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Metrics, Norms, Inner Products, and Operator Theory

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Preface

Mathematics is a very large subject, and many mathematicians consider some particular area of mathematics to be their "main research field." Algebra, topology, probability, and discrete mathematics are only a few examples of areas of mathematics. My own field is *analysis*[1], which (broadly speaking) is the science of functions. *Real analysis* tends to deal with functions whose domain is a set of real numbers or a set of vectors in $\mathbb{R}^d$, while *complex analysis* deals with functions of one or more complex variables. Sometimes we work with individual functions, but even more insight can often be gained by considering *sets* or *spaces* of functions that have some property in common (e.g., the space of all differentiable functions). Functions do not exist in isolation, and by looking at their properties as a family we gain new understanding. The same is true for other objects that we encounter in mathematics—we often learn more by studying spaces of objects than just focusing on the objects themselves. Equally important are how objects in one space can be transformed by some operation into objects in another space, i.e., we want to study *operators* on spaces—functions that map one space (possibly itself a space of functions) into another space. For example, the two premier operators of calculus are differentiation and integration. Integration converts objects (functions) that lie in some appropriate space of integrable functions into objects (functions) that belong to a space of differentiable functions, while differentiation does the reverse. Each of these two *operators* maps one space of functions into another space of functions.

This text is an introduction to the main types of spaces that pervade analysis. These are *metric spaces*, *normed spaces*, and *inner product spaces* (all of which are particular types of an even more general class of sets known as *topological spaces*). Essentially, a metric space is a set on which we can define a notion of "distance," or "metric," between points in the set. Normed spaces are metric spaces, but the concept of a normed space is more restrictive, in

[1] Hence I am an *analyst*, and I like to say that the main thing I do in my job is to listen to functions as they tell me their problems.

two ways. First, a normed space must be a vector space (and hence there is a way to add vectors together, and to multiply a vector by a scalar). Second, the existence of a norm not only provides us with a way to measure the distance between vectors, but also gives each individual vector in the space a "length." Inner product spaces have an even more restrictive definition, because each pair of vectors must be assigned an inner product, which in essence tells us the "angle" between these vectors. The Euclidean space $\mathbb{R}^d$ and its complex analogue $\mathbb{C}^d$ are familiar examples of inner product spaces. We will introduce and study metrics, norms, and inner products in this text, as well as operators that transform one such space into another space.

Audience

This text is aimed at students who have some basic knowledge of undergraduate real analysis, and who have previous experience reading and writing mathematical proofs. The intended reader of this text is a motivated student who is ready to take an upper-level, proof-based undergraduate mathematics course. No knowledge of measure theory or advanced real analysis is required (although a student who has taken some upper-level real analysis will of course be better prepared to handle this material). A brief review of the needed background material is presented in Chapter 1, and can be summarized as follows.

- Real and complex numbers.
- Functions, including 1-1 (or injective) functions and onto (or surjective) functions, inverse functions, direct and inverse images.
- Countable and uncountable sets.
- The definition and properties of the supremum and infimum of a set of real numbers.
- Sequences and series of real numbers, as covered in an undergraduate calculus course.
- Differentiation and integration, as covered in an undergraduate calculus course.
- The definition and basic properties of vector spaces, including independence, spans, and bases.

Aside from the above background material, this text is essentially self-contained, and can be used either as the text for an upper-level undergraduate mathematics course or as a text for independent study. The pace is fast and a considerable amount of material is covered. Most major proofs are included in detail, while certain other proofs are assigned as problems, and references are provided for any proofs that are omitted. Many exercises are included, and the reader should work as many as possible. A solutions manual for instructors is available upon request; instructions for obtaining a copy are given on the Birkhäuser website.

Outline

Chapter 1 is a short, quick-reference guide to notation, terminology, and background information that is assumed throughout the remainder of the text.

In Chapter 2 we introduce our first important class of spaces. These are *metric spaces*, which are sets on which we can define a notion of *distance* between points in the space. We show how the existence of a distance function, or *metric*, allows us to define the limit of a sequence of points, and then show that metric spaces have many properties that are similar to those that we are familiar with from $\mathbb{R}^d$. For example, we can define open and closed sets in a metric space. From this we proceed to define compact sets, and then to study the notion of *continuity* for functions on metric spaces. We end the chapter with two important theorems for metric spaces, namely, Urysohn's Lemma and the Baire Category Theorem.

In Chapters 3 and 4 we focus on a more restrictive class of spaces, the *normed spaces*. These are vector spaces on which we can define a *norm*, or *length function*. Every normed space is a metric space, but not all metric spaces are normed spaces. The more stringent requirements of a normed space mean that these spaces have a "richer structure" that allows us to prove more refined results than we could for generic metric spaces. In particular, we can form *linear combinations* of vectors (because our space is a vector space), but more importantly we can also form *infinite series* because we can both add vectors and take limits. This gives us powerful tools, even when our space is infinite-dimensional. In particular, we study *Schauder bases*, which generalize the idea of a basis for a finite-dimensional vector space to something that is appropriate for infinite-dimensional spaces. We analyze a number of different specific examples, including the sequence spaces ℓ^p and spaces of continuous or differentiable functions. Several important (but optional) results are presented at the end of Chapter 4, including the Weierstrass and Stone–Weierstrass Theorems, the Arzelà–Ascoli Theorem, and the Tietze Extension Theorem.

We impose even more restriction in Chapter 5 by requiring that our space have an *inner product*. Every inner product space is a normed space, but not every normed space is an inner product space. This imparts even more structure to our space. In particular, in an inner product space we can define what it means for two vectors to be *orthogonal*, or perpendicular. This means that we have an analogue of the Pythagorean Theorem, and this in turn leads to the fundamental notion of an *orthogonal projection*, which we can use to find closest points and solve important existence and representation problems. We prove the existence of *orthonormal bases*, which are orthogonal systems that are also Schauder bases. These yield especially simple, and robust, representations of vectors. We construct several important examples of orthonormal bases, including both real and complex versions of the *trigonometric system*, which yields *Fourier series* representations of square-integrable functions.

Issues of *convergence* underlie much of what we do in Chapters 2–5. Closely related to the definition of a *convergent sequence* is the notion of a *Cauchy sequence*. Every convergent sequence is a Cauchy sequence, but a Cauchy sequence need not be convergent. A critical question is whether our space is *complete*, i.e., whether we have the good fortune to be in a space where every Cauchy sequence actually is convergent. A normed space that is complete is called a *Banach space*, and an inner product space that is complete is called a *Hilbert space* (there is no corresponding special name for metric spaces—a metric space that is complete is simply called a *complete metric space*).

Starting with Chapter 6 we focus on *operators*, which are functions (usually linear functions) that map one space into another space. We especially focus on operators that map one normed space into another normed space. For example, if our space X is a set of differentiable functions, then we might consider the differentiation operator, which transforms a function f in X to its derivative f', which may belong to another space Y. Because our spaces may be infinite-dimensional, there are subtleties that we simply do not encounter in finite-dimensional linear algebra. We prove that continuity of linear operators on normed spaces can be equivalently reformulated in terms of a property called *boundedness* that is much easier to analyze. This leads us to consider "meta-spaces" of bounded operators. That is, not only do we have a normed space X as a domain and a normed space Y as a codomain, but we also have a new space $\mathcal{B}(X, Y)$ that contains all of the bounded linear operators that map X into Y. This itself turns out to be a normed space! We analyze the properties of this space, and consider several particularly important examples of *dual spaces* (which are "meta-spaces" $\mathcal{B}(X, Y)$ where the codomain Y is the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$).

Finally, in Chapter 7 we focus on operators that map one Hilbert space into another Hilbert space. The fact that the domain and codomain of these operators both have a rich structure allows us to prove even more detailed (and beautiful) results. We define the *adjoint* of an operator on a Hilbert space, study *compact operators* and *self-adjoint operators*, and conclude with the fundamental *Spectral Theorem* for compact, self-adjoint operators. This theorem gives us a remarkable representation of any compact self-adjoint operator and leads to the *Singular Value Decomposition* of any compact operator and any $m \times n$ matrix.

Some Special Features

We list a few goals and special features of the text.

- Extensive exercises provide opportunities for students to learn the material by working problems. The problems themselves have a range of difficulties, so students can practice on basic concepts but also have the chance to think deeply in the more challenging exercises. Roughly speaking, the problems for each section are approximately ordered in terms of difficultly, with easier problems tending to occur earlier in the problem list and more difficult problems tending to be placed toward the end.

- While many texts present only the real-valued versions of results, in advanced settings and in applications the complex-valued versions are often needed (for example, in harmonic analysis in mathematics, quantum mechanics in physics, or signal processing in engineering). This text presents theorems for both the real and complex settings, but in a manner that easily allows the instructor or reader to focus solely on the real cases if they wish.

- The text presents several important topics that usually do not receive the coverage they deserve in books at this level. These include, for example, unconditional convergence of series, Schauder bases for Banach spaces, the dual of ℓ^p, topological isomorphisms, the Spectral Theorem, the Baire Category Theorem, the Uniform Boundedness Principle, the Spectral Theorem, and the Singular Value Decomposition for operators and matrices.

- The results presented in the text do not rely on measure theory, but an optional online chapter covers related results and extensions to Lebesgue spaces that require measure theory.

Course Outlines

This text is suitable for independent study or as the basis for an upper-level course. There are several options for building a course around this text, three of which are listed below.

Course 1: Metric, Banach, and Hilbert spaces. A one-semester course focusing on the most important aspects of metric, Banach, and Hilbert spaces could present the following material:

Chapter 1: Sections 1.1–1.9
Chapter 2: Sections 2.1–2.9
Chapter 3: Sections 3.1–3.7

Chapter 4: Sections 4.1–4.5
Chapter 5: Sections 5.1–5.10

For classes with better-prepared students, Chapter 1 can be presented as a quick review, while for less-prepared classes more attention and detail can be spent on this material. This one-semester course would not cover the operator theory material covered in Chapters 6 and 7, but interested students who complete the semester would be prepared to go on to study that material on their own.

Course 2: Core material plus operator theory. A fast-paced one-semester course could cover the most important parts of the material on metric, Banach, and Hilbert spaces while still having time to cover selected material on operator theory. Here is one potential outline of such a course.

Chapter 2: Sections 2.1–2.4, 2.6–2.9
Chapter 3: Sections 3.1–3.7
Chapter 4: Sections 4.1–4.2
Chapter 5: Sections 5.1–5.10
Chapter 6: Sections 6.1–6.7
Chapter 7: Sections 7.1–7.8

Course 3: All topics. A two-semester course could cover the background material in Chapter 1, all of the foundations and details of metric, Banach, and Hilbert spaces as presented in Chapters 2–5, and the applications to operator theory that are given in Chapters 6 and 7. Such a course could cover most or all of the entire text in a two-semester sequence.

Further Reading

This text is an introduction to real analysis and operator theory. There are many possible directions for the reader who wishes to learn more, including those listed below.

- *Measure Theory.* We do not use measure theory, Lebesgue measure, or the Lebesgue integral in this text, except where a formal use of these ideas provides context or illumination (such as when we discuss Fourier series in Chapter 5). This does not affect the theory covered in the text and also keeps the text accessible to its intended audience. It does mean that we sometimes needed to be selective in choosing examples or mathematical applications. The reader who wishes to move further in analysis must learn measure theory. There are a number of texts that take different approaches to the subject. For example, the beginning student may benefit from a treatment that begins with Lebesgue measure and integral (before proceeding to abstract measure theory). Some texts that take this approach include Stein and Shakarchi [SS05], Wheeden and Zygmund [WZ77], or the forthcoming text [Heil19]. Some classic comprehensive texts that cover abstract measure theory and more include Folland [Fol99] and Rudin [Rud87]. An optional extra chapter for this text is available online that covers results related to the text that require measure theory.

- *Operator Theory and Functional Analysis.* Chapters 6 and 7 provide an introduction to operator theory and functional analysis. More detailed and extensive development of these topics is available in texts such as those by Conway [Con90], [Con00], Folland [Fol99], Gohberg and Goldberg [GG01], Kreyszig [Kre78], and Rudin [Rud91].

- *Basis and Frame Theory.* A *Schauder basis* is a sequence of vectors in a Banach space that provides representations of arbitrary vectors in terms of unique "infinite linear combinations" of the basis elements. These are studied in some detail in this text. A *frame* is a basis-like system, but one that allows nonuniqueness and redundancy, which can be important in many applications. Frames are a fascinating subject, but are only briefly mentioned in this text. Some introductory texts suitable for readers who wish further study of these topics are the text [Heil11] and the texts by Christensen [Chr16], Han, Kornelson, Larsen, and Weber [HKLW07], and Walnut [Wal02].

- *Topology.* Metric, normed, and inner product spaces are the most common types of spaces encountered in analysis, but they are all special cases of *topological spaces*. The reader who wishes to explore abstract topological spaces in more detail can turn to texts such as Munkres [Mun75] or Singer and Thorpe [ST76].

Additional Material

Additional resources related to this text are available at the author's website,

`http://people.math.gatech.edu/~heil/`

In particular, an optional Chapter 8 is posted online that includes extensions of the results of this text to settings that require measure theory, covering in particular integral operators on Lebesgue spaces and Hilbert–Schmidt operators.

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