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Nonperiodic sampling and the local three squares theorem

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Abstract. This paper presents an elementary proof of the following theorem: *Given* $\{r_j\}_{j=1}^m$ *with* $m=d+1$, *fix* $R\geq\sum_{j=1}^m r_j$ *and let* $Q=[-R,R]^d$. *Then any* $f\in L^2(Q)$ *is completely determined by its averages over cubes of side* r_j *that are completely contained in* Q *and have edges parallel to the coordinate axes if and only if* r_j/r_k *is irrational for* $j\neq k$. When $d=2$ this theorem is known as the local three squares theorem and is an example of a Pompeiu-type theorem. The proof of the theorem combines ideas in multisensor deconvolution and the theory of sampling on unions of rectangular lattices having incommensurate densities with a theorem of Young on sequences biorthogonal to exact sequences of exponentials.

1. Introduction

The *three squares theorem* [1] asserts that any continuous function defined in the plane is completely determined by its averages over all squares of side r_1, r_2 and r_3 with sides parallel to the coordinate axes if and only if r_1/r_2, r_1/r_3, and r_2/r_3 are irrational. The present paper studies a local version of this theorem known as the *local three squares theorem* [2] which asserts that any continuous function defined on a square $Q\subseteq\mathbf{R}^2$ of side $R>r_1+r_2+r_3$ with edges parallel to the coordinate axes is completely determined by its averages over all squares of side r_1, r_2 and r_3 that are completely contained in Q and have edges parallel to the coordinate axes if and only if r_1, r_2, r_3 are pairwise irrationally related, that is, r_1/r_2, r_1/r_3, and r_2/r_3 are irrational.(2)

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(2) In [2], the condition "r_1, r_2, and r_3 are pairwise irrationally related" is erroneously stated as "r_1, r_2, and r_3 are $\mathbf{Q}$ linearly independent". The proof in [2] is of the theorem stated above.

The purpose of this paper is to give a new proof of the local three squares theorem. The new proof allows us to weaken the hypotheses, requiring only $f \in L^2(Q)$ instead of f continuous, and $R \geq r_1+r_2+r_3$ instead of $R > r_1+r_2+r_3$. We also prove the more general theorem in higher dimensions. The novelty of the proof lies in the fact that it uses only the sampling theory of bandlimited functions and a result of Young [18] on biorthogonal systems. The value of this new approach is that (1) the proof is elementary even in higher dimensions, (2) the close relationship of the global and local versions of the three squares theorem (and its higher dimensional generalizations) to the sampling theory of bandlimited functions has not previously been observed, and (3) the present proof suggests an algorithm for the recovery of a function locally from its local averages as indicated above. This recovery is inherently an unstable process, and the proof shows that the instability is related to the instability of recovery of a bandlimited function from its samples on so-called nonperiodic lattices (see Section 5 of [16] and Remark 3.1 below). Efficient algorithms exist in the case where only finitely many samples are available [7].

The three squares theorem is closely related to the *multisensor deconvolution problem* (see [3] and the references cited in [6]): *Given a collection of compactly supported distributions* $\{\mu_j\}_{j=1}^m$ *on* $\mathbf{R}^d$, *find a collection of compactly supported distributions* $\{\nu_j\}_{j=1}^m$ *such that*

$$\sum_{j=1}^{m} \mu_j * \nu_j = \delta. \tag{1}$$

A theorem of Hörmander [8] asserts that (1) has a solution if and only if $\{\mu_j\}_{j=1}^m$ satisfies the *strongly coprime* condition

$$\sum_{j=1}^{m} |\hat{\mu}_j(z)| \geq A(1+|z|)^{-N} e^{-B|\operatorname{Im} z|} \quad \text{for all } z \in \mathbf{C}^d, \tag{2}$$

for some constants $A, B, N > 0$.

If (1) could be solved with $d=2$ and with $\mu_j = \chi_{[-r_j,r_j]^2}$, $j=1, 2, 3$, then the global three squares theorem would follow since

$$\sum_{j=1}^{3} (f*\mu_j)*\nu_j = \sum_{j=1}^{3} f*(\mu_j*\nu_j) = f*\sum_{j=1}^{3} \mu_j*\nu_j = f*\delta = f.$$

However, it turns out that an additional algebraic assumption on the numbers r_j is required in this case, namely that r_j/r_k be poorly approximated by rationals when $j \neq k$ [12] (a number α is poorly approximated by rationals provided that there exist constants $C, N>0$ such that if p and q are integers then $|\alpha-p/q|\geq C|q|^{-N}$). If this condition is satisfied then it is possible to find compactly supported solutions to the equation

(3)
$$\sum_{j=1}^{3}\mu_j*\nu_{j,\varphi}=\varphi$$

for any $\varphi\in C_c^\infty(\mathbf{R}^d)$ supported in $[-R,R]^2$, where $R=r_1+r_2+r_3$, and such that $\operatorname{supp}\nu_{j,\varphi}\subseteq[-R+r_j,R-r_j]^2$ (e.g., [6], [15], [17]). From this, the local three squares theorem follows from the observations that $\widetilde{\varphi}(x)=\overline{\varphi(-x)}$ has the same smoothness and support properties as φ and that

(4)
$$\langle f,\widetilde{\varphi}\rangle=(f*\varphi)(0)=\sum_{j=1}^{3}((f*\mu_j)*\nu_{j,\varphi})(0)=\sum_{j=1}^{3}\langle f*\mu_j,\widetilde{\nu}_{j,\varphi}\rangle.$$

Note that computation of the last set of inner products only requires knowledge of $(f*\mu_j)$ on $[-R+r_j,R-r_j]^2$. Since $\widetilde{\varphi}$ was arbitrary, the local three squares theorem follows.

The idea behind the proof of the local three squares theorem given in this paper is to remove the algebraic requirement of being poorly approximated by rationals by solving (3) with $\widetilde{\varphi}\in\mathcal{B}$ where $\mathcal{B}$ is a suitable complete set in the Hilbert space $L^2[-R,R]^2$. Then by (4), any function for which $(f*\mu_j)=0$ on $[-R+r_j,R-r_j]^2$ must be orthogonal to every element of $\mathcal{B}$ and hence identically zero. It turns out that the most convenient choice for $\mathcal{B}$ is the (countable) set biorthogonal to the complete set of exponentials corresponding to sampling on a nonperiodic lattice.

Section 3 contains a brief summary of the one-dimensional nonperiodic sampling results of [16] required for this paper. Section 4 uses a result of Young [18] to show that the sequence biorthogonal to the set of exponentials corresponding to a nonperiodic sampling set is itself complete. Section 5 contains the proof of a one-dimensional version of the local three squares theorem. This proof is simple and clearly illustrates all of the central ideas of the higher-dimensional proof. Section 6 contains a proof of the higher-dimensional version of the local three squares theorem. Section 7 contains some remarks on the *completeness radius* and *frame radius* of certain nonperiodic sampling sets.

2. Notation and definitions

Given a multiindex $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbf{Z}_+^d$, we write $|\alpha|=\sum_{j=1}^d|\alpha_j|$, and $D^\alpha f=(\partial^{|\alpha|}/\partial^{\alpha_1}\dots\partial^{\alpha_d})f$. Given a set $\mathcal{A}$ and $a,b\in\mathcal{A}$, let $\delta_{a,b}=1$ if $a=b$ and 0 otherwise.

The Fourier transform of a function f is $\hat{f}(\xi)=\int_{\mathbf{R}^d} f(x)e^{-2\pi i x\cdot\xi}\,dx$ for $\xi\in\widehat{\mathbf{R}}^d$ whenever the integral makes sense, and is interpreted distributionally otherwise.

Given $\Omega>0$, $\mathrm{PW}_\Omega(\mathbf{R}^d)$ denotes the Paley–Wiener space

$$\mathrm{PW}_\Omega(\mathbf{R}^d)=\{f\in L^2(\mathbf{R}^d):\operatorname{supp}\hat{f}\subseteq\left[-\tfrac{1}{2}\Omega,\tfrac{1}{2}\Omega\right]^d\}.$$

Let H be a Hilbert space and $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ a collection of vectors in H for some index set $\mathcal{A}$. Then $\{x_\alpha\}$ is *complete* in H provided that any $x\in H$ satisfying $\langle x,x_\alpha\rangle=0$, for all $\alpha\in\mathcal{A}$, must be zero. A collection $\{y_\beta\}_{\beta\in\mathcal{A}}$ is *biorthogonal* to $\{x_\alpha\}$ provided that $\langle x_\alpha,y_\beta\rangle=\delta_{\alpha,\beta}$, and $\{x_\alpha\}$ is *minimal* in H provided that a collection of vectors biorthogonal to $\{x_\alpha\}$ exists. A collection of vectors both minimal and complete in H is said to be *exact* in H.

Given a finite set of positive numbers $\{r_j\}_{j=1}^m$ and an integer $d>0$, μ_j will always denote the function $\chi_{[-r_j,r_j]^d}$, the value of d being clear from the context.

3. Nonperiodic sampling in one dimension

Let $0<r_1<\dots<r_m$ be given such that $r_j/r_k\notin\mathbf{Q}$ for $j\neq k$. Let $R=\sum_{j=1}^m r_j$ and let Λ be defined by

$$\Lambda=\{n/2r_j:n\in\mathbf{Z}\backslash\{0\},\ j=1,\dots,m\}. \tag{5}$$

Our starting point is Theorem 3.1 of [16] which states that *if $f\in\mathrm{PW}_{2R}(\mathbf{R})$, if f vanishes on Λ, and if $f^{(j)}(0)=0$ for $j=0,\dots,m-1$, then $f\equiv0$.* Let

$$\Lambda^*=\{e^{2\pi i\lambda t}:\lambda\in\Lambda\}\cup\{1,2\pi ix,\dots,(2\pi ix)^{m-1}\}. \tag{6}$$

Then Theorem 3.1 of [16] is equivalent to the following result.

Corollary 3.1. *The system Λ^* defined by* (6) *is complete in $L^2[-R,R]$.*

In order to show that Λ^* is exact, recall the following construction from [16]. For $j=0,\dots,m-1$, define $g_j\in\mathrm{PW}_{2R}(\mathbf{R})$ by

$$g_j(t)=\frac{t^j}{j!}\prod_{k=1}^m\frac{\sin(2\pi r_k t)}{2\pi r_k t}. \tag{7}$$

Let $f_{0,m-1}=g_{m-1}$ and define $f_{0,j}\in\mathrm{PW}_{2R}(\mathbf{R})$ recursively for $j=m-2$ down to $j=0$ by

$$f_{0,j}(t)=g_j(t)+\sum_{l=j+1}^{m-1}\left(\frac{d^l}{dt^l}g_j(0)\right)f_{0,l}(t). \tag{8}$$

For $\lambda = n/2r_j \in \Lambda$, define $g_\lambda \in \mathrm{PW}_{2R}(\mathbf{R})$ by

$$(9)\qquad g_\lambda(t) = \frac{\sin(2\pi r_j(t-\lambda))}{2\pi r_j(t-\lambda)} \frac{\displaystyle\prod_{k\neq j} \frac{\sin(2\pi r_k t)}{2\pi r_k t}}{\displaystyle\prod_{k\neq j} \frac{\sin(2\pi r_k \lambda)}{2\pi r_k \lambda}}$$

and define $f_\lambda \in \mathrm{PW}_{2R}(\mathbf{R})$ by

$$(10)\qquad f_\lambda(t) = g_\lambda(t) - \sum_{l=0}^{m-1} \left(\frac{d^l}{dt^l} g_\lambda(0)\right) f_{0,l}(t).$$

Then the following theorem holds (cf. Proposition 3.1 of [16]).

Proposition 3.1. *Let Λ be given by (5), $f_{0,j}$ by (8), and f_λ by (10). Then*

(a) $f_{0,j}(\lambda)=0$ for $0\le j\le m-1$ and $\lambda\in\Lambda$, and $(d^k/dt^k)f_{0,j}(0)=\delta_{j,k}$ for $0\le j,k\le m-1$;

(b) $f_\lambda(\lambda')=\delta_{\lambda,\lambda'}$ for $\lambda,\lambda'\in\Lambda$, and $(d^k/dt^k)f_\lambda(0)=0$ for $0\le k\le m-1$.

Let $F_{0,j}=\hat{f}_{0,j}$, $F_\lambda=\hat{f}_\lambda$, and define the collection $\mathcal{F}$ by

$$(11)\qquad \mathcal{F} = \{F_{0,j}\}_{j=0}^{m-1} \cup \{F_\lambda\}_{\lambda\in\Lambda}.$$

Then $\mathcal{F}$ is biorthogonal to Λ^* and the following holds.

Corollary 3.2. *The system Λ^* defined by (6) is exact in $L^2[-R,R]$.*

Remark 3.1. The biorthogonality of Λ^* and $\mathcal{F}$ means that for $F\in L^2[-R,R]$, we have the formal expansion

$$(12)\qquad F(x) \sim \sum_{j=0}^{m-1} \langle F, F_{0,j}\rangle (2\pi i x)^j + \sum_{\lambda\in\Lambda} \langle F, F_\lambda\rangle e^{2\pi i\lambda x}.$$

However, it was observed in Theorem 5.3 of [16] that the collection Λ^* is not a frame for $L^2[-R,R]$ since it does not have a lower frame bound. This fact follows from the observation that the set $\{g_\lambda\}_{\lambda\in\Lambda}$ is not norm-bounded in $\mathrm{PW}_{2R}(\mathbf{R})$ which implies that $\mathcal{F}$ does not have an upper frame bound. Therefore, the sequence on the right-hand side of (12) need not converge in $L^2[-R,R]$ nor even in some weaker sense.

However, we would like $\mathcal{F}$ to at least be complete in $L^2[-R,R]$ so that F is uniquely determined by the inner products $\langle F, F_{0,j}\rangle$ and $\langle F, F_\lambda\rangle$. It turns out that this statement is not obvious since there are examples of exact sequences in Hilbert spaces whose biorthogonal sequences are not complete (see [18]). We will show in Section 4 that the completeness of $\mathcal{F}$ in $L^2[-R,R]$ follows from a result of Young [18], and in Section 6 that the inner products $\langle F, F_{0,j}\rangle$ and $\langle F, F_\lambda\rangle$ can be easily and stably computed from the convolutions $F*\mu_k$. Hence the instability in recovering F from its local averages comes precisely from the possible nonconvergence of the sum (12). It may still be possible to recover a good approximation to F from (12) if only a finite number of data are used [7].

4. Completeness of $\mathcal{F}$

The goal of this section is to prove the following theorem.

Theorem 4.1. *The collection $\mathcal{F}$ given by* (11) *is complete in $L^2[-R,R]$.*

The proof of this theorem requires the following result of Young.

Theorem 4.2. ([18]) *If a sequence of complex exponentials $\{e^{i\lambda_n t}\}$ is exact in $L^2(-\pi,\pi)$, then its biorthogonal sequence is also exact.*

Since the collection Λ^* contains the monomials $(2\pi ix)^k$ for $1\le k\le m-1$, it is not a collection of complex exponentials. Therefore Theorem 4.2 cannot be applied directly and the following lemmas are required.

Lemma 4.1. *Let Λ be given by* (5) *and let $\omega=\{\omega_j\}_{j=0}^{m-1}$ be a collection of distinct points such that $\Lambda\cap\omega=\emptyset$. Then the system*

$$\Lambda^{\#}=\{e^{2\pi i\lambda t}:\lambda\in\Lambda\cup\omega\}$$

is exact in $L^2[-R,R]$.

Proof. We must show that $\Lambda^{\#}$ is minimal and complete.

To show completeness, let $g\in\mathrm{PW}_{2R}(\mathbf{R})$ be such that $g(\lambda)=0$ for all $\lambda\in\Lambda\cup\omega$. Define h by

$$h(t)=\frac{t^m}{(t-\omega_0)\dots(t-\omega_{m-1})}g(t).$$

Since $g\in\mathrm{PW}_{2R}(\mathbf{R})$, g can be extended to an entire function of exponential type $2R$ on $\mathbf{C}$. Clearly h can also be extended to an entire function of the same exponential type. Moreover, $h\in L^2(\mathbf{R})$ since h has the same decay at infinity as g. Thus, $h\in\mathrm{PW}_{2R}(\mathbf{R})$, h vanishes on Λ, and $h^{(j)}(0)=0$ for $j=0,\dots,m-1$. By Theorem 3.1 of [16] (cf. Corollary 3.1), $h\equiv 0$, whence $g\equiv 0$.

To show minimality, we construct explicitly the sequence biorthogonal to $\Lambda^{\#}$. Define for $\omega_j \in \omega$ the functions $f^{\#}_{\omega_j} \in \mathrm{PW}_{2R}(\mathbf{R})$ by

$$f^{\#}_{\omega_j}(t) = \prod_{k \neq j} \frac{t-\omega_k}{\omega_j-\omega_k} \frac{\displaystyle\prod_{k=1}^{m} \frac{\sin(2\pi r_k t)}{2\pi r_k t}}{\displaystyle\prod_{k=1}^{m} \frac{\sin(2\pi r_k \omega_j)}{2\pi r_k \omega_j}}$$

and for each $\lambda \in \Lambda$ define g_λ by (9) and $f^{\#}_\lambda \in \mathrm{PW}_{2R}(\mathbf{R})$ by

$$f^{\#}_\lambda(t) = g_\lambda(t) - \sum_{l=0}^{m-1} g_\lambda(\omega_l) f^{\#}_{\omega_l}(t). \tag{13}$$

Then

(a) $f^{\#}_{\omega_j}(\lambda)=0$ for $\lambda \in \Lambda$, and $f^{\#}_{\omega_j}(\omega_k)=\delta_{j,k}$ for $\omega_k \in \omega$, and

(b) $f^{\#}_\lambda(\lambda')=\delta_{\lambda,\lambda'}$ for $\lambda, \lambda' \in \Lambda$, and $f^{\#}_\lambda(\omega_j)=0$ for $\omega_j \in \omega$.

Let $F^{\#}_{\omega_j}=\hat{f}^{\#}_{\omega_j}$, $F^{\#}_\lambda=\hat{f}^{\#}_\lambda$, and define the collection $\mathcal{F}^{\#}$ by

$$\mathcal{F}^{\#} = \{F^{\#}_{\omega_j}\}_{j=0}^{m-1} \cup \{F^{\#}_\lambda\}_{\lambda \in \Lambda}. \tag{14}$$

Then $\mathcal{F}^{\#}$ is biorthogonal to $\Lambda^{\#}$, and $\Lambda^{\#}$ is exact. □

Finally, we require the following lemma.

Lemma 4.2. *Let $\mathcal{F}$ be given by* (11) *and $\mathcal{F}^{\#}$ by* (14). *If $\mathcal{F}^{\#}$ is complete in $L^2[-R,R]$ then so is $\mathcal{F}$.*

Proof. Let $G \in L^2[-R,R]$ be given such that $\langle G,F\rangle=0$ for all $F \in \mathcal{F}$. We will show that $\langle G,F^{\#}\rangle=0$ for all $F^{\#} \in \mathcal{F}^{\#}$ which implies $G \equiv 0$ by the assumption of completeness of $\mathcal{F}^{\#}$.

Let $g \in \mathrm{PW}_{2R}(\mathbf{R})$ be such that $\hat{g}=G$. By the Parseval identity, $\langle g, f_{0,j}\rangle=0$ where $f_{0,j}$ is given by (8), and $\langle g, f_\lambda\rangle=0$ where f_λ is given by (10). It will suffice to show that $\langle g, f^{\#}_{\omega_j}\rangle=0$ for $\omega_j \in \omega$ and that $\langle g, f^{\#}_\lambda\rangle=0$ for $\lambda \in \Lambda$.

Since $\langle g, f_{0,j}\rangle=0$, we have by (8) that $\langle g, g_j\rangle=0$ where g_j is given by (7). Therefore, for any polynomial $P(t)$ of degree not greater than $m-1$, g is orthogonal to functions of the form

$$P(t) \prod_{k=1}^{m} \frac{\sin(2\pi r_k t)}{2\pi r_k t}.$$

Since $f^{\#}_{\omega_j}$ has precisely this form, $\langle g, f^{\#}_{\omega_j}\rangle=0$ for $\omega_j \in \omega$. Since g is orthogonal to $f_{0,j}$ and also to f_λ, it follows from (10) that g is orthogonal to g_λ where g_λ is given by (9). Since g is also orthogonal to $f^{\#}_{\omega_j}$ for $\omega_j \in \omega$, it follows from (13) that g is orthogonal to $f^{\#}_{\lambda}$ for $\lambda \in \Lambda$. This completes the proof. □

We can now prove Theorem 4.1.

Proof of Theorem 4.1. Since, by Lemma 4.1, $\Lambda^{\#}$ is an exact system of complex exponentials, Theorem 4.2 implies that $\mathcal{F}^{\#}$ is complete in $L^2[-R,R]$. By Lemma 4.2, $\mathcal{F}$ is also complete in $L^2[-R,R]$, which completes the proof. □

5. Uniqueness from averages on two intervals

In this section, we prove a one-dimensional version of the three squares theorem. The proof contains the fundamental ideas of the proof of the higher-dimensional result (Theorem 6.1). Define $G_j \in L^2[-R,R]$ by $\hat{g}_j = G_j$ where g_j is given by (7), and define $G_\lambda \in L^2[-R,R]$ by $\hat{g}_\lambda = G_\lambda$, where g_λ is given by (9). Now define $\mathcal{G} \subseteq L^2[-R,R]$ by

$$\mathcal{G} = \{G_j\}_{j=0}^{m-1} \cup \{G_\lambda\}_{\lambda\in\Lambda}. \tag{15}$$

Since each $F \in \mathcal{F}$ can be written as a linear combination of elements of $\mathcal{G}$, and since $\mathcal{F}$ is complete in $L^2[-R,R]$, so is $\mathcal{G}$.

Theorem 5.1. *Let $0<r_1<r_2$, let $R=r_1+r_2$ and let Λ be given by* (5). *Then $r_1/r_2 \notin \mathbf{Q}$ if and only if the only $f \in L^2[-R,R]$ satisfying*
(a) *$f * \mu_1(x)=0$ for $x \in [-r_2, r_2]$, and*
(b) *$f * \mu_2(x)=0$ for $x \in [-r_1, r_1]$*
is $f \equiv 0$.

Proof. Suppose that $r_1/r_2 \notin \mathbf{Q}$ and that $f \in L^2[-R,R]$ satisfies (a) and (b). We will show that $\langle f, G\rangle = 0$ for all $G \in \mathcal{G}$, where $\mathcal{G}$ is given by (15) with $m=2$. Since G_0 is a constant multiple of $\mu_1 * \mu_2$, (b) implies that,

$$\begin{aligned}\langle f, G_0\rangle &= c\int_{-R}^{R} f(x)(\mu_1*\mu_2)(x)\,dx = c\int_{-R}^{R} f(x)\int_{-\infty}^{\infty}\mu_1(t)\mu_2(x-t)\,dt\,dx\\ &= c\int_{-\infty}^{\infty}\mu_1(t)\int_{-R}^{R} f(x)\mu_2(t-x)\,dx\,dt = c\int_{-r_1}^{r_1}(f*\mu_2)(t)\,dt = 0.\end{aligned}$$

Since G_1 is a constant multiple of $(d/dt)(\mu_1*\mu_2)(t) = \mu_2(t+r_1) - \mu_2(t-r_1)$, (b) implies that

$$\begin{aligned}\langle f, G_1\rangle &= c\int_{-R}^{R} f(x)(\mu_2(t+r_1)-\mu_2(t-r_1))(x)\,dx\\ &= c((f*\mu_2)(-r_1)-(f*\mu_2)(r_1)) = 0.\end{aligned}$$

If $\lambda=n/2r_j\in\Lambda$ then G_λ is a constant multiple of $(e^{-2\pi i\lambda t}\mu_j)*\mu_k(t)$, where $k\in\{1,2\}$ and $k\neq j$. Therefore, by (a) and (b),

$$\begin{aligned}\langle f,G_\lambda\rangle &= c\int_{-R}^{R} f(x)(e^{-2\pi i\lambda x}\mu_j*\mu_k)(x)\,dx\\ &= c\int_{-R}^{R} f(x)\int_{-\infty}^{\infty} e^{-2\pi i\lambda t}\mu_j(t)\mu_k(x-t)\,dt\,dx\\ &= c\int_{-\infty}^{\infty} e^{-2\pi i\lambda t}\mu_j(t)\int_{-R}^{R} f(x)\mu_k(t-x)\,dx\,dt\\ &= c\int_{-r_j}^{r_j} e^{-\pi int/r_j}(f*\mu_k)(t)\,dt=0.\end{aligned}$$

Since $\mathcal{G}$ is complete in $L^2[-R,R]$ we conclude that $f\equiv 0$.

For the converse, suppose that $r_1/r_2\in\mathbf{Q}$. Then there exist $n,m\in\mathbf{Z}$ such that $n/2r_1=m/2r_2$. Let $\lambda=n/2r_1=m/2r_2$, and let $f(t)=\sin(2\pi\lambda t)\chi_{[-R,R]}(t)$. Since f has period $2r_1$ and $2r_2$ on $[-R,R]$ (that is, if $t_1,t_2\in[-R,R]$ with $t_1-t_2=2r_1$ or $2r_2$, then $f(t_1)=f(t_2)$), and since the integral of f on any interval of length $2r_1$ or $2r_2$ is zero, $(f*\mu_j)(x)=0$ for $x\in[-r_k,r_k]$, $j,k=1,2$, $j\neq k$. However, $f\not\equiv 0$. $\square$

Alternative proofs of Theorems 5.1 and 6.1 are presented in the note by K. Seip following this article [14].

6. Nonperiodic sampling in higher dimensions

Since tensor products of complete sets are complete, the following is an immediate consequence of Corollary 3.1.

Corollary 6.1. *Let $0<r_1<...<r_m$ be given such that $r_j/r_k\notin\mathbf{Q}$ for $j\neq k$. Let $R=\sum_{j=1}^m r_j$ and let Λ^* be defined by (6). Define $\Lambda_d^*\subseteq L^2[-R,R]^d$ by*

$$\Lambda_d^*=\left\{e(x)=\prod_{j=1}^d e^j(x_j):x=(x_1,\dots,x_d),\ e^j\in\Lambda^*\right\}.$$

Then Λ_d^* is complete in $L^2[-R,R]^d$.

In fact, Λ_d^* is exact since its biorthogonal sequence is the collection $\mathcal{F}_d$ defined by

$$\mathcal{F}_d=\left\{F(x)=\prod_{j=1}^d F^j(x_j):x=(x_1,\dots,x_d),\ F^j\in\mathcal{F}\right\},\tag{16}$$

where $\mathcal{F}$ is defined by (11). Also, it follows from Theorem 4.1 that $\mathcal{F}_d$ is complete in $L^2[-R,R]^d$.

Theorem 6.1. *Let* $0<r_1<\ldots<r_m$, *let* $m=d+1$, *and let* $R=\sum_{j=1}^{m} r_j$. *Then the following are equivalent*:

(a) *The collection* $\{r_j\}_{j=1}^{m}$ *satisfies* $r_j/r_k\notin\mathbf{Q}$, $j\neq k$.

(b) *If* $f\in L^2[-R,R]^d$ *satisfies*

$$(f*\mu_j)(x)=0, \quad x\in[-R+r_j,R-r_j]^d,\ j=1,\ldots,m, \tag{17}$$

then $f\equiv 0$.

Proof. (a) $\Rightarrow$ (b) Define the collection $\mathcal{G}_d\subseteq L^2[-R,R]^d$ by

$$\mathcal{G}_d=\left\{G(x)=\prod_{j=1}^{d} G^j(x_j): x=(x_1,\ldots,x_d),\ G^j\in\mathcal{G}\right\},$$

where $\mathcal{G}$ is defined by (15). It follows from the definition of $\mathcal{F}_d$ and $\mathcal{G}_d$ that any element of $\mathcal{F}_d$ can be written as a linear combination of elements of $\mathcal{G}_d$. Therefore, since $\mathcal{F}_d$ is complete in $L^2[-R,R]^d$, so is $\mathcal{G}_d$.

Suppose that the set $\{r_j\}_{j=1}^{m}$ satisfies $r_j/r_k\notin\mathbf{Q}$ if $j\neq k$, and that $f\in L^2[-R,R]^d$ satisfies (17). We will show that f is orthogonal to every element of $\mathcal{G}_d$. To this end, fix $G\in\mathcal{G}_d$. Then there exists $k_0\in\{1,\ldots,m\}$ such that G has the form $G=H*\mu_{k_0}$ where $\operatorname{supp} H\subseteq[-R+r_{k_0},R-r_{k_0}]^d$. To see why this is true, note that for each $l=1,\ldots,d$, $G^l\in\mathcal{G}$ and that, by (15), G^l can be written either as G_j for some $j=0,\ldots,m-1$, or as G_λ for some $\lambda\in\Lambda$. Now, G_j has the form

$$G_j(t)=c_j\frac{d^j}{dt^j}(\mu_1*\ldots*\mu_m),$$

for some constant c_j, and if $\lambda=n/2r_k$ then G_λ has the form

$$G_\lambda(t)=c_\lambda(e^{-2\pi i\lambda t}\mu_k)*(\mu_1*\ldots*\mu_{k-1}*\mu_{k+1}*\ldots*\mu_m).$$

Therefore, $G\in\mathcal{G}$ has the form

$$G(x)=\prod_{l\notin\Gamma} G_{j_l}(x_l)\prod_{l\in\Gamma} G_{\lambda_l}(x_l), \tag{18}$$

where $\Gamma\subseteq\{1,\ldots,d\}$. Since $m=d+1$, $|\Gamma|\leq d<m$ so there exists $k_0\in\{1,\ldots,m\}$ such that $\lambda_l\neq n/2r_{k_0}$ for all $l\in\Gamma$ and $n\in\mathbf{Z}\setminus\{0\}$. Also, for each l we have that $j_l<m$. Hence each term in the product (18) is either of the form

$$G_{j_l}(t)=\left(\left[c_{j_l}\frac{d^{j_l}}{dt^{j_l}}(\mu_1*\ldots*\mu_{k_0-1}*\mu_{k_0+1}*\ldots*\mu_m)\right]*\mu_{k_0}\right)(t)\equiv(H_{j_l}*\mu_{k_0})(t),$$

or of the form

$$G_{\lambda_l}(t) = (c_{\lambda_l}(\mu_1 * \ldots * \mu_{k-1} * (e^{-2\pi i \lambda_l t}\mu_k) * \mu_{k+1} * \ldots * \mu_{k_0-1} * \mu_{k_0+1} * \ldots * \mu_m) * \mu_{k_0})(t)$$
$$\equiv (H_{\lambda_l} * \mu_{k_0})(t).$$

Note that $\operatorname{supp} H_{j_l}$ and $\operatorname{supp} H_{\lambda_l}$ are both contained in $[-R+r_{k_0}, R-r_{k_0}]$. Let

$$H(x) = \prod_{l\notin\Gamma} H_{j_l}(x_l) \prod_{l\in\Gamma} H_{\lambda_l}(x_l).$$

Then $\operatorname{supp} H \subseteq [-R+r_{k_0}, R-r_{k_0}]^d$ and $G = H * \mu_{k_0}$ as required.

By (17),

$$\begin{aligned}\langle f, G\rangle &= \int_{[-R,R]^d} f(x)\overline{G(x)}\,dx = \int_{[-R,R]^d} f(x)(\overline{H} * \mu_{k_0})(x)\,dx \\ &= \int_{[-R,R]^d} f(x) \int_{[-R+r_{k_0}, R-r_{k_0}]^d} \overline{H}(y)\mu_{k_0}(x-y)\,dy\,dx \\ &= \int_{[-R+r_{k_0}, R-r_{k_0}]^d} \overline{H}(y) \int_{[-R,R]^d} f(x)\mu_{k_0}(y-x)\,dx\,dy \\ &= \int_{[-R+r_{k_0}, R-r_{k_0}]^d} \overline{H}(y)(f*\mu_{k_0})(y)\,dy = 0.\end{aligned}$$

Since $\mathcal{G}_d$ is complete in $L^2[-R,R]^d$, $f\equiv 0$.

(a) $\Leftarrow$ (b) Suppose that $r_{j_0}/r_{k_0} \in \mathbf{Q}$ for some $j_0 \neq k_0$. Assume without loss of generality that $j_0=1$ and $k_0=2$. Then there exist $n, k \in \mathbf{Z}\setminus\{0\}$ such that $\lambda = n/2r_1 = k/2r_2$. Let $R_1 = r_1 + r_2$ and let

$$f_1(x) = \sin(2\pi\lambda x_1)\chi_{[-R_1,R_1]^d}(x).$$

Let $R_2 = R - R_1$ and let

$$f_2(x) = \left(\prod_{l=2}^{d} \sin\left(\frac{\pi x_l}{r_{l+1}}\right)\right)\chi_{[-R_2,R_2]^d}(x).$$

Finally, let $f = f_1 * f_2 \in L^2[-R,R]^d$.

Let $j=1$ or 2. Given $x\in[-R,R]^d$, write $x=(x_1,x')$ where $x_1\in[-R,R]$ and $x'\in[-R,R]^{d-1}$. Then

$$\begin{aligned}(f_1 * \mu_j)(x) &= \int_{[-R_1,R_1]^d} f_1(y)\mu_j(x-y)\,dy \\ &= \int_{-R_1}^{R_1} \sin(2\pi\lambda y_1)\chi_{[-r_j,r_j]}(x_1-y_1)\,dy_1 \int_{[-R_1,R_1]^{d-1}} \chi_{[-r_j,r_j]^d}(x'-y')\,dy' \\ &= 0\end{aligned}$$

since the integral of $\sin(2\pi\lambda t)$ over any interval of length $2r_1$ or $2r_2$ is zero. If $j\neq 1$, 2, then

$$\begin{aligned}(f_2*\mu_j)(x) &= \int_{[-R_2,R_2]^d} f_2(y)\mu_j(x-y)\,dy\\ &= \left(\prod_{l=2}^{d}\int_{-R_2}^{R_2}\sin\left(\frac{\pi y_l}{r_{l+1}}\right)\chi_{[-r_j,r_j]}(x_l-y_l)\,dy_l\right)\int_{-R_2}^{R_2}\chi_{[-r_j,r_j]}(x_1-y_1)\,dy_1\\ &= 0\end{aligned}$$

since for $j=l+1$, the integral of $\sin(\pi t/r_{l+1})$ over any interval of length $2r_j$ is zero. Thus, $f*\mu_j=f_1*f_2*\mu_j=f_2*f_1*\mu_j=0$ for $j=1,\dots,m$, but $f\not\equiv 0$. □

7. Further remarks on completeness of Λ^*

The purpose of this section is to examine the *completeness radius* and the *frame radius* of the set Λ given by (5). The completeness radius of a set $S\subseteq\mathbf{C}$ is the supremum of all numbers $r>0$ such that $\{e^{2\pi i\lambda t}\}_{\lambda\in S}$ is complete in $L^2[-r,r]$ (see [5], [11], and [13]). The frame radius of a set $S\subseteq\mathbf{C}$ is the supremum of all numbers $r>0$ such that $\{e^{2\pi i\lambda t}\}_{\lambda\in S}$ is a frame for $L^2[-r,r]$ (see [9]). Assume that the numbers $0<r_1<\dots<r_m$ satisfy $r_j/r_k\notin\mathbf{Q}$ when $j\neq k$, and that $R=\sum_{j=1}^m r_j$. The set Λ is defined by (5), the set Λ^* by (6).

We will prove the following theorem.

Theorem 7.1. (a) If $r<R$ then $\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}$ is a frame for $L^2[-r,r]$.

(b) If $r>R$ then Λ^* is incomplete in $L^2[-r,r]$. In this case, since $\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}\subseteq\Lambda^*$, it follows that $\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}$ is also incomplete in $L^2[-r,r]$.

Corollary 3.1 addresses in part the question of what happens when $r=R$. The answer is that $\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}$ is incomplete in $L^2[-R,R]$. With the addition of a finite number of functions, namely $\{1,2\pi ix,\dots,(2\pi ix)^{m-1}\}$, the resulting collection is complete and in fact exact in $L^2[-R,R]$ (Corollary 3.2). However, it is not a frame for $L^2[-R,R]$ (Remark 3.1).

In order to prove Theorem 7.1, we will need the following definition.

Definition 7.1. Given a set $S\subseteq\mathbf{R}$ and $t>0$, define

$$n^+(t,S)=\sup_{x\in\mathbf{R}}\#(S\cap[x,x+t))\quad\text{and}\quad n^-(t,S)=\inf_{x\in\mathbf{R}}\#(S\cap[x,x+t)).$$

The *upper uniform density* of S is defined by

$$D^+(S)=\limsup_{t\to\infty}\left(\frac{n^+(t,S)}{t}\right)$$

and the *lower uniform density* of S is defined by

$$D^-(S) = \liminf_{t\to\infty}\left(\frac{n^-(t,S)}{t}\right).$$

It is easy to see that $D^+(\Lambda)=D^-(\Lambda)=2R$. We can now prove Theorem 7.1.

Proof. (a) Suppose that $r<R$. In order to prove that $\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}$ is a frame for $L^2[-r,r]$, we use the following result ([4]; [13, Theorem 2.1]; [9, Theorem 3]): *If* $S\subseteq\mathbf{R}$ *is such that* (i) S *is uniformly discrete, that is,* $\inf_{x,y\in S}|x-y|=\delta>0$, *and* (ii) $D^-(S)>2r$, *then* $\{e^{2\pi i\lambda t}\}_{\lambda\in S}$ *is a frame for* $L^2[-r,r]$. Note that the set Λ is not uniformly discrete, that is, there are pairs of points in Λ that are arbitrarily close together. The idea of the proof is to show that Λ contains a uniformly discrete subset with density arbitrarily close to $2R$.

Given $\alpha,\beta\in\{r_j\}_{j=1}^m$ with $\alpha<\beta$ and $\varepsilon>0$, define

$$N_{\alpha,\beta}(\varepsilon)=\left\{\frac{n}{2\beta}:n\in\mathbf{Z}\backslash\{0\} \text{ and } \operatorname{dist}\left(\frac{n}{2\beta},\frac{1}{2\alpha}\mathbf{Z}\right)<\frac{\varepsilon}{2}\right\}.$$

Note that $n/2\beta\in N_{\alpha,\beta}(\varepsilon)$ if and only if $\operatorname{dist}(\alpha n/\beta,\mathbf{Z})<\alpha\varepsilon$ which holds if and only if $\langle\alpha n/\beta\rangle\in[0,\alpha\varepsilon]\cup[1-\alpha\varepsilon,1)\equiv K$ (here $\langle x\rangle=x-[x]$ where $[x]$ is the greatest integer less than or equal to x).

Since α/β is irrational, we know by Weyl's equidistribution theorem ([10, p. 8]) that

$$\lim_{N\to\infty}\frac{1}{N}\left(\sup_{l\in\mathbf{Z}}\sum_{n=l+1}^{N+l}\chi_K\left(\left\langle\frac{\alpha}{\beta}n\right\rangle\right)\right)=|K|=2\alpha\varepsilon.$$

We wish to estimate the upper uniform density of $N_{\alpha,\beta}(\varepsilon)$. Given an interval I with $|I|=t\geq1$, let $m=\min\{n:n/2\beta\in I\}$ and $M=\max\{n:n/2\beta\in I\}$. Then $2\beta t\leq M-m+1\leq2\beta t+1$ and

$$\begin{aligned}\frac{\#(N_{\alpha,\beta}(\varepsilon)\cap I)}{t}&=\frac{1}{t}\sum_{n=m}^{M}\chi_K\left(\left\langle\frac{\alpha}{\beta}n\right\rangle\right)\\&=\left(\frac{M-m+1}{t}\right)\frac{1}{M-m+1}\sum_{n=m}^{M}\chi_K\left(\left\langle\frac{\alpha}{\beta}n\right\rangle\right)\\&\leq\left(2\beta+\frac{1}{t}\right)\frac{1}{M-m+1}\left(\sup_{l\in\mathbf{Z}}\sum_{n=l}^{M-m+l}\chi_K\left(\left\langle\frac{\alpha}{\beta}n\right\rangle\right)\right).\end{aligned}$$

Letting $t\to\infty$ forces $M-m+1\to\infty$ so that

$$\limsup_{t\to\infty}\left(\sup_{|I|=t}\frac{\#(N_{\alpha,\beta}(\varepsilon)\cap I)}{t}\right)\leq2\beta2\alpha\varepsilon=4\alpha\beta\varepsilon.$$

Therefore, $D^+(N_{\alpha,\beta}(\varepsilon)) \le 4\alpha\beta\varepsilon$.

Let $N(\varepsilon) = \bigcup_{j<k} N_{r_j,r_k}(\varepsilon)$. Then

$$D^+(N(\varepsilon)) \le \sum_{j<k} D^+(N_{r_j,r_k}(\varepsilon)) \le 4\varepsilon \sum_{j<k} r_j r_k.$$

Choose $\varepsilon > 0$ so small that $2r + D^+(N(\varepsilon)) < 2R$. Then for each $x \in \mathbf{R}$ and $t > 0$,

$$\begin{aligned} \#((\Lambda \setminus N(\varepsilon)) \cap [x, x+t)) &= \#(\Lambda \cap [x, x+r)) - \#(N(\varepsilon) \cap [x, x+t)) \\ &\ge \#(\Lambda \cap [x, x+r)) - \sup_{a \in \mathbf{R}} \#(N(\varepsilon) \cap [a, a+t)). \end{aligned}$$

Therefore,

$$\inf_{x \in \mathbf{R}} \#((\Lambda \setminus N(\varepsilon)) \cap [x, x+t)) \ge \inf_{x \in \mathbf{R}} \#(\Lambda \cap [x, x+r)) - \sup_{x \in \mathbf{R}} \#(N(\varepsilon) \cap [x, x+t))$$

and

$$\liminf_{t \to \infty} \left(\frac{n^-(t, \Lambda \setminus N(\varepsilon))}{t} \right) \ge \liminf_{t \to \infty} \left(\frac{n^-(t, \Lambda)}{t} \right) - \limsup_{t \to \infty} \left(\frac{n^+(t, N(\varepsilon))}{t} \right)$$

so that

$$D^-(\Lambda \setminus N(\varepsilon)) \ge D^-(\Lambda) - D^+(N(\varepsilon)) = 2R - D^+(N(\varepsilon)) > 2r.$$

Thus $\Lambda \setminus N(\varepsilon)$ satisfies (ii). Also, if $\lambda, \lambda' \in \Lambda \setminus N(\varepsilon)$ and $\lambda \ne \lambda'$ then $|\lambda - \lambda'| \ge \frac{1}{2}\varepsilon$, and $\Lambda \setminus N(\varepsilon)$ satisfies (i). Therefore, $\{e^{2\pi i \lambda t}\}_{\lambda \in \Lambda \setminus N(\varepsilon)}$ is a frame for $L^2[-r, r]$. Hence, the larger set $\{e^{2\pi i \lambda t}\}_{\lambda \in \Lambda}$ has a lower frame bound. To see that it also has an upper frame bound, note that Λ is the union of a finite number of lattices each of which has an upper frame bound. It follows that $\{e^{2\pi i \lambda t}\}_{\lambda \in \Lambda}$ also has an upper frame bound.

(b) Suppose that $r > R$. Let $\varepsilon = r - R$. By Proposition 3.1(a), we know that the function $f_{0,m-1} = g_{m-1} \in \mathrm{PW}_{2R}(\mathbf{R})$ defined by (8) satisfies $f_{0,m-1}(\lambda) = 0$ for $\lambda \in \Lambda$, and $(d^k/dt^k) f_{0,m-1}(0) = 0$ for $k = 0, \dots, m-2$. Let $F_{0,m-1} = \hat{f}_{0,m-1}$ and define $G(t) = F_{0,m-1}(t+\varepsilon) - F_{0,m-1}(t-\varepsilon)$. Then $G \in L^2[-r, r]$ and $\widehat{G}(\gamma) = 2i f_{0,m-1}(\gamma) \sin(2\pi\varepsilon\gamma)$. Therefore $\widehat{G}$ vanishes on Λ and $(d^k/dt^k)\widehat{G}(0) = 0$ for $k = 0, \dots, m-1$. Hence G is orthogonal to every element of Λ^* but clearly $G \ne 0$. Therefore, $\{e^{2\pi i \lambda t}\}_{\lambda \in \Lambda}$ is incomplete in $L^2[-r, r]$. $\square$

References

1. Berenstein, C. and Taylor, B. A., The "three squares" theorem for continuous functions, *Arch. Rational Mech. Anal.* **63** (1977), 253–259.

2. Berenstein, C., Gay, R. and Yger, A., The three squares theorem, a local version, in *Analysis and Partial Differential Equations* (Sadosky, C. ed.), pp. 35–50, Marcel Dekker, New York, 1990.

3. Berenstein, C. and Yger, A., Analytic Bezout identities, *Adv. in Appl. Math.* **10** (1989), 51–74.

4. Beurling, A., Mittag-Leffler lectures on harmonic analysis, in *Collected Works of Arne Beurling,* vol. **2**, pp. 341–365, Birkhäuser, Boston, Mass., 1989.

5. Beurling, A. and Malliavin, P., On the closure of characters and the zeros of entire functions, *Acta. Math.* **118** (1967), 79–93.

6. Casey, S. and Walnut, D., Systems of convolution equations, deconvolution, Shannon sampling and the wavelet and Gabor transforms, *SIAM Rev.* **36** (1994), 537–577.

7. Feichtinger, H. and Strohmer, T., A Kaczmarz-based approach to nonperiodic sampling on unions of rectangular lattices, in *SampTA – Sampling Theory and Applications* (Bilinskis. I. and Marvasti, F., eds.), pp. 32–37, Institute of Electronics and Computer Science, Riga, 1995.

8. Hörmander, L., Generators for some rings of analytic functions, *Bull. Amer. Math. Soc.* **73** (1967), 943–949.

9. Jaffard, S., A density criterion for frames of complex exponentials, *Michigan Math. J.* **38** (1991), 339–348.

10. Kuipers, L. and Niederreiter, H., *Uniform Distributions of Sequences*, Wiley, New York, 1974.

11. Landau, H., Necessary density conditions for sampling and interpolation of certain entire functions, *Acta Math.* **117** (1967), 37–52.

12. Petersen, E. and Meisters, G., Non-Liouville numbers and a theorem of Hörmander, *J. Funct. Anal.* **29** (1978), 142–150.

13. Seip, K., On the connection between exponential bases and certain related sequences in $L^2(-\pi,\pi)$, *J. Funct. Anal.* **130** (1995), 131–160.

14. Seip, K., On Gröchenig, Heil, and Walnut's proof of the local three squares theorem, *Ark. Mat.* **38** (2000), 93–96.

15. Walnut, D., Sampling and the multisensor deconvolution problem, in *Fourier Analysis* (Bray, W. O., Milojević, P. S. and Stanojević, Č. V., eds.), Marcel Dekker, New York, 1994.

16. Walnut, D., Nonperiodic sampling of bandlimited functions on unions of rectangular lattices, *J. Fourier Anal. Appl.* **2** (1996), 435–452.

17. Walnut, D., Solutions to deconvolution equations using nonperiodic sampling, *J. Fourier Anal. Appl.* **4** (1998), 669–709.

18. YOUNG, R., On complete biorthogonal systems, *Proc. Amer. Math. Soc.* **83** (1981), 537–540.

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