acronym *FTC*). The version that we often learn in an undergraduate calculus course is that if a function f is differentiable at every point in a closed finite interval $[a, b]$ and if f' is continuous on $[a, b]$, then the Fundamental Theorem of Calculus holds, and it tells us that

$$\int_a^x f'(t)\,dt \;=\; f(x) - f(a), \qquad \text{for all } x \in [a, b]. \tag{1.1}$$

Since we have assumed that f' is continuous, the integral on the line above exists as a Riemann integral. Does the Fundamental Theorem of Calculus hold if we assume only that f' is Lebesgue integrable? Precisely:

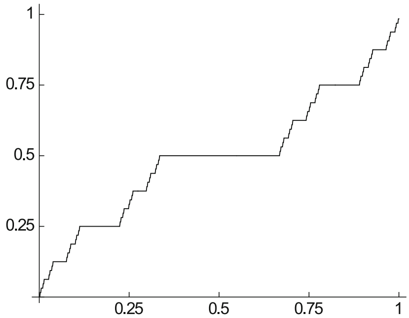

Fig. 1 The Cantor–Lebesgue function (also known as the *Devil's staircase*)

If $f'(x)$ exists for a.e. x and f' is integrable, must Eq. (1.1) *hold?*

The construction of the classical *Cantor–Lebesgue function* (pictured in Fig. 1) shows that the answer to this question is *no* in general. The Cantor–Lebesgue function is *singular* in the following sense.

Definition 1 (Singular Function) A function f on $[a, b]$ is *singular* if f is differentiable at almost every point in $[a, b]$ and $f' = 0$ almost everywhere (a.e.) on $[a, b]$. ◇

For a singular function, we have $\int_a^b f' = 0$, which need not equal $f(b) - f(a)$. For which functions does the FTC hold? We will consider this in the coming pages.

Remark 1 For simplicity of presentation, we will focus on real-valued functions. However, nearly all of the results carry over without change to complex-valued functions. One result where there is a small difference is in the statement of the Banach–Zaretsky Theorem. We will explain the necessary change in hypotheses when we discuss that theorem. ◇

1.2 Notation

We use the standard notations of real analysis. We mention some specific terminology that we will use.

- If E is a subset of the Euclidean space $\mathbb{R}^d$, then $|E|_e$ denotes the exterior Lebesgue measure of E. If E is a Lebesgue measurable set, then we write this measure as $|E|$.
- The Euclidean norm of a point $x \in \mathbb{R}^d$ is denoted by $\|x\|$.
- If E is a measurable subset of $\mathbb{R}^d$ and $1 \le p < \infty$, then $L^p(E)$ denotes the Lebesgue space of all p-integrable functions on E, and $\|f\|_p = \left(\int_E |f(t)|^p \, dt\right)^{1/p}$ denotes the L^p-norm of an element of $L^p(E)$. If $E = [a, b]$, then we denote this Lebesgue space by $L^p[a, b]$.

- If E is a measurable subset of $\mathbb{R}^d$, then $L^\infty(E)$ denotes the Lebesgue space of all essentially bounded functions on E, and $\|f\|_\infty = \operatorname{ess\,sup}_{t\in E} |f(t)|$ denotes the L^∞-norm of an element of $L^\infty(E)$. If $E = [a, b]$, then we denote this Lebesgue space by $L^\infty[a, b]$.

2 Differentiation

2.1 Functions of Bounded Variation

We will briefly review the notion of bounded variation. As with most of the results and definitions that we will discuss, more extensive discussion and motivation can be found in [6]. The idea behind this definition is that if we fix finitely many points $a = x_0 < \cdots < x_n = b$ in the interval $[a, b]$, then the quantity $\sum_{j=1}^n |f(x_j) - f(x_{j-1})|$ is an approximation to the "total variation" in the height of f across the interval. A function has bounded variation if there is an upper bound to such approximations, and the total variation is the supremum of all of those approximations.

Definition 2 (Bounded Variation) Let $f : [a, b] \to \mathbb{R}$ be given. For each finite partition

$$\Gamma = \big\{a = x_0 < \cdots < x_n = b\big\}$$

of $[a, b]$, set

$$S_\Gamma[f] = \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|.$$

The *total variation* of f over $[a, b]$ (or simply the *variation* of f, for short) is

$$V[f] = \sup\big\{S_\Gamma[f] : \Gamma \text{ is a partition of } [a, b]\big\}.$$

We say that f has *bounded variation* on $[a, b]$ if $V[f] < \infty$. We collect the functions that have bounded variation on $[a, b]$ to form the space

$$\mathrm{BV}[a, b] = \big\{f : [a, b] \to \mathbb{R} : f \text{ has bounded variation}\big\}. \qquad \diamondsuit.$$

Letting $\mathrm{Lip}[a, b]$ denote the space of functions that are Lipschitz continuous on $[a, b]$, and $C^1[a, b]$ the space of functions that are differentiable on $[a, b]$ and whose derivative is continuous on $[a, b]$, we have the proper inclusions

$$C^1[a, b] \subsetneq \mathrm{Lip}[a, b] \subsetneq \mathrm{BV}[a, b].$$

The following theorem gives a fundamental characterization of functions in $\mathrm{BV}[a, b]$. The proof is nontrivial, but it is (in our opinion) "clear and enlightening" for the student who works through it, and so we will be content to refer to [6] for details. For us, a *monotone increasing* function on $[a, b]$ is a function f such that $f(x) \le f(y)$ whenever $a \le x \le y \le b$.

Theorem 3 (Jordan Decomposition) *If $f : [a, b] \to \mathbb{R}$, then $f \in \mathrm{BV}[a, b]$ if and only if there exist monotone increasing functions g and h such that $f = g - h$.* $\diamond$

Thus, many questions (though not all) can be reduced to questions about monotone increasing functions.

2.2 *Our Failure: Differentiability of Monotone Increasing Functions*

How hard can it be to understand monotone increasing functions? At first glance, monotone functions seem to be "simple and easy." Indeed, here is one easy fact.

Lemma 4 (Discontinuities of Monotone Functions) *If $f : [a, b] \to \mathbb{R}$ is monotone increasing, then it has at most countably many discontinuities, and they are all jump discontinuities.*

Proof Since f is monotone increasing and takes real values at each point of $[a, b]$, it follows that $f(a) \le f(x) \le f(b)$ for every x, so f is bounded. Further, the one-sided limits

$$f(x-) = \lim_{y \to x^-} f(y) \qquad \text{and} \qquad f(x+) = \lim_{y \to x^+} f(y)$$

exist at every point $x \in (a, b)$. The appropriate one-sided limits also exist at the endpoints a and b. Consequently, each point of discontinuity of f must be a jump discontinuity. Since f is bounded, if we fix a positive integer k, there can be at most finitely many points $x \in [a, b]$ such that

$$f(x+) - f(x-) \ge \frac{1}{k}.$$

Every jump discontinuity must satisfy this inequality for some positive integer k, so there can be at most countably many discontinuities. $\square$

With only countably many discontinuities, all of which are jump discontinuities, just how complicated could a monotone increasing function be? Should not it be differentiable at all but those countably many discontinuities? In fact, it is much more complicated than that. Things would be easier if the discontinuities were separated, but there is no reason that they have to be. Here is a monotone increasing function that is discontinuous at every rational point.

Example Let $\{r_k\}_{k\in\mathbb{N}}$ be an enumeration of the rational points in $(0, 1)$. Then, the function

$$f(x) = \sum_{n=1}^{\infty} 2^{-n}\, \chi_{[r_n,1]}(x), \qquad x \in [0, 1]$$

is monotone increasing on $[0, 1]$, right-continuous at every point in $[0, 1]$, discontinuous at every rational point in $(0, 1)$, and continuous at every irrational point in $(0, 1)$. ◇

Thus, monotone increasing functions are more complicated than we may initially suspect. However, our next theorem states that every monotone increasing function on $[a, b]$ is differentiable *at almost every point*.

Theorem 5 (Differentiability of Monotone Functions) *If a function $f: [a, b] \to \mathbb{R}$ is monotone increasing, then $f'(x)$ exists for almost every $x \in [a, b]$.* ◇

Surprisingly (at least to me), Theorem 5 appears to be a deep and difficult result. I am not aware of any proof that I would call "simple," or even any proof that I would call "enlightening" for the beginning student. I have not been able to construct any such proof myself. There are a number of different standard approaches to the proof; the reader may find it interesting to compare the proofs given in [3, Thm. 7.5], [5, Thm. 3.23], [11, Thm. 3.3.14], or [13, Thm. 7.5], for example. In the end, for the proof presented in my text, I adopted a standard technique based on the *Vitali Covering Lemma*. In my opinion, Theorem 5 has the dubious distinction of being the theorem in [6] whose proof is the least enlightening. I will not present that proof here; for this chapter, we will simply accept Theorem 5 as given and move onward. In comparison to that theorem, the proof of the next corollary is much easier.

Corollary 6 *If $f: [a, b] \to \mathbb{R}$ is monotone increasing on $[a, b]$, then f' is measurable, $f' \geq 0$ a.e., $f' \in L^1[a, b]$, and*

$$0 \leq \int_a^b f' \leq f(b) - f(a).$$

Proof For simplicity of presentation, extend the domain of f to the entire real line by setting $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$.

(a) Since f is differentiable at almost every point, we know that the functions

$$f_n(x) = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}} = n\big(f(x + \tfrac{1}{n}) - f(x)\big), \qquad x \in \mathbb{R}$$

converge pointwise a.e. to $f'(x)$ on $[a, b]$ as $n \to \infty$. Furthermore, each f_n is measurable and nonnegative (because f is monotone increasing), so f' is measurable and $f' \geq 0$ a.e.

(b) Since the functions f_n are nonnegative and converge pointwise a.e. to f', we can apply Fatou's Lemma to obtain

$$\int_a^b f' = \int_a^b \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int_a^b f_n.$$

On the other hand, recalling how we extended the domain of f to $\mathbb{R}$, for each individual n we compute that

$$\begin{aligned}
\int_a^b f_n &= n\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f - n\int_a^b f && \text{(by the definition of } f_n) \\
&= n\int_b^{b+\frac{1}{n}} f - n\int_a^{a+\frac{1}{n}} f \\
&= n\int_b^{b+\frac{1}{n}} f(b) - n\int_a^{a+\frac{1}{n}} f && \text{(since } f \text{ is constant on } [b,\infty)) \\
&\le n\int_b^{b+\frac{1}{n}} f(b) - n\int_a^{a+\frac{1}{n}} f(a) && \text{(since } f \text{ is monotone increasing)} \\
&= f(b) - f(a).
\end{aligned}$$

Therefore,

$$\int_a^b f' \le \liminf_{n\to\infty} \int_a^b f_n \le f(b) - f(a) < \infty,$$

so f' is integrable. □

Since every function with bounded variation is the difference of two monotone increasing functions, an immediate consequence of Corollary 6 is that if f belongs to $\mathrm{BV}[a,b]$ then f is differentiable at almost every point and $f' \in L^1[a,b]$.

2.3 The Simple Vitali Lemma

We mentioned the Vitali Covering Lemma earlier, perhaps implicitly deriding its use in our earlier statements. In fact, this is a fundamental result of great importance. Unfortunately, its proof is (in my opinion) less "enlightening" than many others. However, there is a more restricted version that is sufficient for many proofs, and this "simple" version does have an elegant and enlightening proof that we present here. The proof that we give is close to the one that I learned from Folland's text [5] (which is one of the great reference texts in analysis). Essentially, this *Simple Vitali Lemma* states that if we are given any collection of open balls in $\mathbb{R}^d$, then we can find *finitely many disjoint balls* from the collection that cover a fixed fraction of the measure of the union of the original balls. Up to an ε, this fraction is 3^{-d} (so in dimension $d = 1$, we can choose disjoint open intervals that cover about $1/3$ of the original collection). The proof is an example of a *greedy algorithm*—basically, we choose a ball B_1 that has the largest possible radius, then choose B_2 to be the largest possible ball disjoint from B_1, and so forth.

Theorem 7 (Simple Vitali Lemma) *Let $\mathcal{B}$ be any nonempty collection of open balls in $\mathbb{R}^d$. Let U be the union of all of the balls in $\mathcal{B}$, and fix $0 < c < |U|$. Then there exist finitely many* disjoint *balls $B_1, \dots, B_N \in \mathcal{B}$ such that*

$$\sum_{k=1}^{N} |B_k| > \frac{c}{3^d}.$$

Proof Since $c < |U|$, the number c is finite, and therefore, there exists a compact set $K \subseteq U$ whose measure satisfies $c < |K| < |U|$. Since $\mathcal{B}$ is an open cover of the compact set K, we can find finitely many balls $A_1, \dots, A_m \in \mathcal{B}$ such that

$$K \subseteq \bigcup_{j=1}^{m} A_j.$$

Let B_1 be an A_j ball that has maximal radius.

If there are no A_j balls that are disjoint from B_1, then we set $N = 1$ and stop. Otherwise, let B_2 be an A_j ball with largest radius that is disjoint from B_1 (if there is more than one such ball, just choose one of them). We then repeat this process, which must eventually stop, to select disjoint balls $B_1, \dots, B_N$ from $A_1, \dots, A_m$. These balls need not cover K, but we hope that they will cover an appropriate portion of K.

To prove this, let B_k^* denote the open ball that has the same center as B_k, but with radius three times larger. Suppose that $1 \le j \le m$, but A_j is not one of $B_1, \dots, B_N$.

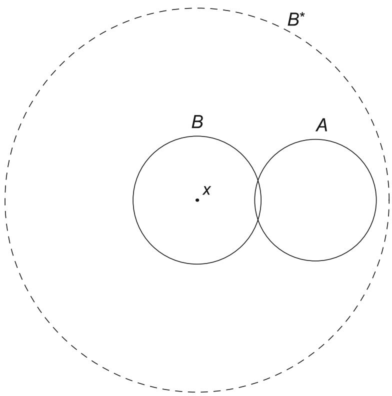

Fig. 2 Circle B has radius 1, circle A has radius 0.95, and circle B^* (which has the same center x as circle B) has radius 3

Then, A_j must intersect at least one of the balls $B_1, \dots, B_N$. Let k be the smallest index such that $A_j \cap B_k \neq \varnothing$. By construction,

$$\operatorname{radius}(A_j) \le \operatorname{radius}(B_k).$$

It follows from this that $A_j \subseteq B_k^*$ (for a "proof of picture," see Fig. 2). Thus, every set A_j that is not one of $B_1, \dots, B_N$ is contained in some B_k^*. Hence,

$$K \subseteq \bigcup_{j=1}^{m} A_j \subseteq \bigcup_{k=1}^{N} B_k^*,$$

and therefore,

$$c < |K| \le \sum_{k=1}^{N} |B_k^*| = 3^d \sum_{k=1}^{N} |B_k|. \qquad \square$$

The Vitali Covering Lemma is a more refined version of this result, and the proof requires a more refined, but still essentially "greedy," approach. One full proof can be found in [6, Thm. 5.3.3].

2.4 The Lebesgue Differentiation Theorem

If a function $f : \mathbb{R}^d \to \mathbb{R}$ is continuous at a point x, then f "does not vary much" over a small ball $B_h(x) = \{t \in \mathbb{R}^d : \|t\| < h\}$ of radius h centered at x. We can write this average in several forms. In one form, it is the convolution $f * k_h$ of f with the normalized characteristic function

$$k_h(t) = \frac{1}{|B_h(0)|}\, \chi_{B_h(0)}(t).$$

This function k_h takes the value $1/|B_h(0)|$ when t belongs to the ball $B_h(0)$, and is zero outside the ball. Letting $\tilde{f}_h(x)$ denote the average of f over the ball ~~$Bh(0)$~~, $B_h(0)$, and noting that the Lebesgue measure of a ball is $|B_h(x)| = C_d h^d$ where C_d is a constant that depends only on the dimension d, we can write the average of f explicitly in several forms:

tilde missing: $\tilde{f}_h(x)$

These should be $B_h(x)$

$$\begin{aligned}
\widetilde{f}_h(x) &= \frac{1}{|B_h(0)|}\int_{B_h(0)} f(t)\,dt && \text{(definition of average)}\\
&= \frac{1}{C_d h^d}\int_{B_h(0)} f(x-t)\,dt && \text{(by change of variable)}\\
&= \int_{\mathbb{R}^d} f(x-t)\,k_h(t)\,dt && \text{(definition of } k_h)\\
&= (f*k_h)(x) && \text{(definition of convolution).}
\end{aligned}$$

An easy exercise shows that if f is continuous at x then $\widetilde{f}_h(x) \to f(x)$ as $h \to 0$. Much more surprising is the following deep and important result, which shows that these averages converge for almost every x even if we only assume that f is locally integrable (i.e., f is integrable on every compact subset of $\mathbb{R}^d$).

Theorem 8 (Lebesgue Differentiation Theorem) *If f is locally integrable on $\mathbb{R}^d$, then for almost every $x \in \mathbb{R}^d$ we have*

$$\lim_{h\to 0} \frac{1}{|B_h(x)|}\int_{B_h(x)} |f(x)-f(t)|\,dt \;=\; 0,$$

and

$$\lim_{h\to 0} \widetilde{f}_h(x) \;=\; \lim_{h\to 0} \frac{1}{|B_h(x)|}\int_{B_h(x)} f(t)\,dt \;=\; f(x).$$

In particular, $\widetilde{f}_h(x) \to f(x)$ at almost every point. ◇

Here is a corollary that we will need.

Corollary 9 *If f is locally integrable on $\mathbb{R}$, then for almost every $x \in \mathbb{R}$ we have*

$$f(x) \;=\; \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt. \qquad \diamond$$

The proof of Theorem 8 does take considerable work, but it is rewarding for the student to read and understand. One approach uses the Simple Vitali Lemma to prove the *Hardy–Littlewood Maximal Theorem*, which deals with the *supremum* of averages over balls instead of the limit of those averages. That proof is too long to include here. On the other hand, it is also true that if f is integrable on $\mathbb{R}^d$, then the averages of f over the balls $B_h(x)$ converge to f *in L^1-norm* as $h \to 0$. We will give the short (and enlightening) proof of this fact, which is actually a special case of more general facts about *approximate identities* for convolution (discussed in more detail in [6, Ch. 9]).

Theorem 10 *If $f \in L^1(\mathbb{R}^d)$, then $\widetilde{f}_h \to f$ in L^1-norm. That is,*

$$\lim_{h\to 0} \|f - \widetilde{f}_h\|_1 \;=\; \lim_{h\to 0} \int_{\mathbb{R}^d} |f(x) - \widetilde{f}_h(x)|\,dx \;=\; 0.$$

Proof The function k_h has been defined so that $\int k_h = 1$ for every h. Using Tonelli's Theorem to interchange the order of integration and noting that k_h is only nonzero on $B_h(0)$, we can therefore estimate the L^1-norm of the difference $f - \widetilde{f}_h$ as follows:

$$\begin{aligned}
\|f - \widetilde{f}_h\|_1 &= \int_{\mathbb{R}^d} |f(x) - \widetilde{f}_h(x)|\,dx \\
&= \int_{\mathbb{R}^d} \left| f(x) \int_{\mathbb{R}^d} k_h(t)\,dt \;-\; \int_{\mathbb{R}^d} f(x-t)\,k_h(t)\,dt \right| dx \\
&\le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x) - f(x-t)|\,k_h(t)\,dt\,dx \\
&= \frac{1}{C_d\,h^d} \int_{B_h(0)} \left(\int_{\mathbb{R}^d} |f(x) - f(x-t)|\,dx \right) dt \\
&= \frac{1}{C_d\,h^d} \int_{\|t\|<h} \|f - T_t f\|_1\,dt,
\end{aligned}$$

where $T_t f(x) = f(x-t)$ denotes the translation of f by t. However, the Lebesgue integral has a "strong continuity" property with respect to translation. Specifically,

$$\lim_{t\to 0} \|f - T_t f\|_1 \;=\; 0.$$

Therefore, if we fix an $\varepsilon > 0$, then there is some $\delta > 0$ such that $\|f - T_t f\|_1 < \varepsilon$ whenever $\|t\| < \delta$. Consequently, for all $0 < h < \delta$, we have

$$\|f - \widetilde{f}_h\|_1 \;\le\; \frac{1}{C_d\,h^d} \int_{\|t\|<h} \|f - T_t f\|_1\,dt \;\le\; \frac{1}{C_d\,h^d} \int_{\|t\|<h} \varepsilon\,dt \;=\; \varepsilon. \qquad \square$$

3 The Fundamental Theorem of Calculus

3.1 The Simple Version of the FTC

Here is a reformulation of the simple version of the FTC. This result follows directly from the fact that all continuous functions on $[a, b]$ are uniformly continuous (and its proof is a good exercise for the student). In the statement of this theorem, *differentiable everywhere on* $[a, b]$ means differentiable in the usual sense at each point of (a, b), differentiable from the right at a, and differentiable from the left at b.

Theorem 11 (Simple Version of the FTC) *If g is a continuous function g on $[a, b]$, then its indefinite integral*

$$G(x) = \int_a^x g(t)\,dt, \qquad x \in [a, b]$$

has the following properties:

(a) *G is differentiable everywhere on $[a, b]$,*

(b) *$G'(x) = g(x)$ for every $x \in [a, b]$,*

(c) *$G \in C^1[a, b]$, and consequently, G is Lipschitz and has bounded variation on $[a, b]$.* ◇

Thus, if g is continuous, then its indefinite integral G is differentiable at every point, and it is an *antiderivative* of g because $G' = g$. What happens if we assume only that the function g is *integrable*? Here is a partial answer. The proof is short but "instructive and enlightening."

Lemma 12 *If $g \in L^1[a, b]$, then its indefinite integral*

$$G(x) = \int_a^x g(t)\,dt, \qquad x \in [a, b]$$

has the following properties:

(a) *G is continuous on $[a, b]$,*

(b) *$G \in \mathrm{BV}[a, b]$, and*

(c) *the total variation of G is bounded by the L^1-norm of g, i.e.,*

$$V[G] \le \int_a^b |g(t)|\,dt = \|g\|_1.$$

Proof

(a) Fix any point $x \in (a, b)$. If $h > 0$ is small enough that $x + h$ belongs to $[a, b]$, then

$$G(x+h) - G(x) = \int_x^{x+h} g(t)\,dt = \int_a^b g(t)\,\chi_{[x,x+h]}(t)\,dt.$$

The integrand $g \cdot \chi_{[x,x+h]}$ is bounded by the integrable function $|g|$, and it converges pointwise a.e. to zero as $h \to 0^+$. The Dominated Convergence Theorem therefore implies that $G(x + h) - G(x) \to 0$ as $h \to 0^+$. Combining this with a similar argument for $h \to 0^-$, we see that G is continuous at x. Similar one-sided arguments show that G is continuous from the right at $x = a$ and continuous from the left at $x = b$, so G is continuous on the interval $[a, b]$.

(b), (c) If $\Gamma = \{a = x_0 < \cdots < x_n = b\}$ is a partition of $[a, b]$, then

$$S_\Gamma[G] = \sum_{j=1}^{n} |G(x_j) - G(x_{j-1})|$$

$$\leq \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |g(t)|\, dt = \int_a^b |g(t)|\, dt = \|g\|_1.$$

Taking the supremum over all such partitions Γ, we see that G has bounded variation and $V[G] \leq \|g\|_1$. □

Unfortunately, Lemma 12 is not very satisfactory when compared to Theorem 11. We are still left with the following questions.

- If $g \in L^1[a, b]$, is the indefinite integral $G(x) = \int_a^x g(t)\, dt$ a differentiable function of x?
- If the indefinite integral G is differentiable, is it the antiderivative of g? That is, is it true that $G' = g$?

My own intuition suggests that the answers to both questions should be *yes*. And we will eventually see that this is true *in a qualified sense*, but not literally true in the same sense as in Theorem 6 . In particular, although the beginning student may not see an obvious counterexample, it is not true that the indefinite integral G must be differentiable *at every point* x (challenge—what is one such function?). Likewise, it is not true that $G'(x)$ must equal $g(x)$ *at every point* x. Instead, we will prove—after some work—that the following statements hold.

- G is an *absolutely continuous* function and, as a consequence, it is differentiable at *almost every* point in $[a, b]$, and
- $G'(x) = g(x)$ for *almost every* $x \in [a, b]$.

3.2 Absolute Continuity

To motivate the definition of absolute continuity, recall that a function $f: [a, b] \to \mathbb{R}$ is *uniformly continuous* on $[a, b]$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Absolutely continuous functions satisfy a similar but more stringent requirement, given next. In the statement of this definition, a collection of intervals is *nonover-*

lapping if any two intervals in the collection intersect at most at their boundaries (hence, their interiors are disjoint).

Definition 13 (Absolutely Continuous Function) We say that a function $f\colon [a,b] \to \mathbb{R}$ is *absolutely continuous on* $[a,b]$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite or countably infinite collection of nonoverlapping subintervals $\big\{[a_j,b_j]\big\}_j$ of $[a,b]$, we have

$$\sum_j (b_j - a_j) \,<\, \delta \implies \sum_j |f(b_j) - f(a_j)| \,<\, \varepsilon.$$

We denote the class of absolutely continuous functions on $[a,b]$ by

$$\mathrm{AC}[a,b] \,=\, \big\{f\colon[a,b]\to\mathbb{R} \,:\, f \text{ is absolutely continuous on } [a,b]\big\}. \qquad \diamond$$

Every absolutely continuous function has bounded variation. Using the spaces we have introduced previously, we have the inclusions

$$C^1[a,b] \subsetneq \mathrm{Lip}[a,b] \subsetneq \mathrm{AC}[a,b] \subsetneq \mathrm{BV}[a,b] \subsetneq L^\infty[a,b] \subsetneq L^1[a,b].$$

The next lemma answers one of the questions that we posed immediately after the proof of Lemma 12.

Lemma 14 *If* $g \in L^1[a,b]$, *then its indefinite integral*

$$G(x) \,=\, \int_a^x g(t)\,dt, \qquad x \in [a,b],$$

has the following properties:

(a) G *is absolutely continuous on* $[a,b]$,

(b) G *is differentiable at almost every point of* $[a,b]$, *and*

(c) $G' \in L^1[a,b]$.

Proof Fix any $\varepsilon > 0$. Since g is integrable, there exists a constant $\delta > 0$ such that $\int_E |g| \,<\, \varepsilon$ for every measurable set $E \subseteq [a,b]$ whose measure satisfies $|E| < \delta$ (this is called the "absolute continuity" property of the Lebesgue integral; see [6, Exer. 4.5.5] for one derivation). Let $\{[a_j,b_j]\}_j$ be any countable collection of nonoverlapping subintervals of $[a,b]$ that satisfies $\sum (b_j - a_j) \,<\, \delta$, and set $E = \cup\,(a_j,b_j)$. Then $|E| < \delta$, so

$$\sum_j |G(b_j) - G(a_j)| \,=\, \sum_j \left|\int_{a_j}^{b_j} g\right| \,\le\, \sum_j \int_{a_j}^{b_j} |g| \,=\, \int_E |g| \,<\, \varepsilon.$$

Thus $G \in \mathrm{AC}[a, b]$. Since G is absolutely continuous, it has bounded variation. Applying the Jordan Decomposition Theorem (Theorem 1), G can be written as the difference of two monotone increasing functions, say $G = u - v$. Each of u and v is differentiable at almost every point, so G is differentiable a.e. Further, u' and v' are both integrable by Corollary 6, so $G' = u' - v'$ is integrable as well. □

3.3 The Growth Lemmas

In Sect. 3.4, we will prove the Banach–Zaretsky Theorem, which gives a reformulation of absolute continuity that is related to the issue of whether a function maps sets with measure zero to sets with measure zero. To do this, we need to understand how much a continuous function can "blow up" the measure of a set. That is, can we estimate the exterior Lebesgue measure $|f(E)|_e$ in terms of the measure $|E|_e$ of E and some properties of f? (Although we are assuming that f is continuous, it is not true that a continuous function must map measurable sets to measurable sets, which is why we are using exterior measure here.)

If we impose enough conditions on f, then it is easy to formulate such a "growth" result. For example, suppose that f is Lipschitz on $[a, b]$ with Lipschitz constant K, i.e.,

$$|f(x) - f(y)| \leq K\,|x - y|, \qquad \text{for all } x,\ y \in [a, b].$$

In this case, if we choose x and y from a subinterval $[c, d]$, then we will always have $|f(x) - f(y)| \leq K\,(d - c)$. Thus, the diameter of the image of $[c, d]$ under f is at most $K\,(d - c)$, and therefore $|f([c, d])|_e \leq K\,|[c, d]|$. That is, the measure of the image of $[c, d]$ under f is at most K times the measure of $[c, d]$. Since (in one dimension) the exterior measure of a set is fundamentally based on intervals, it is not surprising then that we can extend that inequality and show that $|f(E)|_e \leq K\,|E|_e$ for every subset E of $[a, b]$ (this is a nice exercise for the student). In particular, if f is differentiable everywhere on $[a, b]$ and f' is bounded on $[a, b]$, then f is Lipschitz and $K = \|f'\|_\infty$ is a Lipschitz constant, where $\|\cdot\|_\infty$ denotes the L^∞-norm. However, in order to prove the Banach–Zaretsky Theorem, we will need to show that if f' is bounded *on a single subset* E then the estimate $|f(E)|_e \leq K\,|E|_e$ holds *for that set* E (with $K = \sup_{x\in E} |f'(x)|$). We need to obtain this estimate without assuming that f' is bounded on all of $[a, b]$, or that f is Lipschitz on $[a, b]$. Our next (very enlightening!) result takes a more sophisticated approach to derive an elegant estimate.

Remark 2 Naturally, I first learned this "Growth Lemma" from John's book [1]. A proof was published by Varberg in the comparatively "recent" paper [12], and in [6], I state that this is the earliest published proof of which I am aware, (also compare the related paper [10] by Serrin and Varberg). Interestingly enough, Varberg himself comments that this theorem is "an elegant inequality which the author discovered lying buried as an innocent problem in Natanson's book [7]" (but it is not proved in that text). Recently, I took another look at John's book, and to my surprise, I saw that he cites the text by Saks [9] (whose first edition appeared in 1937) for a proof; see [9, Lem. VII.6.3]. Thus, the history of this result seems to trace back farther than I knew. Yet the modern literature seems to be largely unaware of this elegant lemma. One text that does present the result is by Bruckner, Bruckner, and Thomson [3] (though they give no references). In fact, they prove a more general theorem there. Our proof is inspired by the proof given in [3]. ◇

Lemma 15 (Growth Lemma I) *Let E be any subset of $[a, b]$. If $f : [a, b] \to \mathbb{R}$ is differentiable at* every *point of E and if*

$$M_E = \sup_{x \in E} |f'(x)| < \infty,$$

then

$$|f(E)|_e \le M_E \, |E|_e.$$

Proof Choose any $\varepsilon > 0$. If $x \in E$, then

$$\lim_{\substack{y \to x, \\ y \in [a,b]}} \frac{|f(x) - f(y)|}{|x - y|} = |f'(x)| \le M_E.$$

Therefore, there exists an integer $n_x \in \mathbb{N}$ such that

$$y \in [a, b], \ |x - y| < \frac{1}{n_x} \implies |f(x) - f(y)| \le (M_E + \varepsilon) \, |x - y|. \tag{3.1}$$

For each $n \in \mathbb{N}$, let

$$E_n = \big\{x \in E : n_x \le n\big\}.$$

The sets E_n are nested increasing ($E_1 \subseteq E_2 \subseteq \cdots$), and their union is E. We do not know whether E_n is a measurable set, but fortunately continuity from below does hold for *exterior* Lebesgue measure (this is Problem 2.4.8 in [6]; in contrast, continuity from above can fail if the sets are not Lebesgue measurable!). Therefore,

$$|E|_e = \lim_{n \to \infty} |E_n|_e. \tag{3.2}$$

The images $f(E_n)$ are also nested increasing and their union is $f(E)$, so we likewise have

$$|f(E)|_e = \lim_{n \to \infty} |f(E_n)|_e. \tag{3.3}$$

Fix any particular integer n. By the definition of exterior Lebesgue measure, there exists a collection of countably many closed intervals $\{I_n^k\}_k$ such that

$$E_n \subseteq \bigcup_k I_n^k \qquad \text{and} \qquad \sum_k |I_n^k| \le |E_n|_e + \varepsilon. \tag{3.4}$$

By replacing I_n^k with $I_n^k \cap [a,b]$, we may assume that $I_n^k \subseteq [a,b]$ for each n and k. Further, by subdividing if necessary, we may assume that each interval I_n^k has length less than $1/n$.

Suppose that x and y are any two points in $E_n \cap I_n^k$. Then, since $x \in E_n$, we have $n_x \le n$. Also, since x and y belong to I_n^k, whose length is less than $1/n$,

$$|x-y| < \frac{1}{n} \le \frac{1}{n_x}.$$

It therefore follows from Eq. (3.1) that

$$|f(x) - f(y)| \le (M_E + \varepsilon)\,|x-y| \le (M_E+\varepsilon)\,|I_n^k|.$$

Since this is true for all $x,\ y \in E_n \cap I_n^k$, we conclude that

$$\operatorname{diam}\bigl(f(E_n \cap I_n^k)\bigr) = \sup\bigl\{|f(x)-f(y)| : x,\ y \in E_n \cap I_n^k\bigr\} \le (M_E+\varepsilon)\,|I_n^k|.$$

This implies that $f(E_n \cap I_n^k)$ is contained in an interval with length at most $(M_E+\varepsilon)\,|I_n^k|$. Hence,

$$|f(E_n \cap I_n^k)|_e \le (M_E+\varepsilon)\,|I_n^k|. \tag{3.5}$$

Consequently,

$$\begin{aligned}
|f(E_n)|_e &= \Bigl|\textstyle\bigcup_k f(E_n \cap I_n^k)\Bigr|_e && \text{(by Eq. (3.4))}\\
&\le \sum_k |f(E_n \cap I_n^k)|_e && \text{(by subadditivity)}\\
&\le (M_E+\varepsilon) \sum_k |I_n^k| && \text{(by Eq. (3.5))}\\
&\le (M_E+\varepsilon)\bigl(|E_n|_e + \varepsilon\bigr) && \text{(by Eq. (3.4)).}
\end{aligned}$$

Therefore, by applying Eqs. (3.2) and (3.3), we see that

$$\begin{aligned}
|f(E)|_e &= \lim_{n\to\infty} |f(E_n)|_e\\
&\le (M_E+\varepsilon) \lim_{n\to\infty} \bigl(|E_n|_e + \varepsilon\bigr)\\
&= (M_E+\varepsilon)\bigl(|E|_e + \varepsilon\bigr).
\end{aligned}$$

Since ε is arbitrary, the result follows. □

Our second Growth Lemma also appears in Varberg's paper [12], and our proof is based on the one given there. This result relates the exterior measure of $f(E)$ to the integral of $|f'|$ on E. A fun observation is that a key step in the proof is nothing less than the fact that $k = (k-1) + 1$. Sometimes the most elementary facts have large consequences.

Lemma 16 (Growth Lemma II) *Assume that* $f\colon [a,b] \to \mathbb{R}$ *is measurable. If* E *is a measurable subset of* $[a,b]$ *and* f *is differentiable at* every *point of* E, *then*

$$|f(E)|_e \;\le\; \int_E |f'|.$$

Proof An exercise for the student is to show that the derivative $f'\colon E \to \mathbb{R}$ is a measurable function on E. Hence, $\int_E |f'|$ exists, though it could be infinite. Fix any $\varepsilon > 0$, and for each $k \in \mathbb{N}$, define

$$E_k \;=\; \bigl\{x \in E : (k-1)\varepsilon \;\le\; |f'(x)| \;<\; k\varepsilon\bigr\}.$$

The sets E_k are measurable and disjoint, and since f is differentiable everywhere on E, we have $E = \cup E_k$. Since Lebesgue measure is countably additive, it follows that

$$|E| \;=\; \sum_{k=1}^{\infty} |E_k|.$$

Lemma 15 implies that $|f(E_k)|_e \le k\varepsilon\,|E_k|$, so we see that

$$\begin{aligned}
|f(E)|_e \;=\; \biggl|\bigcup_{k=1}^{\infty} f(E_k)\biggr|_e \;&\le\; \sum_{k=1}^{\infty} |f(E_k)|_e \\
&\le\; \sum_{k=1}^{\infty} k\varepsilon\,|E_k| \\
&=\; \sum_{k=1}^{\infty} (k-1)\varepsilon\,|E_k| \;+\; \sum_{k=1}^{\infty} \varepsilon\,|E_k| \\
&\le\; \sum_{k=1}^{\infty} \int_{E_k} |f'| \;+\; \varepsilon\,|E| \\
&=\; \int_E |f'| \;+\; \varepsilon\,|E|.
\end{aligned}$$

Since ε is arbitrary and $|E| < \infty$, the result follows. □

The two Growth Lemmas are striking in their own right and can be used to prove a number of interesting corollaries. We refer to [6] for more details.

3.4 The Banach–Zaretsky Theorem

Now we reach the highlight of this chapter, which is the often-overlooked *Banach–Zaretsky Theorem*. This result tells us what properties that a function $f : [a, b] \to \mathbb{R}$ needs to possess in addition to continuity in order to be absolutely continuous. Specifically, f must map sets with measure zero to sets with measure zero, and we must also know either that f has bounded variation, or that f is differentiable almost everywhere and f' is integrable. (The hypothesis that $|A| = 0$ implies $|f(A)| = 0$ is sometimes referred to as *Luzin's condition*.) The proof is a nice application of the Growth Lemmas.

Theorem 17 (Banach–Zaretsky Theorem) *If $f : [a, b] \to \mathbb{R}$, then the following three statements are equivalent.*

(a) $f \in \mathrm{AC}[a, b]$.

(b) *f is continuous, $f \in \mathrm{BV}[a, b]$, and*

$$A \subseteq [a, b], \ |A| = 0 \implies |f(A)| = 0.$$

(c) *f is continuous, f is differentiable a.e., $f' \in L^1[a, b]$, and*

$$A \subseteq [a, b], \ |A| = 0 \implies |f(A)| = 0.$$

Proof (a) $\Rightarrow$ (b). Every absolutely continuous function is continuous and has bounded variation, so our task is to show that f maps sets with measure zero to sets with measure zero.

Suppose that A is a subset of $[a, b]$ that has measure zero. Since the two-element set $\{a, b\}$ has measure zero and its image $\{f(a), f(b)\}$ also has measure zero, it suffices to assume that A is contained within the open interval (a, b). Fix $\varepsilon > 0$. By the definition of absolute continuity, there exists some $\delta > 0$ such that if $\{[a_j, b_j]\}_j$ is any countable collection of nonoverlapping subintervals of $[a, b]$ that satisfy $\sum (b_j - a_j) < \delta$, then $\sum |f(b_j) - f(a_j)| < \varepsilon$.

By basic properties of Lebesgue measure, there is an open set $U \supseteq A$ whose measure satisfies

$$|U| < |A| + \delta = \delta.$$

By replacing U with the open set $U \cap (a,b)$, we may assume that $U \subseteq (a,b)$. Since U is open, we can write it as a union of countably many disjoint open intervals contained in (a,b), say

$$U = \bigcup_j (a_j, b_j).$$

Fix any particular j. Since f is continuous on the closed interval $[a_j, b_j]$, there is a point in $[a_j, b_j]$ where f attains its minimum value on $[a_j, b_j]$, and another point where f attains its maximum. Let c_j and d_j be points in $[a_j, b_j]$ such that f has a max at one point and a min at the other. By interchanging their roles if necessary, we may assume that $c_j \le d_j$. Because f is continuous, the Intermediate Value Theorem implies that the image of $[a_j, b_j]$ under f is the set of all points between $f(c_j)$ and $f(d_j)$. Hence, the exterior Lebesgue measure of this image is

$$\big|f([a_j, b_j])\big|_e = |f(d_j) - f(c_j)|.$$

Now, $[c_j, d_j] \subseteq [a_j, b_j]$, so $\big\{[c_j, d_j]\big\}_j$ is a collection of nonoverlapping subintervals of $[a,b]$. Moreover,

$$\sum_j |d_j - c_j| \le \sum_j (b_j - a_j) = |U| < \delta.$$

Therefore, $\sum |f(d_j) - f(c_j)| < \varepsilon$, and hence,

$$|f(A)|_e \le |f(U)|_e \le \sum_j \big|f([a_j, b_j])\big|_e = \sum_j \big|f(d_j) - f(c_j)\big| < \varepsilon.$$

Since ε is arbitrary, we conclude that $|f(A)| = 0$.

(b) $\Rightarrow$ (c). This follows from Corollary 6.

(c) $\Rightarrow$ (a). Assume that statement (c) holds, and let D be the set of points where f is differentiable. By hypothesis, $Z = [a,b] \setminus D$ has measure zero, so $D = [a,b] \setminus Z$ is a measurable set.

Let $[c,d]$ be an arbitrary subinterval of $[a,b]$. Since f is continuous, the Intermediate Value Theorem implies that f must take every value between $f(c)$ and $f(d)$. Therefore $f([c,d])$, the image of $[c,d]$ under f, must contain an interval of length $|f(d) - f(c)|$. Define

$$B = [c,d] \cap D \qquad \text{and} \qquad A = [c,d] \setminus D.$$

The set A has measure zero, so $|f(A)| = 0$ by hypothesis. Since f is differentiable at every point of B, we therefore compute that

$$
\begin{aligned}
|f(d) - f(c)| &\le \bigl|f([c,d])\bigr|_e \\
&= \bigl|f(B) \cup f(A)\bigr|_e && \text{(since } [c,d] = B \cup A\text{)} \\
&\le |f(B)|_e + |f(A)|_e && \text{(by subadditivity)} \\
&\le \int_B |f'| + 0 && \text{(by Lemma 16 and hypotheses)} \\
&\le \int_c^d |f'| && \text{(since } B \subseteq [c,d]\text{).}
\end{aligned}
\tag{3.6}
$$

This calculation holds for every subinterval $[c, d]$ of $[a, b]$.

Now fix $\varepsilon > 0$. Because f' is integrable, there is some $\delta > 0$ such that for every measurable set $E \subseteq [a, b]$ we have

$$
|E| < \delta \implies \int_E |f'| < \varepsilon.
$$

Let $\bigl\{[a_j, b_j]\bigr\}_j$ be any countable collection of nonoverlapping subintervals of $[a, b]$ such that $\sum (b_j - a_j) < \delta$. Then, $E = \cup [a_j, b_j]$ is a measurable subset of $[a, b]$ and $|E| < \delta$, so $\int_E |f'| < \varepsilon$. Applying equation (3.6) to each subinterval $[a_j, b_j]$, it follows that

$$
\sum_j \bigl|f(b_j) - f(a_j)\bigr| \le \sum_j \int_{a_j}^{b_j} |f'| = \int_E |f'| < \varepsilon.
$$

Hence, f is absolutely continuous on $[a, b]$. □

Few books seem to mention the Banach–Zaretsky Theorem (also known as the Banach–Zarecki Theorem). Two that do are [1] and [3].

Remark 3 The statement of the Banach–Zaretsky Theorem for complex-valued functions is similar, except that both the real and imaginary parts of f must map sets of measure zero to sets of measure zero (see [6] for details). Specifically, if $f : [a, b] \to \mathbb{C}$ and we write $f = f_r + i f_i$, where f_r and f_i are real-valued, then statements (a)–(c) of Theorem 17 are equivalent if we replace the hypothesis "$|f(A)| = 0$" by "$|f_r(A)| = |f_i(A)| = 0$." ◇

3.5 Corollaries

We give several easy and immediate implications of the Banach–Zaretsky Theorem.

Corollary 18 *Absolutely continuous functions map sets of measure zero to sets of measure zero, and they map measurable sets to measurable sets.*

Proof Continuous functions map compact sets to compact sets. An F_σ-set is a countable union of compact sets, so continuous functions map F_σ-sets to F_σ-sets. If f is absolutely continuous, then it also maps sets of measure zero to sets of measure zero. But every measurable set can be written as the union of an F_σ-set and a set of measure zero, so f maps measurable sets to measurable sets. □

To motivate our second implication, we recall that if f is differentiable everywhere on $[a, b]$ and f' is bounded, then f is Lipschitz and therefore absolutely continuous (for one proof, see [6, Lem. 5.2.5]). What happens if f is differentiable everywhere on $[a, b]$ but we only know that f' is *integrable*? Although such a function need not be Lipschitz, the next corollary shows that f will be absolutely continuous.

Corollary 19 *If* $f: [a, b] \to \mathbb{R}$ *is differentiable everywhere on* $[a, b]$ *and* $f' \in L^1[a, b]$, *then* $f \in \mathrm{AC}[a, b]$.

Proof Let A be any subset of $[a, b]$ that has measure zero. Since f is differentiable everywhere, it is continuous and hence measurable. Because A is a measurable set, we can therefore apply Lemma 16 to obtain the estimate

$$|f(A)|_e \le \int_A |f'| = 0.$$

Consequently, the Banach–Zaretsky Theorem implies that f is absolutely continuous. □

Since all countable sets have measure zero, we immediately obtain the following refinement of Corollary 19.

Corollary 20 *Suppose that* $f: [a, b] \to \mathbb{R}$ *is continuous and is differentiable at all but countably many points of* $[a, b]$. *If* $f' \in L^1[a, b]$, *then* $f \in \mathrm{AC}[a, b]$.

Proof If we let $Z = \{x \in [a, b] : f'(x) \text{ does not exist}\}$, then Z is countable by hypothesis. Suppose that $A \subseteq [0, 1]$ satisfies $|A| = 0$. Then, $|A \setminus Z| = 0$. Since f is measurable and differentiable at every point of $A \setminus Z$, Growth Lemma II implies that

$$|f(A \setminus Z)|_e \le \int_{A \setminus Z} |f'| = 0.$$

On the other hand, the set $A \cap Z$ is countable, so $f(A \cap Z)$ is also countable, and therefore, $|f(A \cap Z)|_e = 0$. Applying countable subadditivity for exterior measure, we see that

$$|f(A)|_e \le |f(A \setminus Z)|_e + |f(A \cap Z)|_e = 0.$$

The Banach–Zaretsky Theorem therefore implies that $f \in \mathrm{AC}[a, b]$. □

Considering the Cantor–Lebesgue function, we see that we cannot extend Corollary 20 to functions that are differentiable at almost every point. Indeed, the Cantor–Lebesgue function φ is continuous and differentiable a.e., but it is not absolutely continuous.

Remark 4 We easily obtained Corollary 20 from the Banach–Zaretsky Theorem. In contrast, in the notes to Chapter 3 of [5], Folland writes,

> It is a highly nontrivial theorem that if F is continuous on $[a, b]$, $F'(x)$ exists for every $x \in [a, b] \setminus A$ where A is *countable*, and $F' \in L^1$, then F is absolutely continuous and hence can be recovered from F' by integration. A proof can be found in Cohn [4, Sec. 6.3]; see also Rudin [8, Theorem 7.26] for the somewhat easier case when $A = \varnothing$.

To us, this highlights the fundamental but overlooked nature of the Banach–Zaretsky Theorem. $\diamond$

Our final implication uses the Banach–Zaretsky Theorem (and the Growth Lemmas) to show that the only functions that are both absolutely continuous and singular are constant functions.

Corollary 21 (AC + Singular Implies Constant) *If $f : [a, b] \to \mathbb{R}$ is both absolutely continuous and singular, then f is constant.*

Proof Suppose that $f \in \mathrm{AC}[a, b]$ and $f' = 0$ a.e., and define

remove this equality symbol

$$E = \big\{x \cancel{=} \in [a, b] : f'(x) = 0\big\} \qquad \text{and} \qquad Z = [a, b] \setminus E.$$

Since $|Z| = 0$, the Banach–Zaretsky Theorem implies that $|f(Z)| = 0$. Since E is measurable and f is differentiable on E, Growth Lemma II implies that

$$|f(E)|_e \le \int_E |f'| = 0.$$

Therefore, the range of f has measure zero because

$$|\mathrm{range}(f)|_e = \big|f([a, b])\big|_e = |f(E) \cup f(Z)|_e \le |f(E)|_e + |f(Z)|_e = 0.$$

However, f is continuous and $[a, b]$ is compact, so the Intermediate Value Theorem implies that the range of f is either a single point or a closed interval $[c, d]$. Since $\mathrm{range}(f)$ has measure zero, we conclude that it is a single point, and therefore f is constant. $\square$

One standard proof of Corollary 21 uses the Vitali Covering Lemma (e.g., see the exposition in [13, Thm. 7.28]). By using the Banach–Zaretsky Theorem, we obtain a much simpler and more enlightening proof. However, the Vitali Covering Lemma does still play a role, since we use it to prove that monotone increasing functions are differentiable a.e. (Theorem 5).

3.6 The Fundamental Theorem of Calculus

Following Lemma 12, we asked two questions: First, is the indefinite integral G of an integrable function g differentiable? Second, if G is differentiable, does $G' = g$? The first question was answered affirmatively in Lemma 14, and the next lemma will show that $G' = g$ a.e.

Lemma 22 *If $g \in L^1[a,b]$, then its indefinite integral*

$$G(x) = \int_a^x g(t)\,dt, \qquad x \in [a,b],$$

is absolutely continuous and satisfies $G' = g$ a.e.

Proof Lemma 14 implies that G is absolutely continuous. Applying Corollary 9 (extend g by zero outside of $[a,b]$, so that it is locally integrable on $\mathbb{R}$), we also see that, for almost every $x \in [a,b]$,

$$\frac{G(x+h)-G(x)}{h} = \frac{1}{h}\int_x^{x+h} g(t)\,dt \to g(x) \quad \text{as } h \to 0.$$

Therefore, G is differentiable and $G'(x) = g(x)$ for almost every x. □

Now we give a simple proof of the Fundamental Theorem of Calculus.

Theorem 23 (Fundamental Theorem of Calculus) *If $f : [a,b] \to \mathbb{R}$, then the following three statements are equivalent.*

(a) $f \in \mathrm{AC}[a,b]$.

(b) *There exists a function $g \in L^1[a,b]$ such that*

$$f(x) - f(a) = \int_a^x g(t)\,dt, \qquad \textit{for all } x \in [a,b].$$

(c) *f is differentiable almost everywhere on $[a,b]$, $f' \in L^1[a,b]$, and*

$$f(x) - f(a) = \int_a^x f'(t)\,dt, \qquad \textit{for all } x \in [a,b].$$

Proof (a) $\Rightarrow$ (c). Suppose that f is absolutely continuous on $[a,b]$. Then f has bounded variation, so we know that f' exists a.e. and is integrable. Lemma 22 implies that the indefinite integral

$$F(x) = \int_a^x f'(t)\,dt$$

is absolutely continuous and satisfies $F' = f'$ a.e. Hence, $(F - f)' = 0$ a.e., so the function $F - f$ is both absolutely continuous and singular. Applying Corollary 21, we conclude that $F - f$ is constant. Consequently, for all $x \in [a, b]$,

$$F(x) - f(x) = F(a) - f(a) = 0 - f(a) = -f(a).$$

(c) $\Rightarrow$ (b). This follows by taking $g = f'$.

(b) $\Rightarrow$ (a). This follows from Lemma 22. □

Combining Theorem 23 with the Banach–Zaretsky Theorem gives us a remarkable list of equivalent characterizations of absolute continuity of functions on $[a, b]$. Many other results follow from this, and we refer to [6] for a continuation of this story.

Acknowledgments This chapter draws extensively from Chapters 5 and 6 of [6]. That material is used with permission of Springer. Many classic and recent volumes influenced the writing, the choice of topics, the proofs, and the selection of problems in [6]. We would like to explicitly acknowledge those tests that had the most profound influence. These include Benedetto and Czaja [2], Bruckner, Bruckner, and Thomson [3], Folland [5], Rudin [8], Stein and Shakarchi [11], and Wheeden and Zygmund [13]. I greatly appreciate all of these texts and encourage the reader to consult them. Many additional texts and papers are listed in the references to [6].

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