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Metrics, Norms, Inner Products and Operator Theory

Miscellaneous Extra Material and Problems

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Miscellaneous Extra Material and Problems

We collect here miscellaneous problems and results that didn't make it into the text, but have some relevance and may be useful. Several notes and caveats are in order.

- This is a somewhat random collection of results and problems that I happen to have in typed form that have some relation to the main text. This extra material has not been organized to be in "book form," nor has it been thoughtfully developed to supplement the main text. It is simply a collection of notes and problems that happen to be related to the text.
- There may be some duplication (sometimes considerable duplication) of facts stated in the main text.
- Some of the ordering of the material may be off, e.g., Chapter 2 extra problems might contain material about normed spaces, which are not defined until Chapter 3, and so forth.
- This extra material has not been proofread as well as the material in the main text, and hence the probability of errors is higher.
- The numbering of the extra material begins with 50 in each section, so any reference to a theorem, lemma, definition, or problem whose number is 50 or greater must be from the extra material.

CHAPTER 2: METRIC SPACES

2.1 Metrics

On the higher dimensional Euclidean spaces $\mathbb{F}^d$, the metric that we usually think of first is the *Euclidean metric* (which in $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$ is simply the ordinary physical distance between points). Here is the precise definition.

Definition 2.1.50 (The Euclidean Metric). The Euclidean metric on $\mathbb{F}^d$ is defined by

$$\mathrm{d}_2(x,y) \;=\; \bigl(|x_1-y_1|^2+\cdots+|x_d-y_d|^2\bigr)^{1/2}, \tag{2.14}$$

where $x=(x_1,\dots,x_d)$ and $y=(y_1,\dots,y_d)$ are vectors in $\mathbb{F}^d$. $\diamondsuit$

Of course, so far we have only *called* the Euclidean metric a metric—we have not yet *proved* that it actually is a metric on $\mathbb{F}^d$! It is easy to see that d_2 satisfies the nonnegativity, uniqueness, and symmetry requirements, and the following exercise outlines one method for showing that d_2 also satisfies the Triangle Inequality. This exercise uses the *dot product* of vectors in $\mathbb{F}^d$, which is defined by

$$x\cdot y \;=\; x_1\overline{y_1}+\cdots+x_d\overline{y_d}, \qquad x,y\in\mathbb{F}^d. \tag{2.15}$$

Note that if $\mathbb{F}=\mathbb{R}$, then the complex conjugate that appears in equation (2.15) is superfluous. That is, for vectors in $\mathbb{R}^d$ we can write the dot product simply as

$$x\cdot y \;=\; x_1y_1+\cdots+x_dy_d, \qquad x,y\in\mathbb{R}^d. \tag{2.16}$$

The following exercise is slightly easier to state for real scalars, so we outline the argument for $\mathbb{F}=\mathbb{R}$ first, then ask for an extension to $\mathbb{F}=\mathbb{C}$.

Exercise 2.1.51. Given vectors x, y, $z\in\mathbb{R}^d$, prove the following statements.

(a) $\mathrm{d}_2(x,z)^2 = \mathrm{d}_2(x,y)^2+2\,(x-y)\cdot(y-z)+\mathrm{d}_2(y,z)^2$.

(b) $|x\cdot y| \;\le\; \bigl(|x_1|^2+\cdots+|x_d|^2\bigr)^{1/2}\bigl(|x_1|^2+\cdots+|x_d|^2\bigr)^{1/2}$.

(c) $(x-y)\cdot(y-z) \;\le\; \mathrm{d}_2(x-y)\,\mathrm{d}_2(y-z)$.

(d) $\mathrm{d}_2(x-z) \;\le\; \mathrm{d}_2(x-y)+\mathrm{d}_2(y-z)$.

Use this to prove that d_2 is a metric on $\mathbb{R}^d$. Then determine how to modify the argument to prove that d_2 is also a metric on $\mathbb{C}^d$. $\diamondsuit$

The Euclidean metric is the metric on $\mathbb{F}^d$ that we encounter most often, and therefore we sometimes refer to it as the "standard" or "default" metric for $\mathbb{F}^d$. However, it is not the only metric on $\mathbb{F}^d$. The following exercise, which is much easier to solve than Exercise 2.1.51, defines two additional metrics on $\mathbb{F}^d$.

Exercise 2.1.52. Given $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ in $\mathbb{F}^d$, set

$$\mathrm{d}_1(x, y) \;=\; |x_1 - y_1| + \cdots + |x_d - y_d|$$

and

$$\mathrm{d}_\infty(x, y) \;=\; \max\bigl\{|x_1 - y_1|, \dots, |x_d - y_d|\bigr\}.$$

Prove that d_1 and d_∞ are each metrics on $\mathbb{F}^d$. ♢

More generally, we will see in Section 3.2 that if p is any number in the range $1 \le p < \infty$, then

$$\mathrm{d}_p(x, y) \;=\; \bigl(|x_1 - y_1|^p + \cdots + |x_d - y_d|^p\bigr)^{1/p} \tag{2.17}$$

defines a metric on $\mathbb{F}^d$. The usual Euclidean metric corresponds to the choice $p = 2$, and the cases $p = 1$ and $p = \infty$ give the metrics defined in Exercise 2.1.52. If the dimension is $d = 1$ then $\mathrm{d}_1 = \mathrm{d}_2 = \mathrm{d}_\infty$, but when $d \ge 2$ these three metrics do not coincide (even so, we will see in Section 3.6 that these metrics are *equivalent* in the sense that they all generate the same topologies on $\mathbb{F}^d$).

What happens if we take $0 < p < 1$ in equation (2.17)? In this case we do not obtain a metric on $\mathbb{F}^d$ (except for the trivial case $d = 1$). To see why, consider $p = \frac{1}{2}$, which corresponds to

$$\mathrm{d}_{1/2}(x, y) \;=\; \bigl(|x_1 - y_1|^{1/2} + \cdots + |x_d - y_d|^{1/2}\bigr)^2. \tag{2.18}$$

This function $\mathrm{d}_{1/2}$ does not satisfy the Triangle Inequality, because the choice $x = (1, 0, 0, \dots, 0)$, $y = (1, 1, 0, \dots, 0)$, and $z = (0, 1, 0, \dots, 0)$ yields

$$\mathrm{d}_{1/2}(x, z) \;=\; 4 \;>\; 2 \;=\; \mathrm{d}_{1/2}(x, y) + \mathrm{d}_{1/2}(y, z).$$

Consequently, equation (2.18) does not define a metric. On the other hand, the following exercise shows how to create a related but slightly different function $\mathrm{d}_{1/2}$ that is a metric on $\mathbb{F}^d$.

Exercise 2.1.53. Given $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ in $\mathbb{F}^d$, set

$$\mathrm{d}_{1/2}(x, y) \;=\; |x_1 - y_1|^{1/2} + \cdots + |x_d - y_d|^{1/2}.$$

Prove the following statements.

(a) $(a + b)^{1/2} \le a^{1/2} + b^{1/2}$ for all $a,\, b \ge 0$.

(b) If $x,\, y,\, z \in \mathbb{F}^d$, then $\mathrm{d}_{1/2}(x, z) \le \mathrm{d}_{1/2}(x, y) + \mathrm{d}_{1/2}(y, z)$.

(c) $\mathrm{d}_{1/2}$ is a metric on $\mathbb{F}^d$. ♢

In summary, we have constructed four metrics on $\mathbb{F}^d$. These are the familiar Euclidean metric, which we have denoted by d_2, and the less-familiar metrics d_1, d_∞, and $\mathrm{d}_{1/2}$. We will see some additional metrics on $\mathbb{F}^d$ in Section

3.2. Problem 2.1.13 shows that is possible to define a metric on *any* set X. However, in practice we usually want to do more than just define an arbitrary metric on X. Typically, our set X has some kind of interesting properties, and we seek a metric that somehow takes those properties into account. We will see many examples of such metrics later. In particular, we will define metrics on some spaces whose elements are infinite sequences in Section 3.2, and metrics on some spaces whose elements are functions in Section 3.5.

Extra Problems

2.1.54. (a) Can you define a space of sequences ℓ^2 and a corresponding metric d_2 that is analogous to the Euclidean metric on $\mathbb{F}^d$?

(b) Can you define a metric on $C[0,1]$ that is analogous in some way to the Euclidean metric on $\mathbb{F}^d$?

2.1.55. Determine which of the following is a metric on $\mathbb{R}$.

(a) $\mathrm{d}(x,y) = |x-2y|$.

(b) $\mathrm{d}(x,y) = \dfrac{|x-y|}{1+|x-y|}$.

2.2 Convergence and Completeness

Definition 2.2.50 (Complete Sets and Complete Metric Spaces). Let X be a metric space.

(a) If S is a subset of X and every Cauchy sequence in X converges to an element of S, then we say that S is a *complete subset* of X.

(b) If X itself has the property that every Cauchy sequence in X converges to an element of X, then we say that X is a *complete metric space*, or simply that X is *complete*. ◇

Extra Problems

2.2.51. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of vectors in $\mathbb{F}^d$. For each n, write

$$x_n \;=\; \big(x_n(1),\dots,x_n(d)\big).$$

That is, $x_n(k)$ denotes the kth component of x_n. Let y be a vector in $\mathbb{F}^d$, and write $y = \big(y(1),\dots,y(d)\big)$. Prove that

$$\lim_{n\to\infty} \mathrm{d}_2(x_n, y) = 0 \quad \Longleftrightarrow \quad \lim_{n\to\infty} x_n(k) = y(k) \text{ for each } k = 1, \dots, d.$$

Also prove that the same result holds if we replace the Euclidean metric d_2 by d_1, d_∞, or $\mathrm{d}_{1/2}$.

2.2.52. Prove that $\mathbb{F}^d$ is complete with respect to each of the metrics d_1, d_2, d_∞, and $\mathrm{d}_{1/2}$.

2.2.53. Do you think that the space $C[0,1]$ is complete with respect to the uniform metric d_u or the L^1-metric d_1? Try to give a variety of different examples of Cauchy sequences and determine whether they converge in $C[0,1]$. We will consider the completeness of $C[0,1]$ in more detail later, but try to prove now that it is complete or incomplete with respect to each of these two metrics.

2.3 Topology in Metric Spaces

Neighborhoods

A topology consists of open sets, but sometimes it is useful to consider sets that are not open but which *contain* an open set. For example, the interval $[0,1)$ is not open, but it contains the open interval $(0,1)$. We give the following special name to a set that contains an open set that contains a point x.

Definition 2.3.50 (Neighborhood). Let X be a metric space.

(a) A *neighborhood* of a point $x \in X$ is any set A such that there exists an open set U that satisfies
$$x \in U \subseteq A.$$

(b) An *open neighborhood* of x is a neighborhood U of x that is an open set. Equivalently, an open neighborhood of x is simply an open set that contains x. ♢

A neighborhood need not be an open set. For example, if we consider $X = \mathbb{R}$, the interval $[0,3)$ is a neighborhood of the point $x = 1$, but it is not an open set. On the other hand, $(0,3)$ is an open set that is a neighborhood of $x = 1$, so $(0,3)$ is an open neighborhood of $x = 1$. In an arbitrary metric space, the open ball $B_r(x)$ is an open neighborhood of x, but not every open neighborhood need be an open ball.

Extra Problems

2.3.51. Let X be a metric space and fix $x \in X$. Prove that if y and z are any two points in $B_r(x)$, then $\mathrm{d}(x,y) < 2r$.

2.3.52. Let X be a metric space, and let $x \in X$ and $E \subseteq X$ be fixed. Prove the following statements.

(a) Every open ball $B_r(x)$ is an open subset of X.

(b) $A \subseteq X$ is a neighborhood of x if and only if there exists some $r > 0$ such that $B_r(x) \subseteq A$.

(c) E is open if and only if $E = E^\circ$.

(d) E° is the union of all open balls that are contained in E.

2.3.53. Let X be a metric space, and let x_n, $x \in X$ be given. Prove that the following three statements are equivalent.

(a) $x_n \to x$.

(b) For every open neighborhood U of x, there is an $N > 0$ such that $x_n \in U$ for all $n > N$.

(c) For every neighborhood A of x, there is an $N > 0$ such that $x_n \in A$ for all $n > N$.

2.4 Closed Sets

Extra Problems

2.4.50. Exhibit a metric space X that contains a subset E such that $E \neq \varnothing$, $E \neq X$, and E is both open and closed.

2.4.51. Consider $C[0,1]$ with respect to the uniform metric. Prove that

$$E \;=\; \bigl\{f \in C[0,1] : |f(x)| < 1 \text{ for all } x \in [0,1]\bigr\}$$

is open but not closed, and therefore its complement is closed but not open. Given an explicit direct description of E^{C}, i.e., $f \in E^{\mathrm{C}}$ if and only if what?

2.4.52. Let X be a metric set. Given A, $B \subseteq X$, define

$$\mathrm{D}(A,B) \;=\; \sup\bigl\{\mathrm{dist}(x,B) : x \in A\bigr\} \;+\; \sup\bigl\{\mathrm{dist}(y,A) : y \in B\bigr\}.$$

Let $\mathcal{B}$ be the set of all closed, bounded, nonempty subsets of X. Determine, with proof or a counterexample, which of the metric space requirements that D satisfies on the set $\mathcal{B}$. Why do we restrict $\mathcal{B}$ to closed, bounded, and nonempty sets?

2.5 Accumulation Points and Boundary Points

Accumulation Points

It is often helpful to reformulate a definition in equivalent ways, or to see its implications. For example, suppose that x is an accumulation point of a set E, and let us see what this tells us about the neighborhoods of x. If A is a neighborhood of x, then, by the definition of a neighborhood, there is an open set U such that $x \in U \subseteq A$. Consequently, by the definition of an open set, there is some $r > 0$ such that $B_r(x) \subseteq U$. Now, since x is an accumulation point, we know that there exist points $x_n \in E$, with every x_n different from x, such that $x_n \to x$. By definition of convergence, there exists some integer N such that

$$n > N \quad \Longrightarrow \quad \mathrm{d}(x_n, x) \; < \; r.$$

This tells us that for every $n > N$ we have

$$x_n \;\in\; B_r(x) \;\subseteq\; U \;\subseteq\; A.$$

Thus the neighborhood A contains at least one point (and maybe more, if the x_n are not all equal) that belongs to E but is not equal to x. Another way to say this is that if x is an accumulation point of E, then

$$A \text{ is a neighborhood of } x \quad \Longrightarrow \quad (E \cap A) \setminus \{x\} \neq \varnothing. \tag{2.19}$$

According to the next exercise, it is also possible to argue in the converse direction. That is, we can show that if equation (2.19) holds, then x must be an accumulation point of X. Thus, equation (2.19) gives us an equivalent way to view accumulation points. Several other equivalent reformulations are also given in the following exercise (which is exactly the same as Problem 2.5.8 except that it includes wording about neighborhoods instead of just open sets).

Exercise 2.5.50. Let E be a subset of a metric space X. Given $x \in X$, prove that the following five statements are equivalent.

(a) x is an accumulation point of E.

(b) If A is any neighborhood of x, then $(E \cap A) \setminus \{x\} \neq \varnothing$.

(c) If U is an open set that contains x, then there exists a point $y \in E \cap U$ such that $y \neq x$.

(d) If $r > 0$, then exists a point $y \in E$ such that $0 < \mathrm{d}(x, y) < r$.

(e) Every neighborhood A of x contains *infinitely many* points of E. $\diamondsuit$

We give the set of accumulation points of E the following name.

Definition 2.5.51 (Derived Set). Let E be a subset of a metric space X. The *derived set* of E is the set E' that consists of all of the accumulation points of E, i.e.,

$$E' = \{x \in X : x \text{ is an accumulation point of } E\}. \qquad \diamondsuit$$

Using this terminology, Theorem 2.4.2 tells us that

$$E \text{ is closed} \quad \Longleftrightarrow \quad E' \subseteq E.$$

Here are some examples of derived sets for subsets of $X = \mathbb{R}$.

Example 2.5.52. (a) The set

$$A = [0,1] \cup \{2\}$$

is a closed subset of $\mathbb{R}$. We observed in Section 2.5 that x is an accumulation point of A if and only if $x \in [0,1]$. Hence the derived set of A is $A' = [0,1]$.

(b) The set

$$S = \bigl\{\tfrac{1}{n}\bigr\}_{n\in\mathbb{N}} = \bigl\{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots\bigr\}.$$

is not a closed subset of the real line. In particular, 0 is an accumulation point of S even though 0 does not belong to S. Since 0 is the *only* accumulation point of S, its derived set is $S' = \{0\}$.

(c) The set

$$T = S \cup \{0\} = \bigl\{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots\bigr\}$$

contains all of its accumulation points, and therefore is closed. However, 0 is the only accumulation point of T, so its derived set is $T' = \{0\}$.

(d) It is possible for a nonempty closed set to have no accumulation points. For example, $E = \{0\}$ is a closed subset of $\mathbb{R}$ because its complement $\mathbb{R} \setminus \{0\}$ is open. However, there is no way to choose points $x_n \in E = \{0\}$ with $x_n \neq 0$ such that $x_n \to 0$, so 0 is not an accumulation point of E. In fact, $E' = \varnothing$. Likewise, even though the set of integers is an infinite set, its derived set is $\mathbb{Z}' = \varnothing$.

(e) The interval $I = [0,1]$ is a closed subset of $\mathbb{R}$, and *every* point $x \in [0,1]$ is an accumulation point of I. Hence I is a closed set that has the property that all of its elements are accumulation points of I. Moreover, no point outside of I is an accumulation point of I (why?), so $I' = I$. $\diamondsuit$

We saw in Theorem 2.5.3 that a closed set must contain all of its accumulation points. However, not every point in a closed set need be an accumulation point of the set. We give the following special name to closed sets that have the property that every point in the set is an accumulation point.

Definition 2.5.53 (Perfect Set). Let E be a subset of a metric space X. We say that E is *perfect* if E is nonempty and $E = E'$. $\diamondsuit$

The interval $[0,1]$ is a perfect set. However, perfect sets can be quite unusual. For example, we will see in Problem 2.7.3 that the Cantor set C is perfect and *has no interior*. The Cantor set is a closed, uncountable subset of $[0,1]$ that contains no intervals, yet every point $x \in C$ is an accumulation point of C!

According to Problem 2.8.59, every perfect subset of $\mathbb{R}^d$ is uncountable. Although we will not prove it, the *Cantor–Bendixson Theorem* implies that every nonempty closed set $F \subseteq \mathbb{R}^d$ can be uniquely written as $F = E \cup Z$ where E is perfect and Z is countable.

Boundary Points

Our next goal is to characterize closed sets in terms of their boundary points. Here is the definition of an interior point, exterior point, and boundary point of a set E.

Definition 2.5.54 (Interior, Exterior, and Boundary Points). Let E be a subset of a metric space X, and let x be any point in X.

(a) We say that x is an *interior point* of E if there exists a neighborhood A of x such that $A \subseteq E$.

(b) We say that x is an *exterior point* of E if there exists a neighborhood A of x such that $A \subseteq E^{\mathrm{C}}$.

(c) We say that x is a *boundary point* of E if for every neighborhood A of x we have both $A \cap E \neq \varnothing$ and $A \cap E^{\mathrm{C}} \neq \varnothing$. The set of all boundary points of E is called the *boundary* of E, and it is denoted by

$$\partial E \;=\; \bigl\{x \in X : x \text{ is a boundary point of } E\bigr\}. \qquad \diamondsuit$$

Boundary points and accumulation points are similar in some respects, but they are not identical. For example, 0 is a boundary point of the singleton $\{0\}$, but it is not an accumulation point. This shows that boundary points need not be accumulation points. On the other hand, only 0 and 1 are boundary points of the interval $[0,1]$, yet every point in $[0,1]$ is an accumulation point. Therefore accumulation points need not be a boundary points.

According to Theorem 2.5.5, a subset of a metric space is closed if and only if it contains all of its boundary points, i.e., $\partial E \subseteq E$.

Can we ever have $\partial E = E$? If L is the line segment

$$L \;=\; \{(x,x) : 0 \le x \le 1\}$$

in $\mathbb{R}^2$ then $\partial L = L$. However, if $L = [a,b]$ is a line segment in $\mathbb{R}$ then ∂L only contains the two endpoints a and b:

$$\partial L \;=\; \{a,b\} \;\neq\; [a,b] \;=\; L.$$

It is hard to imagine an infinite subset of the real line that equals its own boundary! Yet Problem 2.7.3 shows that the Cantor set is an example of an *uncountable* subset of the real line that has this unusual property.

Thus C is a rather strange set. It is uncountable, yet its interior is empty. It equals its own boundary, and every point in C is both an accumulation point of C and an accumulation point of its complement $\mathbb{R} \setminus C$. Following Definition 2.5.53, since every point in C is an accumulation point of C, we say that C is a *perfect set*. It is also *totally disconnected*, which means that it contains no nontrivial connected subsets (in one dimension, a nontrivial connected set is simply an interval).

Closed Sets versus Complete Sets

Suppose that E is a subset of a metric space X. Theorem 2.4.2 showed that E is *closed* if and only if E contains every limit of points from E. On the other hand, Definition 2.2.50 states that E is a *complete set* if and only if every Cauchy sequence in E converges to an element of E. These concepts are quite similar, but the following example shows that they are not identical.

Example 2.5.55. Let $X = \mathbb{Q}$, where the metric on X is the restriction of the absolute value metric to $\mathbb{Q}$. That is,

$$\mathrm{d}(x,y) \;=\; |x-y|, \qquad x,y \in \mathbb{Q}.$$

Let E be the set of all rational numbers that are less than π, i.e.,

$$E \;=\; (-\infty,\pi) \cap \mathbb{Q} \;=\; \{r \in \mathbb{Q} : r < \pi\}.$$

Let $x_1 = 3.1$, $x_2 = 3.14$, $x_3 = 3.141$, $x_4 = 3.1415$, and so forth. Then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$ (why?), but it does not converge to an element of $\mathbb{Q}$. Therefore E is not a complete subset of $\mathbb{Q}$.

On the other hand, we will show that E is a closed subset of $\mathbb{Q}$. Suppose that $\{y_n\}_{n\in\mathbb{N}}$ is any sequence of points from E that converges to some point $y \in \mathbb{Q}$. By definition of E, we have $y_n < \pi$ for every n. Since y_n converges to y, it follows that

$$y \;=\; \lim_{n\to\infty} y_n \;\le\; \pi.$$

But y is rational by hypothesis, so it follows that $y < \pi$ and therefore $y \in E$. We have shown that any limit of points from E belongs to E, so Theorem 2.4.2 tells us that E is a closed subset of $\mathbb{Q}$. $\diamondsuit$

Thus, a closed set need not be a complete set in general. Problem 2.4.8 shows that *if X itself is complete then the closed subsets of X coincide with the complete subsets of X.*

Extra Problems

2.5.56. For this problem, use any of the metrics d_1, d_2, d_∞, or $\mathrm{d}_{1/2}$ on $\mathbb{R}^2$. The distance $\mathrm{dist}(x, A)$ from a point x to a set A is defined in Problem 2.4.7.

(a) Give an example of a point $x \in \mathbb{R}^2$ and a set $A \subseteq \mathbb{R}^2$ such that $x \notin A$ but $\mathrm{dist}(x, A) = 0$.

(b) Exhibit disjoint sets A, $B \subseteq \mathbb{R}^2$ such that $\mathrm{dist}(A, B) = 0$.

(c) Give an example of a set $A \subseteq \mathbb{R}^2$ such that $\mathrm{d}(x, y) < \mathrm{diam}(A)$ for every choice of x, $y \in A$.

2.5.57. Modify the Cantor middle-thirds set construction as follows. Fix a parameter $0 < \theta < 1$, and at stage n remove intervals of relative length θ from F_n to form F_{n+1}. Show that the generalized Cantor set $C_\theta = \cap F_n$ is perfect, has no interior, and equals its own boundary.

2.5.58. Let X be a metric space, and suppose that x is an accumulation point of a set $E \subseteq X$. Show that x is an accumulation point of every set F that contains E.

2.5.59. Given a subset E of a metric space X, prove the following statements.

(a) E° is the set of all interior points of E, and $(E^{\mathrm{C}})^\circ$ is the set of all exterior points of E.

(b) E°, $(E^{\mathrm{C}})^\circ$, and ∂E are disjoint, and their union is X. Consequently, every point $x \in X$ is precisely one of an interior point, an exterior point, or a boundary point of E.

(c) If $x \in \partial E$ but $x \notin E$, then x is an accumulation point of E.

(d) ∂E is a closed set.

2.5.60. Let A, B be subsets of a metric space X.

(a) Prove that $\partial(A \cup B) \subseteq \partial A \cup \partial B$.

(b) Show by example that $\partial(A \cup B)$ need not equal $\partial A \cup \partial B$.

2.5.61. This problem continues Problem 2.3.18. Let Y be a subset of a metric space (X, d). We call the restriction of d to Y ithe *inherited metric* on Y.

(a) Given a set $F \subseteq Y$, prove that F is a closed subset of Y (with respect to the inherited metric) if and only if there exists a closed set $E \subseteq X$ such that $F = E \cap Y$.

(b) Let $X = \mathbb{R}$ with the usual metric, and set $S = \{r \in \mathbb{Q} : 0 \leq r \leq \pi\}$. Prove that S is a closed set in $\mathbb{Q}$, but it is not a complete set.

2.6 Closure, Density, and Separability

Density

Observe that Theorem 2.6.5 implies that the Banach space ℓ^1 contains *proper, dense subspaces*. This is quite different from our experience with the Euclidean space $\mathbb{R}^d$, whose proper subspaces are closed sets and therefore are not dense in $\mathbb{R}^d$. Indeed, Problem 3.7.5 shows that no proper subspace of any finite-dimensional normed space can be dense.

Separability

In $\mathbb{R}^d$, we can find up to $d+1$ vectors that are each a distance δ from each other (for example, consider the four vertices of a regular tetrahedron in $\mathbb{R}^3$). The following lemma implies that in a separable space there can be at most countably many vectors that are each a distance δ from each other.

Lemma 2.6.50. *Let X be a metric space. If there exists an uncountable set $S \subseteq X$ and a constant $\delta > 0$ such that $\mathrm{d}(x,y) \ge \delta$ for every $x \ne y \in S$, then X is not separable.*

Proof. We will prove the contrapositive statement. Suppose that X is separable, and suppose that S is a subset of X that satisfies $\mathrm{d}(x,y) \ge \delta$ for every $x \ne y \in S$, where δ is a fixed positive constant. We must prove that S is countable.

Since X is separable, there exists a countable set A that is dense in X. Applying the equivalent formulation of density given in Corollary 2.6.4, for each $x \in S$ there must exist a point $a_x \in A$ such that

$$\mathrm{d}(x, a_x) \;<\; \frac{\delta}{2}.$$

If x, y are two distinct elements of S, then

$$\delta \;\le\; \mathrm{d}(x,y) \;\le\; \mathrm{d}(x,a_x) \;+\; \mathrm{d}(a_x,a_y) \;+\; \mathrm{d}(a_y,y) \;<\; \frac{\delta}{2} \;+\; \mathrm{d}(a_x,a_y) \;+\; \frac{\delta}{2}.$$

Consequently $\mathrm{d}(a_x,a_y) > 0$, which implies that $a_x \ne a_y$. This shows that each point $x \in S$ determines a unique point $a_x \in A$. In other words, the function $f(x) = a_x$ is an injective mapping of S into the countable set A. Consequently S must be countable. □

Since ℓ^p is separable when p is finite, a consequence of Lemma 2.6.50 is that if p is finite, then there can exist at most countably many vectors in ℓ^p each a distance δ apart. In contrast, Theorem 2.6.8 shows that ℓ^∞ is nonseparable and we can find *uncountably* many vectors in ℓ^∞ that are each a distance 1 from each other!

Problem 2.6.20 shows that ℓ^1 and c_0 are separable, and Theorem 2.6.8 shows that ℓ^∞ is not separable. his may seem to be related to the fact that

$$\ell^1 \subseteq c_0 \subseteq \ell^\infty,$$

but this is misleading, because these spaces do not all share the same norms. Indeed, ℓ^1 is separable with respect to the ℓ^1-norm, c_0 is separable with respect to the ℓ^∞-norm, and ℓ^∞ is not separable with respect to the ℓ^∞-norm. Problem 2.6.22 does show that if Y is a subspace of a separable space X and we use the norm on Y inherited from X, then Y must be separable. However, if Y is given a different norm, then it is entirely possible that Y could be nonseparable even though X is separable. Although we will not prove it here, an example from measure theory is given by the *Lebesgue spaces* $L^1[0,1]$ and $L^\infty[0,1]$. The space $L^\infty[0,1]$ is nonseparable, even though it is a proper subset of the separable space $L^1[0,1]$ (for precise definitions and proof, see [Heil18]). Again, the norms on these two spaces are different. If we replace the standard norm on $L^\infty[0,1]$ with the norm of $L^1[0,1]$, then $L^\infty[0,1]$ is a separable space.

Extra Problems

2.6.51. Let E be a subset of a metric space X. Must E and $\overline{E}$ always have the same interior?

2.6.52. Let

$$A = \bigl\{x = (x_k)_{k\in\mathbb{N}} \in \ell^1 : x_1 = 0\bigr\}.$$

(a) Prove that A is not a dense subset of ℓ^1 by showing that there exists some point $z \in \ell^1$ such that there do not exist any points $x_n \in A$ that converge to z in ℓ^1-norm.

(b) Prove that A is a proper, closed subspace of ℓ^1, and use this to give another proof that A is not dense in ℓ^1.

2.6.53. Let S be the set defined in equation (2.20).

(a) Is S a dense subset of ℓ^∞ (with respect to the ℓ^∞-metric)? Is this by itself enough to prove that ℓ^∞ is not separable?

(b) Let c_0 be the subspace of ℓ^∞ defined in equation (2.21). Prove that when we use the ℓ^∞-norm, S is a countable dense subset of c_0 and therefore c_0 is separable with respect to the sup-norm.

2.7 The Cantor Set

Extra Problems

2.7.50. The sum of two sets A, $B \subseteq \mathbb{R}$ is $A + B = \{x + y : x \in A, y \in B\}$.

(a) Prove that $[a, b] + [c, d] = [a + c, b + d]$.

(b) Let C be the Cantor set, and let

$$D = \biggl\{\sum_{n=1}^{\infty} \frac{c_n}{3^n} : c_n = 0, 1\biggr\}.$$

Show that $D + D = [0, 1]$, and use this to show that $C + C = [0, 2]$.

2.8 Compact Sets in Metric Spaces

We know that a closed set contains all of its accumulation points. In fact, Theorem 2.4.2 tells us that if $\{x_n\}_{n\in\mathbb{N}}$ is *any* convergent sequence that is contained in a closed set F, then the limit of this sequence must belong to F. Even so, it is not true that *every* infinite sequence $\{x_n\}_{n\in\mathbb{N}}$ in F must converge, or even have an accumulation point or a convergent subsequence. For example, the set of integers $\mathbb{Z}$ is a closed subset of the real line $\mathbb{R}$, but if we set $x_n = n$ then $\{x_n\}_{n\in\mathbb{N}}$ does not converge, and no subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converges. One of the equivalences proved in Theorem 2.8.9 shows that any infinite sequence in a *compact* set must contain a convergent subsequence.

We will use Theorem 2.8.9 to give an example of a closed and bounded set that is compact.

Lemma 2.8.50. *The interval* $[0, 1]$ *is a compact subset of* $\mathbb{R}$.

Proof. Implicitly, we are assuming that the metric on $\mathbb{R}$ is its standard metric (the absolute value metric).

The interval $[0, 1]$ is a closed subset of $\mathbb{R}$. Since $\mathbb{R}$ is complete, Problem 2.4.8 implies that $[0, 1]$ is complete.

Fix any $r > 0$, and let n be an integer larger than $\frac{1}{r}$. For each $k = 0, \dots, n$, let

$$B_k = \bigl(\tfrac{k}{n} - \tfrac{r}{2},\, \tfrac{k}{n} + \tfrac{r}{2}\bigr).$$

Then B_k is an open interval of radius r. Since $r > \frac{1}{n}$, it follows that

$$[0, 1] \subseteq \bigcup_{k=0}^{n} U_k.$$

Thus $[0,1]$ is covered by finitely many open balls of radius r. Since we can do this for every $r > 0$, we conclude that $[0,1]$ is totally bounded.

Thus $[0,1]$ is both complete and totally bounded, so Theorem 2.8.9 implies that $[0,1]$ is a compact subset of $\mathbb{R}$. $\square$

The Heine–Borel Theorem for $\mathbb{C}^d$

The proof of Theorem 2.8.4 shows that a subset of $\mathbb{R}^d$ is compact if and only if it is closed and bounded. We could use essentially the same proof to show that a subset of $\mathbb{C}^d$ is compact if and only if it is closed and bounded. Instead, we will indicate a different (and possibly more difficult) approach based on the relationship between the topologies of $\mathbb{R}^{2d}$ and $\mathbb{C}^d$. Our intuition suggests that $\mathbb{C}$ "looks just like" $\mathbb{R}^2$, and indeed the following exercise will show that there is a natural identification between the open subsets of $\mathbb{C}^d$ and the open subsets of $\mathbb{R}^{2d}$. We assume in this exercise that the norms on $\mathbb{C}^d$ and $\mathbb{R}^{2d}$ are the Euclidean norms.

Exercise 2.8.51. Let U be an open subset of $\mathbb{R}^2$, and let V be an open subset of $\mathbb{C}$. Show that

$$\widetilde{U} \;=\; \bigl\{a+ib \in \mathbb{C} : (a,b) \in U\bigr\} \text{ is an open subset of } \mathbb{C},$$

and

$$\widetilde{V} \;=\; \bigl\{(a,b) \in \mathbb{R}^2 : a+ib \in V\bigr\} \text{ is an open subset of } \mathbb{R}^2.$$

Formulate and prove a corresponding result for $\mathbb{C}^d$ and $\mathbb{R}^{2d}$. $\diamondsuit$

Using the language that we will introduce in Definition 3.52.51, Exercise 2.8.51 says that $\mathbb{C}^d$ and $\mathbb{R}^{2d}$ are *homeomorphic* spaces—in essence, they have the "same" topologies. As a consequence, the Heine–Borel Theorem also holds for As a consequence, the Heine–Borel Theorem also holds for subsets of $\mathbb{C}^d$. We assign the details of the argument as the following exercise.

Exercise 2.8.52. Let K be a compact subset of $\mathbb{C}^d$. Prove that

$$\widetilde{K} \;=\; \bigl\{(x_1,y_1,\dots,x_d,y_d) \,:\, (x_1+iy_1,\dots,x_d+iy_d) \in K\bigr\}$$

is a compact subset of $\mathbb{R}^{2d}$ and is therefore closed and bounded in $\mathbb{R}^{2d}$. Use this to prove that K is a closed and bounded set in $\mathbb{C}^d$. $\diamondsuit$

Extra Problems

2.8.53. Prove directly that $\mathbb{R}^d$ is not compact.

2.8.54. Assume that $\{[a_n, b_n]\}_{n\in\mathbb{N}}$ is a nested decreasing sequence of closed finite intervals. Prove that

$$\bigcap_{n=1}^{\infty} [a_n, b_n] \;=\; [a, b]$$

where $a = \sup_n a_n$ and $b = \inf_n b_n$.

2.8.55. For each of the following statements, exhibit a metric space X that has that property.

(a) X is compact.

(b) X is not compact.

(c) X is infinite, but no infinite subset of X is compact.

(d) X is infinite but not compact, yet there exists an infinite subset of X that is compact.

2.8.56. Let X be a metric space.

(a) Prove that an arbitrary intersection of compact subsets of X is compact, and a finite union of compact subsets of X is compact.

(b) Is it possible for an intersection of finitely many open subsets of X to be a nonempty compact set?

(c) Is it possible for an intersection of infinitely many open subsets of X to be a nonempty compact set? What if X is a normed space?

2.8.57. Let X be a metric space. Show that if x_n, $x \in X$ and $x_n \to x$, then $K = \{x_n\}_{n\in\mathbb{N}} \cup \{x\}$ is compact.

2.8.58. Let K be a compact subset of a metric space X. Show that every infinite set $S \subseteq K$ has an accumulation point, and that accumulation point belongs to K.

2.8.59. A set $S \subseteq \mathbb{R}$ is *totally disconnected* if it contains no intervals. It is *perfect* if every point $x \in S$ is an accumulation point of S.

(a) Show that the interval $I = [0, 1]$ is perfect but not totally disconnected.

(b) Show that the Cantor set C is both perfect and totally disconnected.

(c) This part will show that any perfect subset of $\mathbb{R}$ must be uncountable. Suppose that $S = \{x_1, x_2, \dots\}$ is perfect. Let $n_1 = 1$ and $r_1 = 1$, and let $U_1 = B_{r_1}(x_{n_1})$. Let n_2 be the first integer greater than n_1 such that $x_{n_2} \in U_1$, and let $U_2 = B_{r_2}(x_{n_2})$ be such that $U_2 \subseteq \overline{U_2} \subseteq U_1$ but $x_{n_1} \notin U_2$. Continue in this way, and then define $K = \bigcap (\overline{U_n} \cap S)$. Show that the sets $\overline{U_n} \cap S$ are compact and nested decreasing. The *Cantor Intersection Theorem* therefore implies that K is nonempty. Show that no element of S can belong to K.

2.8.60. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle in the complex plane. Fix $z \in S^1$, and define $T \colon S^1 \to S^1$ by $T(x) = zx$. Given $x \in S^1$, the set $\mathcal{O}(x) = \{z^n x\}_{n\geq 0}$ is called the *forward orbit* of x under T, and the *cluster set* of x under T is

$$\mathcal{A}(x) \;=\; \bigcap_{k\geq 0} \overline{\mathcal{O}(z^k x)} \;=\; \bigcap_{k\geq 0} \overline{\{z^n x\}_{n\geq k}}.$$

Prove the following statements.

(a) $\overline{\mathcal{O}(x)} = \mathcal{O}(x) \cup \mathcal{A}(x)$.

(b) T maps $\mathcal{A}(x)$ into itself. That is, if $y \in \mathcal{A}(x)$, then $T(y) \in \mathcal{A}(x)$.

For the remainder of this problem, assume that $\mathcal{O}(x)$ is compact. Then from part (a) we obtain $\mathcal{A}(x) \subseteq \mathcal{O}(x)$, so there is a smallest nonnegative integer n_0 such that $z^{n_0} x \in \mathcal{A}(x)$. Prove the following statements.

(c) $\mathcal{O}(z^{n_0} x) = \mathcal{A}(x) = \mathcal{A}(z^{n_0} x)$.

(d) If $\mathcal{A}(x)$ is infinite then it is perfect.

(e) $\mathcal{O}(x)$ is finite.

2.9 Continuity for Functions on Metric Spaces

Continuity at a Point

Definition 2.9.50 (Continuity at a Point). Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f \colon X \to Y$ be a function that maps X into Y. We say that f is *continuous at a point* $x \in X$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\forall\, y \in X, \qquad \mathrm{d}_X(x,y) < \delta \implies \mathrm{d}_Y\big(f(x),\, f(y)\big) < \varepsilon. \qquad \diamondsuit \tag{2.20}$$

We give some equivalent formulations of continuity at a point in terms of how f acts on sequences, and in terms of inverse images of open sets. Recall that the *inverse image* of a set $V \subseteq Y$ is

$$f^{-1}(V) \;=\; \big\{x \in X : f(x) \in V\big\}.$$

Theorem 2.9.51. *Let X, Y be metric spaces. Given a function $f \colon X \to Y$ and a point $x \in X$, the following three statements are equivalent.*

(a) *f is continuous at the point x.*

(b) *For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X,*

$$x_n \to x \text{ in } X \qquad \Longrightarrow \qquad f(x_n) \to f(x) \text{ in } Y. \tag{2.21}$$

(c) *For each open neighborhood V of $f(x)$ in Y, there exists an open neighborhood U of x in X such that $U \subseteq f^{-1}(V)$.* $\diamondsuit$

Proof. (a) $\Rightarrow$ (b). Assume that f is continuous at x, and suppose that $x_n \to x$. Given $\varepsilon > 0$, there is a $\delta > 0$ such that equation (2.20) holds. Also, by definition of convergent sequences, since $x_n \to x$ there is an $N > 0$ such that $\mathrm{d}_X(x_n, x) < \delta$ for all $n > N$. Therefore $\mathrm{d}_Y(f(x_n), f(x)) < \varepsilon$ for all $n > N$, so we have $f(x_n) \to f(x)$ in Y.

(b) $\Rightarrow$ (c). Suppose that statement (c) fails, i.e., there exists some open neighborhood V of $f(x)$ such that no open neighborhood U of x is contained in $f^{-1}(V)$. Since $U = B_{1/n}(x)$ is an open neighborhood of x, there must be a point $x_n \in B_{1/n}(x)$ that is not in $f^{-1}(V)$. Hence $x_n \to x$, but $f(x_n) \notin V$ for any n so $f(x_n)$ cannot converge to $f(x)$. Therefore statement (b) fails.

(c) $\Rightarrow$ (a). Suppose that statement (c) holds, and fix any $\varepsilon > 0$. Then $V = B_\varepsilon(f(x))$ is an open neighborhood of $f(x)$ in Y, so statement (c) implies that there exists an open neighborhood U of x in X such that $U \subseteq f^{-1}(V)$. Since $x \in U$ and U is open, there is a $\delta > 0$ such that $B_\delta(x) \subseteq U$. If y is any point in X that satisfies $\mathrm{d}_X(x, y) < \delta$, then we have

$$y \in B_\delta(x) \subseteq U \subseteq f^{-1}(V).$$

Consequently $f(y) \in V = B_\varepsilon(f(x)$, so $\mathrm{d}_Y(f(x), f(y)) < \varepsilon$. Therefore f is continuous at the point x. $\square$

Here is an analogous exercise that gives equivalent conditions for continuity of a function on all of X.

Exercise 2.9.52. Let X, Y be metric spaces. Given a function $f \colon X \to Y$, prove that the following three statements are equivalent.

(a) f is continuous, i.e., if V is any open subset of Y, then its inverse image $f^{-1}(V)$ is an open subset of X.

(b) f is continuous at each point $x \in X$.

(c) Given any point $x \in X$ and any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X,

$$x_n \to x \text{ in } X \quad \Longrightarrow \quad f(x_n) \to f(x) \text{ in } Y. \qquad \diamondsuit$$

The equivalent condition given in statement (b) of Exercise 2.9.52 tells us that a continuous function *preserves convergent sequences.* This is often the most convenient way to think of continuity of functions on metric spaces, but the definition, given in statement (a) is just as important, and even more fundamental in many respects. That statement tells us that a continuous function *respects topological structure* in the sense that the *inverse image of every open set is open.*

Although the *inverse image* of an open set under a continuous function is open, the following example shows that it is not true that the *direct image* of an open set under a continuous function must be open.

Example 2.9.53. We know from undergraduate calculus class that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is continuous. If V is any open subset of the real line, then its inverse image $f^{-1}(V)$ is open. For example, the inverse image of the interval $V = (0,2)$ is

$$f^{-1}(V) \;=\; \cdots \cup (-2\pi, \pi) \,\cup\, (0,\pi) \,\cup\, (\pi, 2\pi) \,\cup\, \cdots,$$

which is open. However, the *direct image* of the open interval $U = (0, 2\pi)$ is

$$f(U) \;=\; [-1, 1],$$

which is not open. ♢

Examples

These examples use some function spaces that are introduced in Chapter 3.

We will illustrate continuity by considering a function that maps the normed space $C_b^1(\mathbb{R})$ into the space $C_b(\mathbb{R})$. The space $C_b(\mathbb{R})$ consists of all bounded, continuous functions, while $C_b^1(\mathbb{R})$ is the space of all bounded, differentiable functions f such that f' is bounded and continuous. As discussed in Section 3.5, the standard norm for $C_b(\mathbb{R})$ is the uniform norm, while the standard norm for $C_b^1(\mathbb{R})$ is $\|f\|_{C_b^1} = \|f\|_{\mathrm{u}} + \|f'\|_{\mathrm{u}}$. Unless we specify otherwise, we always assume that these are the norms on these two spaces.

Example 2.9.54. Given $g \in C_b^1(\mathbb{R})$, let $Dg = g'$ be the derivative of g. By definition of $C_b^1(\mathbb{R})$, the derivative g' belongs to $C_b(\mathbb{R})$. Therefore we can think of D as a function whose domain is $C_b^1(\mathbb{R})$ and whose codomain is $C_b(\mathbb{R})$. We input a function g from $C_b^1(\mathbb{R})$, and we output the function $Dg = g' \in C_b(\mathbb{R})$. Normally we would write $D(g)$ for the output of the function D, but in this context it is traditional, and simpler, to just write Dg.

We claim that $D\colon C_b^1(\mathbb{R}) \to C_b(\mathbb{R})$ is continuous. Note that the elements of $C_b^1(\mathbb{R})$ are functions. Hence a "point" in $C_b^1(\mathbb{R})$ is a function g. The *function* that we are trying to show is continuous is D, not g. In order to show that D is continuous at a point g, we must interpret equation (2.21) correctly. In our situation, "g" in that equation is our function D, while "x" is our point g in the domain $C_b^1(\mathbb{R})$. To prove that D is continuous at g, we must show that if $\{g_n\}_{n\in\mathbb{N}}$ is a sequence of points in $C_b^1(\mathbb{R})$ that converge to g, then Dg_n converges to Dg. That is, we must prove that

$$g_n \to g \text{ in } C_b^1(\mathbb{R}) \quad \Longrightarrow \quad Dg_n \to Dg \text{ in } C_b(\mathbb{R}).$$

This is easy! To see why, suppose that $g_n \to g$ in $C_b^1(\mathbb{R})$. This means that g_n converges to g with respect to the norm of $C_b^1(\mathbb{R})$, or in other words

$$\lim_{n\to\infty} \|g - g_n\|_{C_b^1} \;=\; 0.$$

But then we can compute that

$$\begin{aligned}\|Dg - Dg_n\|_{\mathrm{u}} &= \|g' - g_n'\|_{\mathrm{u}} \\ &\le \|g - g_n\|_{\mathrm{u}} + \|g' - g_n'\|_{\mathrm{u}} \\ &= \|g - g_n\|_{C_b^1} \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus Dg_n converges to Dg with respect to the uniform norm, which is the norm of $C_b(\mathbb{R})$. In summary, we have shown that if $g_n \to g$ in $C_b^1(\mathbb{R})$, then $Dg_n \to Dg$ in $C_b(\mathbb{R})$. Therefore D is continuous at the point g. Since g is an arbitrary function in $C_b^{(}\mathbb{R})$, we conclude that D is continuous on the domain $C_b^1(\mathbb{R})$. ◇

We emphasize that continuity depends on the choice of metric or norm for both the domain and codomain. For example, if in Example 2.9.54 we replace the standard norm for $C_b^1(\mathbb{R})$ with the uniform norm, then D is no longer continuous (see Problem 3.52.59).

Example 2.9.55. Let us point out a few other properties of the derivative function D considered in Example 2.9.54.

First, D is not injective, because we can find two different functions f, g in the domain $C_b^1(\mathbb{R})$ such that $Df = Dg$. In fact, if f is any function in $C_b^1(\mathbb{R})$ and c is a constant, then $g = f + c$ has the same derivative as f.

Second, D is not surjective, because it is not true that the range of D equals the codomain $C_b(\mathbb{R})$. In particular, if g is the constant function 1, then there is no function $f \in C_b^1(\mathbb{R})$ that satisfies $Df = 1$. In fact, the only functions whose derivative is the constant 1 are $f(x) = x + c$ where c is a constant, but none of these functions belong to $C_b^1(\mathbb{R})$ because they are unbounded.

Third, D is a *linear function*, because if f, g belong to $C_b^1(\mathbb{R})$ and a, b are scalars, then

$$D(af + bg) = (af + bg)' = af' + bg' = a\,Df + b\,Dg.$$

A linear function "respects" the operations of addition and scalar multiplication. ◇

Uniform Continuity

Example 2.9.56. Let $D\colon C_b^1(\mathbb{R}) \to C_b(\mathbb{R})$ be the derivative operator discussed in Example 2.9.55. We will prove that D is uniformly continuous. Given $\varepsilon > 0$, let $\delta = \varepsilon$. Our "points" x, y from Definition 2.9.5 are now elements of $C_b^1(\mathbb{R})$, i.e., they are functions in $C_b^1(\mathbb{R})$. So, suppose that f, g are any two elements of $C_b^1(\mathbb{R})$ such that $\|f-g\|_{\mathrm{u}} < \delta$. Then

$$\begin{aligned} \|Df - Dg\|_{\mathrm{u}} &= \|f' - g'\|_{\mathrm{u}} \\ &\le \|f-g\|_{\mathrm{u}} + \|f'-g'\|_{\mathrm{u}} \\ &= \|f-g\|_{C_b^1} \\ &< \delta = \varepsilon. \end{aligned}$$

Thus $\|Df - Dg\|_{\mathrm{u}} < \varepsilon$ simultaneously for all f and g in $C_b^1(\mathbb{R})$. This shows that D is uniformly continuous on $C_b^1(\mathbb{R})$. ♢

In Example 2.9.56 it was easy to find δ given ε, because D is linear and we have the relationship

$$\|Dg\|_{\mathrm{u}} \le \|g\|_{C_b^1}. \tag{2.22}$$

Using the language that we will introduce in Section 6.2, equation (2.22) says that D is a *bounded operator*. We will prove in Section 6.2 that all linear, bounded operators are continuous.

The following exercise characterizes uniform continuity for scalar-valued functions whose domain is the Euclidean space $\mathbb{F}^d$.

Exercise 2.9.57. Prove that a function $f\colon \mathbb{F}^d \to \mathbb{F}$ is uniformly continuous if and only if

$$\lim_{a \to 0} \|T_a f - f\|_{\mathrm{u}} = 0,$$

where $T_a f(x) = f(x-a)$ denotes the translation of f by $a \in \mathbb{R}^d$. ♢

According to Problem 2.9.17, a *uniformly continuous* function must map Cauchy sequences to Cauchy sequences. This fact can be used to prove the following theorem, which states that it is possible to extend a uniformly continuous function from a dense subset of the domain to the entire domain if the codomain is a complete metric space. We assign the proof of this result as Problem 2.9.65.

Theorem 2.9.58. *Let X and Z be metric spaces such that Z is complete. Suppose that Y is a proper dense subset of X and $f\colon Y \to Z$ is uniformly continuous. Then there exists a uniformly continuous function $g\colon X \to Z$ such that $g(x) = f(x)$ for every $x \in Y$.*

Extra Problems

2.9.59. Let X, Y, and Z be metric spaces, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions.

(a) Show that if f is continuous at a point $x \in X$ and g is continuous at the point $f(x)$, then their composition $g \circ f \colon X \to Z$ is continuous at the point x.

(b) Show that if f and g are each continuous, then their composition $g \circ f \colon X \to Z$ is continuous.

2.9.60. (a) Suppose that X is a normed space, and define $g \colon X \to \mathbb{R}$ by $g(x) = \|x\|$. Prove that g is continuous.

(b) Let X be a metric space, and suppose that $f \colon X \to \mathbb{F}$ is continuous. Let $h = |f|$, i.e., $h(x) = |f(x)|$ for $x \in X$, and prove that $h \colon X \to \mathbb{R}$ is continuous.

2.9.61. Let $f \colon X \to Y$ be a function that maps one metric space X into another metric space Y. Prove that the following two statements are equivalent.

(a) f is continuous.

(b) For each compact set $K \subseteq X$, the restriction of f to K is a continuous mapping of K into Y.

2.9.62. Suppose that $f \colon \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following two conditions.

(a) If $K \subseteq \mathbb{R}^d$ is compact, then $f(K)$ is compact.

(b) If $\{K_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of compact subsets of $\mathbb{R}^d$, then

$$f\biggl(\bigcap_{n=1}^{\infty} K_n\biggr) \;=\; \bigcap_{n=1}^{\infty} f(K_n).$$

Prove that f is continuous.

2.9.63. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative continuous functions on a compact metric space X. Show that if the sequence monotonically decreases to the zero function, i.e., $f_1(x) \ge f_2(x) \ge \cdots$ and $f_n(x) \to 0$ for every x, then $f_n \to 0$ *uniformly*.

2.9.64. Let X be a metric space with metric d_X and let Y be a metric space with metric d_Y.

(a) Prove that

$$\mathrm{d}\bigl((x_1, y_1), (x_2, y_2)\bigr) \;=\; \mathrm{d}_X(x_1, x_2) \;+\; \mathrm{d}_Y(y_1, y_2)$$

defines a metric on $X \times Y$.

(b) Suppose that $f \colon X \to Y$ is continuous, and prove that

$$\mathrm{graph}(f) \;=\; \bigl\{(x, f(x)) : x \in X\bigr\}$$

is a closed subset of $X \times Y$.

(c) Suppose that X is compact. Prove that $f \colon X \to Y$ is continuous if and only if

$$\mathrm{graph}(f) \;=\; \big\{(x, f(x)) : x \in X\big\}$$

is a compact subset of $X \times Y$.

2.9.65. (a) Prove Theorem 2.9.58.

(b) Let S be the set of rational numbers in $(0,1)$, and note that S is dense in the closed interval $[0,1]$. Exhibit a function $f\colon S \to \mathbb{R}$ that is continuous on S, but which cannot be extended to a continuous function on $[0,1]$.

CHAPTER 3: NORMS AND BANACH SPACES

3.1 The Definition of a Norm

The simplest vector space, aside from the trivial space $X = \{0\}$, is the field of scalars $\mathbb{F}$. The "standard" norm on $\mathbb{F}$ is absolute value. However, this is not the only norm, for if $\lambda > 0$ is a fixed positive real number then

$$\|x\| \;=\; \lambda\,|x|, \qquad x \in \mathbb{F},$$

defines another norm on $\mathbb{F}$. Are there any other norms on $\mathbb{F}$? Yes, but we can identify all of them—according to Problem 3.1.8, a function $\|\cdot\|$ is a seminorm on $\mathbb{F}$ if and only if there exists a scalar $\lambda \ge 0$ such that $\|x\| = \lambda\,|x|$ for every $x \in \mathbb{F}$; furthermore, this seminorm is a norm if and only if $\lambda > 0$.

We often summarize this by saying that *up to multiplication by a positive scalar, absolute value is the only norm on* $\mathbb{F}$, *and the only seminorm on* $\mathbb{F}$ *that is not a norm is the zero function.*

The situation in higher dimensions is more complicated. One seminorm on $\mathbb{F}^d$ is the zero function, defined by $\|x\| = 0$ for all $x \in \mathbb{F}^d$. However, when $d \ge 2$ there are many nonzero functions on $\mathbb{F}^d$ that are seminorms, such as

$$\|x\| \;=\; |x_2| + \cdots + |x_d|, \qquad x = (x_1, x_2, \dots, x_d) \in \mathbb{F}^d.$$

Recall that if d is a metric on a *set* X, then we can create a metric on any *subset* S of X simply by restricting d to S. In contrast, if $\|\cdot\|$ is a norm on a *vector space* X and S is a subset of X, then the restriction of $\|\cdot\|$ to S need not define a norm on S. This is because Definition 3.1.1 requires that the domain of a norm be a *vector space*. Therefore, only when S is a *subspace* of X will it be true that the restriction of $\|\cdot\|$ to S is a norm on S.

Definition 3.1.50. Let $(X, \|\cdot\|)$ be a normed vector space. If S is a subspace of X, then the norm on S obtained by restricting $\|\cdot\|$ to S is called the *norm on S inherited from X*, or simply the *inherited norm* on S. We usually use the same symbol $\|\cdot\|$ to denote the inherited norm on S. $\diamondsuit$

For example, if $X = \mathbb{F}^3$ with respect to the Euclidean norm $\|\cdot\|_2$, then the inherited norm on the subspace $S = \{x = (x_1, x_2, 0) : x_1, x_2 \in \mathbb{F}\}$ is given by

$$\|x\|_2 \;=\; \bigl(|x_1|^2 + |x_2|^2\bigr)^{1/2}, \qquad x = (x_1, x_2, 0) \in S.$$

Extra Problems

3.1.51. (a) Fix a dimension $d \geq 1$. Show that there exist positive constants A_d, B_d such that

$$A_d \|x\|_1 \ \leq \ \|x\|_\infty \ \leq \ B_d \|x\|_1 \qquad \text{for all } x \in \mathbb{F}^d. \tag{3.23}$$

The constants A_d, B_d can depend on the dimension d, but not on x. Find the *optimal* constants A_d, B_d, i.e., find the largest constant A_d and the smallest constant B_d such that equation (3.23) holds for every $x \in \mathbb{F}^d$.

(b) Repeat part (a), but with $\|\cdot\|_2$ in place of $\|\cdot\|_\infty$.

(c) Repeat part (a), but with $\|\cdot\|_2$ in place of $\|\cdot\|_1$.

Remark: Using the terminology of Definition 3.6.1, this problem says that $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ are all *equivalent norms* on $\mathbb{F}^d$.

3.2 Examples: The ℓ^p Spaces

We write the components of an infinite sequence in two ways. If x is a sequence of scalars, then we usually write x as

$$x \ = \ (x_k)_{k\in\mathbb{N}} \ = \ (x_1, x_2, \dots).$$

Using this notation, the symbol x_k denotes the kth component of x. However, it is sometimes preferable to write

$$x \ = \ \big(x(k)\big)_{k\in\mathbb{N}} \ = \ \big(x(1), x(2), \dots\big).$$

That is, it is sometimes more clear to let $x(k)$ denote the kth component of x. In any particular situation we will use whichever of these two notations is more convenient.

Remark 3.2.50. By making appropriate changes in the definitions above, we can consider spaces of sequences that are indexed by sets other than the natural numbers $\mathbb{N}$. For example, if I is a countable index set, then we say that a sequence $x = (x_k)_{k\in I}$ is p-summable if and only if

$$\sum_{k\in I} |x_k|^p \ < \ \infty.$$

We let $\ell^p(I)$ be the space of all p-summable sequences indexed by I, and let $\ell^\infty(I)$ be the space of all bounded sequences indexed by I. If $I = \mathbb{N}$, then this reduces to the definition of ℓ^p that we gave before.

A common choice of index set is $I = \mathbb{Z}$. A sequence indexed by $\mathbb{Z}$ is a bi-infinite sequence of the form

$$x = (x_k)_{k\in\mathbb{Z}} = (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots).$$

Hence $\ell^p(\mathbb{Z})$ denotes the set of all bi-infinite sequences that are p-summable (if p is finite) or bounded (if $p=\infty$). For example, the bi-infinite sequence

$$x = \bigl(2^{-|k|}\bigr)_{k\in\mathbb{Z}} = \bigl(\dots, \tfrac{1}{4}, \tfrac{1}{2}, 1, \tfrac{1}{2}, \tfrac{1}{4}, \dots\bigr)$$

belongs to $\ell^p(\mathbb{Z})$ for every index $0<p\le\infty$.

Going in a different direction, we can let the index set be finite. For example, if $I=\{1,\dots,d\}$ then a sequence indexed by I is simply a vector $x=(x_1,\dots,x_d)\in\mathbb{F}^d$. Every such sequence is p-summable and bounded, so for $I=\{1,\dots,d\}$ we have $\ell^p(I)=\mathbb{F}^d$, no matter what index p that we choose. In contrast to equation (3.5), when I is a finite set we have $\ell^p(I)=\ell^q(I)$ for every p and q.

It is possible to consider uncountable index sets I, but this requires more care, see [Heil11, Exercise 1.15] for details. ♢

We will see that if $1\le p\le\infty$, then $\|\cdot\|_p$ is an norm on ℓ^p (see Theorem 3.4.3). Therefore we refer to $\|\cdot\|_p$ as the "ℓ^p-norm" when $p\ge 1$. For two specific values of p we have additional names: We call $\|\cdot\|_2$ the *Euclidean norm*, and we refer to $\|\cdot\|_\infty$ as the *sup-norm.*

Before we can address the question of whether $\|\cdot\|_p$ is a norm on ℓ^p, we need to prove that ℓ^p is a vector space. We know how to add sequences (just add corresponding components), and how to multiply a sequence by a scalar. It is clear that ℓ^p is closed under multiplication by scalars, and the following lemma shows that ℓ^p is closed under addition when p is finite (even for $0<p<1$).

Lemma 3.2.51. *Fix $0<p<\infty$. If x, $y\in\ell^p$, then the sequence $x+y=(x_k+y_k)_{n\in\mathbb{N}}$ satisfies*

$$\|x+y\|_p^p \le 2^p\bigl(\|x\|_p^p+\|y\|_p^p\bigr), \tag{3.24}$$

and therefore $x+y\in\ell^p$.

Proof. Given any *scalars* a, $b\in\mathbb{F}$, we have

$$\begin{aligned}
|a+b|^p &\le \bigl(|a|+|b|\bigr)^p \\
&\le \Bigl(\max\{|a|,|b|\} + \max\{|a|,|b|\}\Bigr)^p \\
&= 2^p\max\{|a|^p,|b|^p\} \\
&\le 2^p\bigl(|a|^p+|b|^p\bigr).
\end{aligned}$$

Therefore, if $x=(x_k)_{k\in\mathbb{N}}$ and $y=(y_k)_{k\in\mathbb{N}}$ are sequences in ℓ^p, then

$$\|x+y\|_p^p \;=\; \sum_{k=1}^{\infty} |x_k+y_k|^p \;\le\; 2^p \sum_{k=1}^{\infty} \bigl(|x_k|^p + |y_k|^p\bigr) \;=\; 2^p \bigl(\|x\|_p^p + \|y\|_p^p\bigr).$$

Since the right-hand side is finite, we conclude that $x+y \in \ell^p$. □

Equation (3.24) is not necessarily the most useful inequality that relates $\|x+y\|_p$ to $\|x\|_p$ and $\|y\|_p$, but it is sufficient for the purpose of proving that ℓ^p is closed under addition for every finite index p. The reader should verify that ℓ^∞ is also closed under addition and scalar multiplication and therefore is a vector space.

The zero vector in ℓ^p is the *zero sequence* whose components are all zero. We use the same symbol 0 to denote both the zero sequence and the number zero. That is, we write the zero sequence as

$$0 \;=\; (0,0,0,\dots).$$

As illustrated in the following example, it should usually be clear from context whether the symbol 0 is meant to represent the number zero or the zero sequence.

Example 3.2.52. Let $x = (x_k)_{k\in\mathbb{N}}$ be a sequence of scalars. If we write "$x = 0$," then, since x is a sequence, we infer that 0 in this context must also represent a sequence and therefore stands for the zero sequence. In other words, "$x = 0$" is equivalent "x is the zero sequence," which is itself equivalent to "$x_k = 0$ for every k."

Similarly "$x \neq 0$" means that x is not the zero sequence, and consequently there exists *at least one* index k such that $x_k \neq 0$. We say that *x is a nonzero sequence*, or simply that *x is nonzero*, if $x \neq 0$. That is, a nonzero sequence is a sequence that has at least one nonzero component. For example, $y = (1,0,0,\dots)$ is a nonzero sequence. ♢

The Standard Basis Vectors

Let

$$\delta_n \;=\; (0,\dots,0,1,0,0,\dots)$$

denote the sequence that has a 1 in the nth component and zeros elsewhere. We call δ_n the nth *standard basis vector*, and refer to $\mathcal{E} = \{\delta_n\}_{n\in\mathbb{N}}$ as the *sequence of standard basis vectors*, or simply the *standard basis*.

Note that $\mathcal{E} = \{\delta_n\}_{n\in\mathbb{N}}$ is a sequence, and each element δ_n of this sequence is itself a sequence of scalars. Thus $\mathcal{E}$ is a "sequence of sequences," but that is not necessarily the best way to think about $\mathcal{E}$. Instead, it is better to think of $\mathcal{E}$ as being a *sequence of vectors*, where it just so happens that "vector" here means a sequence of scalars. Each δ_n is a vector in ℓ^p, and $\mathcal{E}$ is just the sequence of vectors $\mathcal{E} = \{\delta_1, \delta_2, \dots\}$ in ℓ^p.

Since $\mathcal{E}$ is a set of vectors, we would like to know its properties. Is it a linearly independent set? What is its span? Before we address these questions, we consider an even more fundamental question: What does a linear combination of elements of $\mathcal{E}$ look like? By definition, a linear combination is a sum of *finitely many* scalar multiples of elements of $\mathcal{E}$. If we choose finitely many δ_n, say $\delta_{n_1}, \dots, \delta_{n_k}$, then a linear combination of these vectors is

$$x \;=\; c_{n_1}\delta_{n_1} + \cdots + c_{n_k}\delta_{n_k},$$

where $c_{n_1}, \dots, c_{n_k} \in \mathbb{F}$. If we let N be the largest of $n_1, \dots, n_k$, then by inserting terms with $c_k = 0$, we can write

$$x \;=\; \sum_{k=1}^{N} c_k\delta_k \;=\; c_1\delta_1 + \cdots + c_N\delta_N \;=\; (c_1, \dots, c_N, 0, 0, \dots). \tag{3.25}$$

Every linear combination of elements of $\mathcal{E}$ has the form given in equation (3.25) for some integer $N > 0$ and scalars $c_1, \dots, c_N \in \mathbb{F}$. For example, if $n_1 = 2$, $n_2 = 3$, $n_3 = 5$ and $c_{n_1} = 6$, $c_{n_2} = -\pi$, $c_{n_3} = 4$, then $N = 5$ and

$$\begin{aligned} x \;&=\; c_{n_1}\delta_{n_1} + c_{n_2}\delta_{n_2} + c_{n_3}\delta_{n_3} \\ &=\; 6\delta_2 - \pi\delta_3 + 4\delta_5 \\ &=\; 0\delta_1 + 6\delta_2 - \pi\delta_3 + 0\delta_4 + 4\delta_5 \\ &=\; (0, 6, -\pi, 0, 5, 0, 0, \dots). \end{aligned}$$

Now we determine whether $\mathcal{E}$ is linearly independent. Suppose that some finite linear combination of elements of $\mathcal{E}$ equals the zero vector in ℓ^p. That is, suppose that

$$\sum_{k=1}^{N} c_k\delta_k \;=\; 0 \tag{3.26}$$

for some N and some scalars $c_1, \dots, c_N$. Rewriting equation (3.26) in terms of components, we obtain

$$(c_1, \dots, c_N, 0, 0, \dots) \;=\; (0, 0, 0, \dots).$$

Hence $c_1 = \cdots = c_N = 0$. Therefore the only linear combination that equals the zero vector is the trivial combination, so $\mathcal{E}$ is a linearly independent set in ℓ^p.

However, the span of $\mathcal{E}$ is not ℓ^p, and therefore $\mathcal{E}$ is *not* a vector space basis for ℓ^p. To see why, recall that $\operatorname{span}(\mathcal{E})$ is the set of all finite linear combinations of elements of $\mathcal{E}$. Every linear combination of standard basis vectors has the form given in equation (3.25). Consequently, if x is a vector in $\operatorname{span}(\mathcal{E})$ then x has only *finitely many* nonzero components. Any vector that has infinitely many nonzero components cannot belong to the span of

$\mathcal{E}$. For example,

$$z = \left(2^{-n}\right)_{n\in\mathbb{N}} = \left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \dots\right)$$

belongs to ℓ^p for every p, yet $z \notin \mathrm{span}(\mathcal{E})$, because it is not a *finite* linear combination of $\delta_1, \delta_2, \dots$. Therefore

$$\mathrm{span}(\mathcal{E}) \subsetneq \ell^p, \qquad 0 < p \le \infty.$$

Indeed, $\mathrm{span}(\mathcal{E})$ is precisely the set of all sequences that have finitely many nonzero components. We will study this space, which we often denote by $c_{00} = \mathrm{span}(\mathcal{E})$, in more detail in Example 2.2.13.

The fact that ℓ^p contains an infinite linearly independent set implies that it must be infinite dimensional. This is because a basic result from linear algebra tells us that if a vector space X has finite dimension d, then any set of more than d vectors must be dependent. A finite-dimensional vector space cannot contain an infinite linearly independent set.

Hölder's Inequality

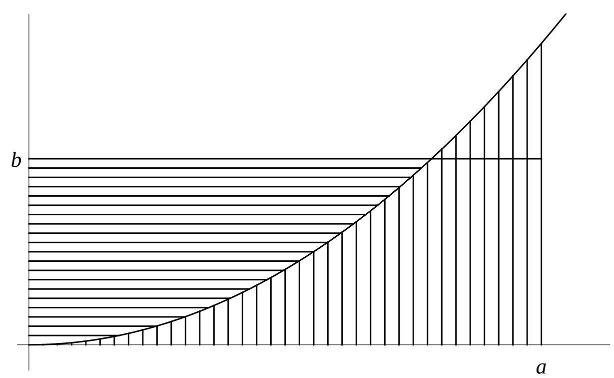

Fig. 3.50 The curved line is the graph of $y = x^{p-1}$. The area of the vertically hatched region is $\int_0^a x^{p-1}\,dx$, the area of the horizontally hatched region is $\int_0^b y^{\frac{1}{p-1}}\,dy$, and the area of the rectangle $[0,a] \times [0,b]$ is ab.

The method of Problem 3.2.11 is not the only way to prove equation (3.8), which stated that if $1 < p < \infty$ then

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}, \qquad a, b \ge 0.$$

For another approach, note that x^{p-1} is continuous and strictly increasing on the interval $[0,a]$, and its inverse function is $y^{\frac{1}{p-1}}$. Figure 3.50 gives a "proof by picture" that

$$ab \;\le\; \int_0^a x^{p-1}\,dx + \int_0^b y^{\frac{1}{p-1}}\,dy. \tag{3.27}$$

Evaluating the right-hand side of equation (3.27), we obtain equation (3.8). Yet another proof of equation (3.8) can be given based on a theorem known as *Jensen's inequality*.

Here is one application of Hölder's Inequality. Let $x = y = \bigl(\frac{1}{k}\bigr)_{k\in\mathbb{N}}$. We have $x \in \ell^p$ for $1 < p \le \infty$. Taking $p = 3/2$, Hölder's Inequality therefore implies the interesting inequality

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \;=\; \|xy\|_1 \;\le\; \|x\|_{3/2}\,\|y\|_3 \;=\; \biggl(\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\biggr)^{2/3} \biggl(\sum_{n=1}^{\infty} \frac{1}{n^3}\biggr)^{1/3}. \qquad \diamondsuit$$

Euler's formula (see Problem 5.10.11) states that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \;=\; \frac{\pi^2}{6}.$$

Extra Problems

3.2.53. Given $0 < p < 1$, prove that ℓ^p is complete with respect to the metric defined in Problem 3.3.14.

3.3 The Induced Metric

Convexity

If a metric space contains an open ball that is not convex, then there is an important further conclusion that we can draw. This is spelled out in the following lemma.

Lemma 3.3.50. *Let X be a vector space that has a metric d. If there exists an open ball $B_r(x)$ in X that is not convex, then the metric d is not induced from any norm on X. That is, there is no norm $\|\cdot\|$ on X such that $\mathrm{d}(y,z) = \|y-z\|$ for all $y,\ z \in X$.*

Proof. Suppose that the open ball $B_r(x)$ is convex. Suppose also that there is a $\|\cdot\|$ on X that satisfies $\|y-z\| = \mathrm{d}(y,z)$ for all $y,\ z \in X$. By Problem

3.3.13, open balls defined with respect to a norm are convex. Yet the open ball centered at x with radius r that is defined with respect to $\|\cdot\|$ is precisely $B_r(x)$:

$$B_r(x) \;=\; \bigl\{y \in X : \mathrm{d}(y,x) < 1\bigr\} \;=\; \bigl\{y \in X : \|y-x\| < 1\bigr\}.$$

Since $B_r(x)$ is not convex, this is a contradiction. Hence no such norm $\|\cdot\|$ can exist. □

Bounded Sets

Before giving the next property of convergent and Cauchy sequences, we recall from Definition 2.3.2 that a subset of a metric space is bounded if it is contained in some open ball. The following (easy) exercise gives some equivalent formulations for boundedness of subsets of a normed space. In particular, statement (b) of this exercise says that a subset of a normed space is bounded if it is contained in some open ball that is centered at the origin (note that a generic metric space need not be a vector space, and hence there need not be an "origin" in a metric space, as there is in a normed space).

Exercise 3.3.51. Let E be a subset of a normed space X. Prove that the following three statements are equivalent.

(a) E is bounded, i.e., $E \subseteq B_r(x)$ for some $x \in X$ and $r > 0$.

(b) $E \subseteq B_r(0)$ for some $r > 0$.

(c) $\sup_{x\in E} \|x\| < \infty$. ◇

Separability

There is an interesting difference between ℓ^∞ and the other ℓ^p spaces. Problem 2.6.20 showed that ℓ^1 is *separable*, i.e., it contains a countable dense subset. A very similar argument shows that ℓ^p is separable for *any* finite p, but we saw in Theorem 2.6.8 that ℓ^∞ is not separable. In this sense ℓ^∞ is "much larger" than the other ℓ^p spaces.

The Standard Topology for $\mathbb{F}^d$

Many sets have a "standard" or "default" metric or norm associated with them. The topology induced from this standard metric or norm is often referred to as the *standard topology* for that space. For example, the standard norm for the space ℓ^p when $1 \le p \le \infty$ is $\|\cdot\|_p$, so the topology induced from this norm is the standard topology for ℓ^p. The standard norm for $C_b(\mathbb{R})$ is the uniform norm $\|\cdot\|_u$.

The standard topology for $\mathbb{F}^d$ (also called the *usual topology* for $\mathbb{F}^d$) is the topology induced from the Euclidean norm on $\mathbb{F}^d$. However, we will show that exactly the same topology is induced from the norm $\|\cdot\|_1$.

Lemma 3.3.52. *If $U \subseteq \mathbb{F}^d$, then U is open with respect to the Euclidean norm if and only if U is open with respect to the norm $\|\cdot\|_1$.*

Proof. For this proof, let $B_r^1(x)$ denote an open ball with respect to $\|\cdot\|_1$, i.e.,

$$B_r^1(x) \;=\; \bigl\{y \in \mathbb{F}^d : \|x-y\|_1 < r\bigr\}.$$

Similarly, let $B_r^2(x)$ denote an open ball with respect to the Euclidean norm $\|\cdot\|_2$. These balls are illustrated in Figure 2.5 for dimension $d=2$ and $\mathbb{F}=\mathbb{R}$. As we see in that figure, $B_r^2(x)$ is a ball or disk in the usual sense of the word, while $B_r^1(x)$ is a diamond. Figure 2.5 suggests that the diamond fits entirely within the disk. Indeed, by direct calculation we can see that

$$B_r^1(x) \subseteq B_r^2(x). \tag{3.28}$$

Suppose now that U is any set that is open with respect to the Euclidean norm. By definition, this means that for each $x \in U$ there exists some $r_x > 0$ such that $B_{r_x}^2(x) \subseteq U$. Hence,

$$\begin{aligned} U \;=\; \bigcup_{x\in U}\{x\} \;&\subseteq\; \bigcup_{x\in U} B_{r_x}^1(x) && (\text{since } x \in B_{r_x}^1(x)) \\ &\subseteq\; \bigcup_{x\in U} B_{r_x}^2(x) && (\text{by equation (3.28)}) \\ &\subseteq\; U && (\text{since } B_r^2(x) \subseteq U). \end{aligned}$$

This implies that U is the union of the open balls $B_{r_x}^1(x)$ over $x \in U$, i.e.,

$$U \;=\; \bigcup_{x\in U} B_{r_x}^1(x).$$

Since each such ball $B_{r_x}^1(x)$ is open with respect to $\|\cdot\|_1$, it follows that U is open with respect to $\|\cdot\|_1$.

If we attempt to use the same argument to prove the converse implication, then we encounter a problem because $B_r^2(x)$ is not contained in $B_r^1(x)$. On the other hand, by applying Hölder's Inequality with the index $p=2$ we see that

$$\|x\|_1 \;=\; \sum_{k=1}^d |1\cdot x_k| \;\le\; \biggl(\sum_{k=1}^d 1^2\biggr)^{1/2} \biggl(\sum_{k=1}^d |x_k|^2\biggr)^{1/2} \;=\; d^{1/2}\,\|x\|_2,$$

and it follows from this that

$$B_r^2(x) \subseteq B^1_{d^{1/2}r}(x).$$

That is, every open disk is contained in a diamond whose radius has been enlarged by a factor of $d^{1/2}$. We can then argue just as before to show that if U is open with respect to the norm $\|\cdot\|_1$, then it is also open with respect to $\|\cdot\|_2$. □

In summary, the topology on $\mathbb{F}^d$ induced from the norm $\|\cdot\|_1$ is the same as the topology induced by $\|\cdot\|_2$. An entirely similar argument shows that the norm $\|\cdot\|_\infty$ also induces this same topology. We will prove in Theorem 3.7.2 that *every* norm on $\mathbb{F}^d$ induces the same topology. Thus, there is only one topology on $\mathbb{F}^d$ that is induced by a norm, and that is the standard topology. There are other topologies on $\mathbb{F}^d$, such as the discrete topology, but these are not induced from any norm on $\mathbb{F}^d$.

Extra Problems

3.3.53. Given a set E in a normed space X and given $t \ge 0$, define $tE = \{tx : x \in E\}$.

(a) Prove that $tB_r(0) = B_{rt}(0)$.

(b) Given an arbitrary point $x \in X$, must $tB_r(x) = B_{rt}(x)$?

3.3.54. Show that if X is a nontrivial normed space, then every open ball $B_r(x)$ is an infinite, proper subset of X, and $\cup_{r>0} B_r(x) = X$.

In contrast, construct examples of metric spaces X_1, X_2, X_3 that have the following properties.

(a) $B_r(x) = X_1$ for some $x \in X_1$ and some $r > 0$.

(b) X_2 is infinite, yet $B_r(x)$ is a finite set for every $x \in X_2$ and $r > 0$.

(c) For every integer $n \in \mathbb{N}$ there exists some $x \in X_3$ and some $r > 0$ such that $B_r(x)$ contains exactly n elements.

3.3.55. (a) Show that a set $B \subseteq \mathbb{R}$ is an open ball in $\mathbb{R}$ if and only if B is a bounded open interval.

(b) Identify all of the convex subsets of the real line.

3.3.56. Suppose that X is a normed vector space. Show that if U is an open subset of X, then the translated set $U + x = \{u + x : u \in U\}$ is also open for every $x \in X$.

3.3.57. Let X be a nontrivial normed space. Prove that every subspace of X is convex, but no nontrivial subspace of X is a bounded set.

3.3.58. (a) Show that if E is a subset of a *normed* space X and

$$\delta \;=\; \inf\bigl\{\|x-y\| : x,y \in E, x \neq y\bigr\} \;>\; 0,$$

then $\partial E = E$. In particular, show that $\partial E = E$ for every finite set $E \subseteq \mathbb{R}^d$.

(b) Show by example that the conclusion of part (a) can fail if $\delta = 0$.

(c) Show by example that the conclusion of part (a) can fail if X is a metric space, even if $\delta = \inf\bigl\{\mathrm{d}(x,y) : x,y \in E, x \neq y\bigr\} > 0$.

3.4 Banach Spaces

Complete Subsets

Sometimes we need to refer to a *subset* of a normed space that has a completeness property.

Definition 3.4.50 (Complete Sets in Normed Spaces). If S is a subset of a normed vector space X and every Cauchy sequence in X converges to an element of S, then we say that S is a *complete subset* of X. $\diamondsuit$

If S is a complete *subspace* of a normed space X, then S is itself a normed space, and so in this case we would call S a *Banach space*. However, if S is a subset that is complete but is not a subspace, then we refer to S as a *complete subset* of X. In Lemma 3.4.4 we will see that a subspace S of a Banach space X is complete if and only if S is closed.

Remark 3.4.51. As we have noted before, the term “complete” has a number of distinct mathematical meanings. In this volume, the two main uses are *complete sets* in the sense of Definition 2.2.9, and *complete sequences* in the sense of Definition 4.4.4. The reader should pause to consider context whenever the term “complete” is encountered. $\diamondsuit$

Here is an example of an incomplete subset of $\mathbb{R}$.

Example 3.4.52. Let S be the open interval $S = (0,1)$ in the real line. Then $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence of elements of S, but this sequence does not converge to an element of S. The sequence does converge, but it does not converge to a point in S. Therefore S is not a complete subset of $\mathbb{R}$. $\diamondsuit$

Another example is the set of rationals $\mathbb{Q}$, which Example 2.2.7 shows is an incomplete subset of $\mathbb{R}$. A modification of the argument given in Example 2.2.7 can be used to show that

$$\mathbb{Q}^d \;=\; \bigl\{r = (r_1, \dots, r_d) : r_1, \dots, r_d \in \mathbb{Q}\bigr\}$$

is an incomplete subset of $\mathbb{R}^d$ with respect to each of the norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$, and that

$$\mathbb{Q}^d + i\mathbb{Q}^d \;=\; \bigl\{z = r + is : r, s \in \mathbb{Q}^d\bigr\}$$

is an incomplete subset of $\mathbb{C}^d$ with respect to $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$.

Here is a direct proof that c_{00} is an incomplete subset of ℓ^∞.

Theorem 3.4.53. *The sup-norm is a norm on c_{00}, but c_{00} is not complete with respect to $\|\cdot\|_\infty$.*

Proof. Since c_{00} is a subspace of ℓ^∞, it is a normed space with respect to the sup-norm $\|\cdot\|_\infty$.

For each $n \in \mathbb{N}$, let x_n be the sequence

$$x_n \;=\; \bigl(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{n}, 0, 0, \dots\bigr),$$

and consider the sequence of vectors $\{x_n\}_{n\in\mathbb{N}}$ in c_{00}. If $m < n$, then

$$x_n - x_m \;=\; \bigl(0, \dots, 0, \tfrac{1}{m+1}, \dots, \tfrac{1}{n}, 0, 0, \dots\bigr),$$

so

$$\|x_n - x_m\|_\infty \;=\; \frac{1}{m+1}.$$

Therefore, if we fix $\varepsilon > 0$ then we have $\|x_n - x_m\|_\infty < \varepsilon$ for all m, $n > \frac{1}{\varepsilon}$. Hence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in c_{00}.

Suppose that there was a sequence $x \in c_{00}$ such that $x_n \to x$ with respect to the sup-norm. We know that ℓ^∞-norm convergence implies componentwise convergence, so x_n must converge componentwise to x as $n \to \infty$. If we fix any particular k, then

$$x_n(k) \;=\; \frac{1}{k} \qquad \text{for all } n > k.$$

Therefore we must have $x(k) = \frac{1}{k}$. That is, x is the sequence

$$x \;=\; \bigl(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\bigr). \tag{3.29}$$

However, this x does not belong to c_{00}, so we have reached a contradiction. Therefore there is no vector $x \in c_{00}$ such that $x_n \to x$ with respect to the sup-norm.

In summary, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in c_{00}, but this sequence does not converge to an element of c_{00}. Therefore c_{00} is incomplete with respect to the sup-norm. □

Theorem 3.4.6 shows that an infinite-dimensional Banach space (such as c_0) can contain an incomplete subspace (such as c_{00}). This is quite different from what happens in finite-dimensions! Every subspace of a finite-dimensional normed space is complete. In particular, every subspace of $\mathbb{F}^d$ is complete.

Closed Subspaces of Banach Spaces

It is important to note that Lemma 3.4.4 is only applicable if the subspace Y has the same norm as X. For example, although $C_b^1(\mathbb{R})$ is a subspace of $C_b(\mathbb{R})$, the standard norms for these two spaces are different. For $C_b(\mathbb{R})$ the standard norm is the uniform norm, while for $C_b^1(\mathbb{R})$ the standard norm is $\|f\|_{C_b^1} = \|f\|_{\mathrm{u}} + \|f'\|_{\mathrm{u}}$. Therefore, even though we know that $C_b(\mathbb{R})$ is complete with respect to the uniform norm, we cannot use Lemma 3.4.4 to prove that $C_b^1(\mathbb{R})$ is complete with respect to its standard norm, because that norm is not the uniform norm.

Extra Problems

3.4.54. Given an arbitrary (possibly uncountable) index set I, let $\ell^\infty(I)$ be the space of all bounded sequences $x = (x_i)_{i\in I}$, and set $\|x\|_\infty = \sup_{i\in I} |x_i|$. For $1 \le p < \infty$ let $\ell^p(I)$ consist of all sequences $x = (x_i)_{i\in I}$ *with at most countably many nonzero components* such that $\|x\|_p^p = \sum |x_i|^p < \infty$. Prove that each of these spaces $\ell^p(I)$ is a Banach space with respect to $\|\cdot\|_p$.

3.5 Uniform Convergence of Functions

Remark 3.5.50. The uniform norm is the analogue for functions of the sup-norm

$$\|x\|_\infty \;=\; \|(x_k)_{k\in\mathbb{N}}\|_\infty \;=\; \sup_{k\in\mathbb{N}} |x_k| \;<\; \infty$$

that is defined for sequences. It would therefore be natural to use the notation $\|f\|_\infty$ instead of $\|f\|_{\mathrm{u}}$ to denote the uniform norm of f. However, the symbols $\|f\|_\infty$ traditionally denote the L^∞*-norm* of a function f. If f is a continuous function on an interval I, then there is no difference between its L^∞-norm and its uniform norm. However, for discontinuous functions there can be a difference, because the L^∞-norm "ignores sets of measure zero" (in a sense that is made precise by using measure theory; see [Heil18]). For this reason, we will always write $\|f\|_{\mathrm{u}}$ to denote the uniform norm of a function. $\diamondsuit$

$C_0(\mathbb{R})$ *and* $C_c(\mathbb{R})$

Now we look at continuous functions whose domain is the entire real line $\mathbb{R}$. We will define two important subspaces of $C_b(\mathbb{R})$. To define the first of these, we declare that a function $f\colon \mathbb{R} \to \mathbb{F}$ *vanishes at infinity* if we have both

$$\lim_{x\to\infty} f(x) \;=\; 0 \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) \;=\; 0. \tag{3.30}$$

For simplicity, we often write $\lim_{x\to\pm\infty} f(x) = 0$ or $\lim_{|x|\to\infty} f(x) = 0$ to mean that the two conditions in equation (3.30) both hold.

The *space of all continuous functions that vanish at infinity* is

$$C_0(\mathbb{R}) \;=\; \Big\{f \in C(\mathbb{R}) : \lim_{x\to\pm\infty} f(x) = 0\Big\}.$$

Every function $f \in C_0(\mathbb{R})$ is bounded (why?), so $C_0(\mathbb{R})$ is a subset of $C_b(\mathbb{R})$. Further, $C_0(\mathbb{R})$ is closed under addition of functions and multiplication of a function by a scalar, so it is a subspace of $C_b(\mathbb{R})$. Since the uniform norm is a norm on $C_b(\mathbb{R})$, it is therefore a norm on $C_0(\mathbb{R})$ as well.

Here are some examples.

Example 3.5.51. (a) The *Gaussian function*

$$\phi(x) \;=\; e^{-x^2}, \qquad \in \mathbb{R},$$

belongs to $C_0(\mathbb{R})$. Since $\phi(x) \neq 0$ for any x, this shows that an element of $C_0(\mathbb{R})$ need not be zero at any point.

(b) The function $f(x) = e^{-x}$ does not belong to $C_0(\mathbb{R})$ because $f(x)$ does not converge to zero as $x \to -\infty$.

(c) The *two-sided exponential* $g(x) = e^{-|x|}$ is an element of $C_0(\mathbb{R})$, and it satisfies $g(x) \neq 0$ for every x.

(d) The *sinc function* is

$$\mathrm{sinc}(x) \;=\; \frac{\sin \pi x}{\pi x}, \qquad x \neq 0.$$

The sinc function has a removable discontinuity at the origin, so if we define $\mathrm{sinc}(0) = 1$ then it belongs to $C_0(\mathbb{R})$. The sinc function has the property that $\mathrm{sinc}(x) = 0$ if and only if x is a nonzero integer (this plays an important role in the *Classical* (or Shannon) *Sampling Theorem*; see [Heil11]). $\diamondsuit$

Another example of a function in $C_0(\mathbb{R})$ is any continuous function that is identically zero outside of some finite interval. We have a special name for functions that have this property.

Definition 3.5.52 (Compactly Supported Function). We say that a continuous function $f\colon \mathbb{R} \to \mathbb{F}$ is *compactly supported* if there exists some finite interval $[a,b]$ such that $f(x) = 0$ for all $x \notin [a,b]$. $\diamondsuit$

For example, $f(x) = \sin x$ is not compactly supported, but the continuous function

$$g(x) \;=\; (\sin x)\,\chi_{[0,2\pi]}(x) \;=\; \begin{cases} \sin x, & x \in [0,2\pi], \\ 0, & x \notin [0,2\pi], \end{cases}$$

is compactly supported because it is identically zero outside of $[0,2\pi]$.

We denote the *space of compactly supported continuous functions* by

$$C_c(\mathbb{R}) \;=\; \Bigl\{ f \in C(\mathbb{R}) \,:\, f \text{ is compactly supported} \Bigr\}.$$

Each function in $C_c(\mathbb{R})$ belongs to $C_0(\mathbb{R})$, and $C_c(\mathbb{R})$ is closed under addition and multiplication by scalars, so it is a subspace of $C_0(\mathbb{R})$. Each of the functions in parts (a), (c), and (d) of Example 3.5.51 belong to $C_0(\mathbb{R})$ but not $C_c(\mathbb{R})$. Hence $C_c(\mathbb{R})$ is a *proper* subspace of $C_0(\mathbb{R})$. More generally, we have the inclusions

$$C_c(\mathbb{R}) \;\subsetneq\; C_0(\mathbb{R}) \;\subsetneq\; C_b(\mathbb{R}) \;\subsetneq\; C(\mathbb{R}).$$

Since $C_b(\mathbb{R})$ is a normed space and $C_0(\mathbb{R})$ is a subspace of $C_b(\mathbb{R})$, the uniform norm defines a norm on $C_0(\mathbb{R})$. We take the uniform norm to be the standard, or "default," norm for $C_0(\mathbb{R})$, so unless we specify otherwise, we always assume that the norm on $C_0(\mathbb{R})$ is the uniform norm.

According to Problem 3.5.19, $C_0(\mathbb{R})$ is complete with respect to the uniform norm. Problem 3.5.20 shows that $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.

In contrast, although $C_c(\mathbb{R})$ is a subspace of $C_b(\mathbb{R})$, it is not dense in $C_b(\mathbb{R})$. To see why, let f be the constant function $f = 1$. If $f_n \in C_c(\mathbb{R})$, then $f_n(x) = 0$ for some x. Hence $|f(x) - f_n(x)| = 1$ for that x, and therefore

$$\|f - f_n\|_{\mathrm{u}} \;=\; \sup_{x \in \mathbb{R}} |f(x) - f_n(x)| \;\ge\; 1.$$

No matter what functions $f_n \in C_c(\mathbb{R})$ that we choose, $\|f - f_n\|_{\mathrm{u}}$ *does not* converge to zero. Therefore there is no sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ that converges uniformly to the constant function $f = 1$. This implies that $C_c(\mathbb{R})$ is not dense in $C_b(\mathbb{R})$.

Example 3.5.53. Let $X = C_0(\mathbb{R})$ and let $E = C_c(\mathbb{R})$. The "standard" or "default" norm on $C_0(\mathbb{R})$ is the uniform norm, and that is the norm that we will use in this example. Let f be any function in $C_0(\mathbb{R})$ that does not belong to $C_c(\mathbb{R})$. For example, we could take $f(x) = e^{-|x|}$. By Problem 3.5.20, there exist functions $f_n \in C_c(\mathbb{R})$ that converge uniformly to f. We have $f_n \ne f$ since f_n belongs to $C_c(\mathbb{R})$ while f does not. Thus:

- $f_n \in C_c(\mathbb{R})$ for every n,

- $f_n \neq f$ for every n, and
- $f_n \to f$ (with respect to the norm on $C_0(\mathbb{R})$, which is the uniform norm).

According to Definition 2.5.1, this says that f is an accumulation point of $C_c(\mathbb{R})$. Since f does not belong to $C_c(\mathbb{R})$, this implies that $C_c(\mathbb{R})$ is not a closed subset of $C_0(\mathbb{R})$. In fact, a modification of the argument given above shows that $C_c(\mathbb{R})$ is a dense but proper subset of $C_0(\mathbb{R})$. $\diamondsuit$

Example 3.5.54. Suppose that instead of $C_c(\mathbb{R})$, we consider the set S that consists of those continuous, compactly supported functions that are identically zero outside of the interval $[0,1]$, i.e.,

$$S \;=\; \bigl\{g \in C_c(\mathbb{R}) : \operatorname{supp}(g) \subseteq [0,1]\bigr\}.$$

We will show that this set S is a closed subset of $C_0(\mathbb{R})$.

To do this, suppose that f is any function in $C_0(\mathbb{R})$ that is a limit of functions from S. By limit, we mean here a limit with respect to the norm of $C_0(\mathbb{R})$. In other words, suppose that functions $f_n \in S$ are such that $f_n \to f$ *uniformly* as $n \to \infty$. Since f_n belongs to S, it is supported within the interval $[0,1]$. Now, uniform convergence implies pointwise convergence, so for every $x \in \mathbb{R}$ we have

$$f(x) \;=\; \lim_{n\to\infty} f_n(x).$$

In particular, if we choose x outside of $[0,1]$, then $f_n(x) = 0$ for every n and therefore $f(x) = 0$ as well. Hence f is compactly supported and is identically zero outside of $[0,1]$, so f belongs to S. Applying Theorem 2.4.2, we conclude that S is a closed subset of $C_0(\mathbb{R})$. $\diamondsuit$

Expanded Discussion of the Cantor–Lebesgue Function

We expand a little on Problem 3.5.26, which constructs the *Cantor–Lebesgue Function*.

To illustrate uniform convergence, we will construct a striking function called the Cantor–Lebesgue Function. The construction of this function is related to the middle-thirds Cantor set, which was constructed in Section 2.7.

The next exercise constructs the Cantor–Lebesgue function φ. As in the construction of the Cantor set, we do not define φ directly, but instead obtain it as a limit, in this case the limit of a uniformly convergent sequence of continuous functions.

Exercise 3.5.55 (Cantor–Lebesgue Function). Consider the two functions φ_1, φ_2 pictured in Figure 3.51. The function φ_1 takes the constant value $\frac{1}{2}$ on the interval $(\frac{1}{3}, \frac{2}{3})$ that is removed from $[0,1]$ in the first stage of

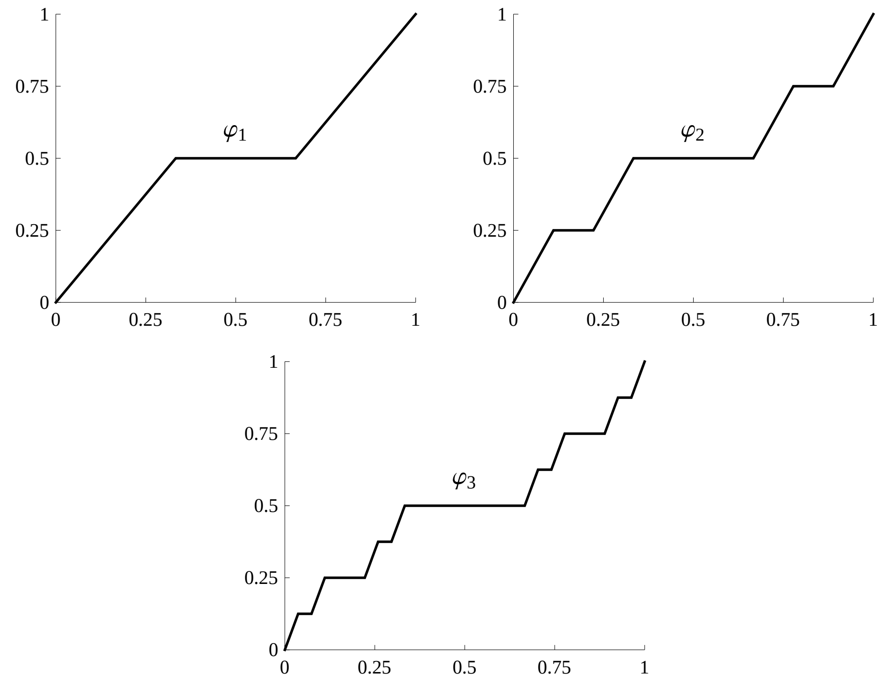

Fig. 3.51 First stages in the construction of the Cantor–Lebesgue function.

the construction of the Cantor set, and it is linear on the remaining subintervals of $[0,1]$. The function φ_2 takes the same constant $\frac{1}{2}$ on the interval $(\frac{1}{3}, \frac{2}{3})$ but additionally is constant with values $\frac{1}{4}$ and $\frac{3}{4}$ on the two intervals that are removed during the second stage of the construction of the Cantor set. We continue this process and define $\varphi_3, \varphi_4, \dots$ in a similar fashion. Each function φ_n is continuous, and φ_n is constant on each of the open intervals that are removed during the nth stage of the construction of the Cantor set.

Looking at the graphs of φ_1 and φ_2, observe that $\varphi_1 = \varphi_2$ on the interval $\left[\frac{1}{3}, \frac{2}{3}\right]$, and if x belongs to either $\left[0, \frac{1}{3}\right]$ or $\left[\frac{2}{3}, 1\right]$ then $|\varphi_1(x) - \varphi_2(x)|$ is at most $\frac{1}{2}$. Hence $\|\varphi_1 - \varphi_2\|_{\mathrm{u}} \le \frac{1}{2}$. Continue this argument, and prove the following statements.

(a) For each $n \in \mathbb{N}$, φ_n is continuous and monotone increasing on $[0,1]$, i.e., if $0 \le x \le y \le 1$, then $\varphi_n(x) \le \varphi_n(y)$.

(b) For each $n \in \mathbb{N}$,

$$\|\varphi_n - \varphi_{n+1}\|_{\mathrm{u}} = \sup_{x \in [0,1]} |\varphi_n(x) - \varphi_{n+1}(x)| \le 2^{-n}.$$

(c) $\{\varphi_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $C[0,1]$ with respect to the uniform norm.

We know that $C[0,1]$ is a Banach space with respect to the uniform norm. Therefore, since $\{\varphi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C[0,1]$, there must exist a function $\varphi \in C[0,1]$ such that φ_n converge uniformly to φ as $n \to \infty$! Since uniform convergence implies pointwise convergence, we have

$$\varphi(x) \;=\; \lim_{n\to\infty} \varphi_n(x), \qquad x \in [0,1].$$

This continuous function φ is called the *Cantor–Lebesgue function* or, more picturesquely, the *Devil's staircase* on $[0,1]$.

If we like, we can extend φ to a continuous function that is defined on the entire real line $\mathbb{R}$ by reflecting its graph about the point $x = 1$ and declaring φ to be zero outside of $[0,2]$. This reflected Devil's staircase function is pictured in Figure 3.52. ♢

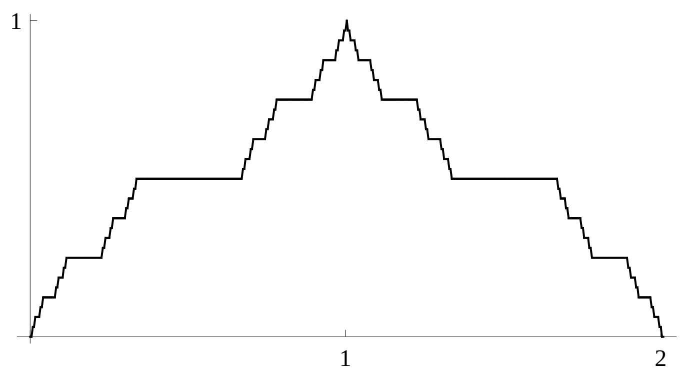

Fig. 3.52 The reflected Devil's staircase (Cantor–Lebesgue function).

The Cantor–Lebesgue function is differentiable at some points, but not all. Suppose that $x \in [0,1] \setminus C$, i.e., x is in the complement of the Cantor set in $[0,1]$. In this case, x belongs to one of the open intervals that is removed during some stage of the construction of the Cantor set. Let us say it is stage n, and let I denote that interval that is removed. Then φ_n is constant on I, and $\varphi_{n+1} = \varphi_n$ on I. Taking the limit, the Cantor–Lebesgue function φ must be constant on the open interval I, which contains the point x. Hence φ is differentiable at x, and $\varphi'(x) = 0$.

Thus φ is differentiable at every point $x \in [0,1] \setminus C$, and $\varphi'(x) = 0$ at each such x. If it was the case that $\varphi'(x) = 0$ for *every* $x \in [0,1]$, then φ would be a constant function. However, while φ is constant on many subintervals of $[0,1]$, those constants are different for each subinterval, and φ is not constant on $[0,1]$.

In summary, φ is differentiable at all points in the interval $[0,1]$ except for points $x \in C$. As discussed above, the measure of the Cantor set is zero.

Hence φ is differentiable except on a set whose measure is zero. A function that has this property is called a *singular function*; see [Heil18, Chap. 5].

Expanded discussion of Lipschitz Continuity

Recall that given an interval I in the real line, we let $C_b(I)$ be the set of all functions $f\colon I \to \mathbb{F}$ that are both bounded and continuous. The space $C_b^1(I)$ consists of all differentiable functions $f\colon I \to \mathbb{F}$ such that both f and f' are bounded (we focused on $C_b^1(\mathbb{R})$ in Section 3.50, but the same ideas apply to functions on an interval I). By definition, we have

$$C_b^1(I) \;\subsetneq\; C_b(I).$$

However, we often encounter functions that are "better than continuous" but are "not quite differentiable." The idea of Lipschitz continuity is one way of quantifying a measure of smoothness that lies between continuity and differentiability. We will define a study this notion in this section, and will define a space of of functions called $\mathrm{Lip}(I)$ that satisfies

$$C_b^1(I) \;\subsetneq\; \mathrm{Lip}(I) \;\subsetneq\; C_b(I).$$

To motivate the definition of Lipschitz continuity, let f be a real-valued function in $C_b^1(I)$. Then f is differentiable at every point of I and both f and f' are bounded, so the quantity

$$K \;=\; \|f'\|_{\mathrm{u}} \;=\; \sup_{t\in I} |f'(t)|$$

is finite. Choose any two points $x < y$ in I. Because f is real-valued, the Mean Value Theorem (Theorem 1.9.1) implies that there exists a point c between x and y such that

$$f'(c) \;=\; \frac{f(y)-f(x)}{y-x}.$$

Rearranging, we see that

$$|f(y)-f(x)| \;=\; |f'(c)|\,|y-x| \;\le\; K\,|y-x|. \tag{3.31}$$

Hence the distance between $f(x)$ and $f(y)$ is never more than K times the distance between x and y. Such a function does not expand the distance between points by more than the factor K (technically the word "compression" may be more appropriate if $K<1$, but the words "expand" or "dilate" are often used in this context regardless of whether K is larger or smaller than 1). Every real-valued differentiable function whose derivative is bounded has this property of limited expansion. On the other hand, the following exercise shows that there exist non-differentiable functions that have the same property.

Exercise 3.5.56. The *hat function* $W(x) = \max\{1 - |x|, 0\}$ is piecewise linear. Prove that W satisfies

$$|W(y) - W(x)| \le |y - x|, \qquad \text{all } x, y \in \mathbb{R},$$

even though W is not differentiable at the points -1, 0, and 1. $\diamondsuit$

We introduce the following terminology for functions that have this kind of behavior.

Definition 3.5.57 (Lipschitz Continuous Function). Let I be an interval in the real line. We say that a function $f \colon I \to \mathbb{F}$ is *Lipschitz continuous on I* if there exists a constant $K \ge 0$ such that

$$\forall\, x, y \in I, \quad |f(x) - f(y)| \le K\,|x - y|. \tag{3.32}$$

Any number K such that equation (3.32) holds is called a *Lipschitz constant* for f. $\diamondsuit$

The Lipschitz constant is not unique, for if K is a Lipschitz constant for f then so is any number larger than K. The smallest, or "optimal," Lipschitz constant for f is

$$K = \sup_{x \ne y} \frac{|f(y) - f(x)|}{|x - y|}.$$

If f is Lipschitz and we choose any $\varepsilon > 0$, then $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta = \varepsilon/K$. Hence every Lipschitz constant is uniformly continuous.

We collect the bounded Lipschitz functions to form the following space.

Definition 3.5.58. Given an interval I, we set

$$\mathrm{Lip}(I) = \{f \in C_b(I) : f \text{ is Lipschitz continuous on } I\}. \qquad \diamondsuit$$

We try to avoid extra parentheses where possible, so if I is given explicitly as $I = [a, b]$ then we write $\mathrm{Lip}[a, b]$ instead of $\mathrm{Lip}([a, b])$, and similarly for other types of intervals.

We have already seen that if f is a real-valued function in $C_b^1(I)$, then f is Lipschitz. The proof required the Mean-Value Theorem, which does not hold for complex-valued functions. However, by applying the Mean-Value Theorem to the real and imaginary parts of a complex-valued function, we can derive an analogous result for complex-valued functions in $C_b^1(I)$.

Lemma 3.5.59. *If f is differentiable everywhere on an interval I and both f and f' are bounded on I, then f is Lipschitz on I. If f is real-valued, then $K = \|f'\|_{\mathrm{u}}$ is a Lipschitz constant, while if f is complex-valued, then $K = 2^{1/2}\|f'\|_{\mathrm{u}}$ is a Lipschitz constant.* $\diamondsuit$

As a corollary, we see that $C_b^1(I) \subseteq \mathrm{Lip}(I)$. The hat function W is Lipschitz but not differentiable on the interval $[-1,1]$. If I is an arbitrary interval, then by appropriately dilating and translating the function so that it is supported within I, we obtain a function that is Lipschitz on I but does not belong to $C_b^1(I)$. Hence $C_b^1(I)$ is a *proper* subspace of $\mathrm{Lip}(I)$. Another very interesting example is constructed in part (c) of Problem 3.5.74, which shows that there exists a function that is *Lipschitz and differentiable everywhere on* I but does not belong to $C_b^1(I)$ because its derivative is *not continuous*.

By definition, $\mathrm{Lip}(I) \subseteq C_b(I)$. Are there bounded continuous functions that are not Lipschitz? Yes; here is an example.

Example 3.5.60. The Cantor–Lebesgue function φ takes the value 2^{-k} at the point $x = 3^{-k}$. Therefore,

$$|\varphi(3^{-k}) - \varphi(0)| \;=\; |2^{-k} - 0| \;=\; 2^{-k} \;=\; |3^{-k} - 0|^{\log_3 2}.$$

Set $\alpha = \log_3 2 \approx 0.63093$. Since $1 - \alpha > 0$,

$$\begin{aligned}
\lim_{k\to\infty} \frac{|\varphi(3^{-k}) - \varphi(0)|}{|3^{-k} - 0|} &= \lim_{k\to\infty} \frac{|\varphi(3^{-k}) - \varphi(0)|}{|3^{-k} - 0|^{\alpha}}\, |3^{-k} - 0|^{1-\alpha} \\
&= \lim_{k\to\infty} |3^{-k} - 0|^{1-\alpha} \\
&= \infty.
\end{aligned}$$

It follows that there is no finite constant K such that

$$|\varphi(3^{-k}) - \varphi(0)| \;\le\; K\, |3^{-k} - 0|$$

for every k, so φ is not Lipschitz on the interval $[0,1]$. $\diamondsuit$

If I is an arbitrary interval, then by appropriately dilating and translating the Cantor–Lebesgue function φ so that it is supported within I, we obtain a function that is continuous but not Lipschitz. Thus, $\mathrm{Lip}(I)$ is a proper subspace of $C_b(I)$. Combining this with the fact that $C_b^1(I)$ is a proper subspace of $\mathrm{Lip}(I)$, we see that

$$C_b^1(I) \;\subsetneq\; \mathrm{Lip}(I) \;\subsetneq\; C_b(I).$$

In this sense, the space of Lipschitz functions is intermediate between $C_b^1(I)$ and $C_b(I)$.

Problem 3.5.76 shows that although the Cantor–Lebesgue function is not Lipschitz, it does satisfy a weaker *Hölder continuity* condition. Problem 4.2.15 shows that the space of Lipschitz functions is a Banach space with respect to an appropriate norm, and Problem 3.5.77 does the same for the space of Hölder continuous functions.

Extra Problems

3.5.61. Show that if I is an interval in the real line other than $[a,b]$, then $C_b(I) \subsetneq C(I)$.

3.5.62. Give an example of a Banach space X that contains a dense subspace S and a closed subspace M such that $S \cap M = \{0\}$.

3.5.63. (a) Let X, Y be metric spaces. Given a function $f\colon X \to Y$, prove that f is continuous if and only if the following implication holds:

$$F \text{ is closed subset of } Y \implies f^{-1}(F) \text{ is a closed subset of } X.$$

(b) Show by example that the direct image of a closed set under a continuous function need not be closed.

(c) Show by example that the inverse image of a compact set under a continuous function need not be compact.

3.5.64. We say that a function $f\colon \mathbb{R} \to \mathbb{F}$ is *periodic* if there exists some $a \in \mathbb{R}$ such that $f(x-a) = f(x)$ for every $x \in \mathbb{R}$. Prove that every periodic function $f \in C(\mathbb{R})$ is bounded and uniformly continuous on $\mathbb{R}$.

3.5.65. Let X be a metric space, and suppose $f_n,\ f \in C_b(X)$. Show that if $f_n \to f$ uniformly and $x_n \to x$ in X, then $f_n(x_n) \to f(x)$.

3.5.66. (a) Suppose that $f_n,\ f \in C_b(\mathbb{R})$, and f_n converges pointwise to f, i.e., $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$. Show that

$$\|f\|_{\mathrm{u}} \le \liminf_{n\to\infty} \|f_n\|_{\mathrm{u}} \stackrel{\text{def}}{=} \sup_{n\in\mathbb{N}} \inf_{k\ge n} \|f_k\|_{\mathrm{u}}.$$

Give an example that shows that strict inequality is possible.

(b) Prove that if $f_n \to f$ uniformly in $C_b(\mathbb{R})$, then $\|f\|_{\mathrm{u}} = \lim_{n\to\infty} \|f_n\|_{\mathrm{u}}$.

3.5.67. Let $C(\mathbb{R}^d)$ be the set of all scalar-valued functions $f\colon \mathbb{R}^d \to \mathbb{F}$. How should the spaces $C_b(\mathbb{R}^d)$, $C_0(\mathbb{R}^d)$, and $C_c(\mathbb{R}^d)$ be defined? How should the uniform norm for functions on $\mathbb{R}^d$ be defined? Prove that $C_b(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$ are complete with respect to the uniform norm, but $C_c(\mathbb{R}^d)$ is not.

3.5.68. For this problem we take $X = C_0(\mathbb{R})$, under the uniform norm. Let $f(x) = e^{-|x|}$ and set

$$E = \{cf : c \in \mathbb{R}\}.$$

Prove the following statements.

(a) If $g = cf \in E$ and $r > 0$, then there exists a function $h \in C_c(\mathbb{R})$ such that $\|g-h\|_{\mathrm{u}} < r$.

(b) E contains no open subsets of $C_0(\mathbb{R})$, and therefore $E^\circ = \varnothing$.

(c) E is a closed subset of $C_0(\mathbb{R})$.

(d) $E' = E$, and therefore E is a perfect subset of $C_0(\mathbb{R})$.

3.5.69. Suppose that there was a norm $\|\cdot\|$ on $C[0,1]$ such that $\|f - f_n\| \to 0$ if and only $f_n \to f$ pointwise. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of Shrinking Triangles from Example 3.5.2.

(a) Explain why $\|f_n\| \neq 0$.

(b) Let $g_n = f_n/\|f_n\|$. Show that there exists a function g such that $g_n \to g$ pointwise.

(c) What is $\|g_n\|$? What is $\|g\|$? Why is this a contradiction?

Conclude that no such norm exists. (For this reason, we say that pointwise convergence of functions on $[0,1]$ is not a *normable* convergence criterion.)

3.5.70. Recall that $C(\mathbb{R})$ is the space of all continuous scalar-valued functions on $\mathbb{R}$. This space includes unbounded functions (such as $f(x) = x^2$), so the uniform norm $\|\cdot\|_{\mathrm{u}}$ is not a norm on $C(\mathbb{R})$ (why not?).

For each $N \in \mathbb{N}$, define

$$|||f|||_N \;=\; \sup_{x\in[-N,N]} |f(x)|, \qquad f \in C(\mathbb{R}).$$

Prove the following statements.

(a) If a, b, $c \geq 0$ and $a \leq b + c$, then $\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}$.

(b) The following is a metric on $C(\mathbb{R})$:

$$\mathrm{d}(f,g) \;=\; \sum_{N=1}^{\infty} 2^{-N} \frac{|||f-g|||_N}{1 + |||f-g|||_N}, \quad f,\, g \in C(\mathbb{R}).$$

3.5.71. Given a metric space X, let $C(X)$ be the space of all continuous, scalar-valued functions $f\colon X \to \mathbb{F}$, and let $C_b(X)$ be the subspace of $C(X)$ that contains all bounded, continuous functions $f\colon X \to \mathbb{F}$.

(a) Prove that $C_b(X)$ is a vector space, the uniform norm

$$\|f\|_{\mathrm{u}} \;=\; \sup_{x\in X} |f(x)|, \qquad f \in C_b(X),$$

is a norm on $C_b(X)$, and $C_b(X)$ is a Banach space with respect to the uniform norm.

(b) Prove that $C_b(X)$ is closed with respect to products of functions, i.e., if f, $g \in C_b(X)$ then $fg \in C_b(X)$. Further, show that the following *submultiplicative norm inequality* is satisfied:

$$\|fg\|_{\mathrm{u}} \;\leq\; \|f\|_{\mathrm{u}}\, \|g\|_{\mathrm{u}}, \qquad f, g \in C_b(X).$$

(c) Suppose that $g \in C_b(X)$ and there exists an $\varepsilon > 0$ such that $|g(x)| \geq \varepsilon$ for every $x \in X$. Prove that $1/g \in C_b(X)$.

(d) Show that if X is compact, then

$$C(X) = C_b(X)$$

and $C(X)$ is a Banach space with respect to the uniform norm.

(e) Let $X = C_b(\mathbb{R})$, and define $\delta\colon X \to \mathbb{F}$ by

$$\delta(f) = f(0), \qquad f \in X = C_b(\mathbb{R}).$$

For example, if $f(x) = \cos x$, then $\delta(f) = \cos 0 = 1$. Prove that δ is continuous. Does δ belong to $C_b(X)$?

3.5.72. Let $\|\cdot\|_1$ be the norm on $C[0,1]$ defined in Theorem 3.2.6.

(a) Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of "Shrinking Triangles" constructed in Example 3.5.2. For each $n \in \mathbb{N}$ define $g_n(x) = nf_n(x)$, and set $g(x) = 0$ (see Figure 3.53). Show that $g_n(x) \to g(x)$ for every $x \in [0,1]$, but $\|g - g_n\|_1$ does not converge to zero. Thus, pointwise convergence *does not* imply convergence with respect to the norm $\|\cdot\|_1$ in general.

Fig. 3.53 Graph of the function g_n from Problem 3.5.72(a).

(b) Given an interval $[c,d] \subseteq [0,1]$, let $W_{[c,d]}$ be the hat function of height 1 on the interval $[c,d]$. Explicitly,

$$W_{[c,d]} = \begin{cases} 0, & a \le x \le c, \\ \text{linear}, & c < x < \frac{c+d}{2}, \\ 1, & x = \frac{c+d}{2}, \\ \text{linear}, & \frac{c+d}{2} < x < d, \\ 0, & d \le x \le b. \end{cases}$$

Define

$$h_1 = W_{[0,1]},$$
$$h_2 = W_{[0,\frac{1}{2}]}, \qquad h_3 = W_{[\frac{1}{2},1]},$$
$$h_4 = W_{[0,\frac{1}{3}]}, \qquad h_5 = W_{[\frac{1}{3},\frac{2}{3}]}, \quad h_6 = W_{[\frac{2}{3},1]},$$

$$h_7 = W_{[0,\frac{1}{4}]}, \qquad h_8 = W_{[\frac{1}{4},\frac{1}{2}]}, \quad h_9 = W_{[\frac{1}{2},\frac{3}{4}]}, \qquad h_{10} = W_{[\frac{3}{4},1]},$$

$$h_{11} = W_{[0,\frac{1}{5}]}, \quad h_{12} = W_{[\frac{1}{5},\frac{2}{5}]}, \quad h_{13} = W_{[\frac{2}{5},\frac{3}{5}]}, \quad h_{14} = W_{[\frac{3}{5},\frac{4}{5}]}, \quad h_{15} = W_{[\frac{4}{5},1]},$$

and so forth. Picturing the graphs of these functions as triangles, the triangles march from left to right across the interval $[0,1]$, then shrink in size and march across the interval again, and do this over and over (see Figure 3.54).

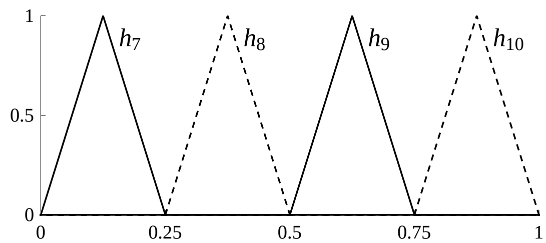

Fig. 3.54 Graphs of the functions h_7, h_8, h_9, and h_{10} from Problem 3.5.72(b).

Let $h = 0$ (the zero function). Prove that $\lim_{n\to\infty} \|h - h_n\|_1 = 0$, but it is *not* true that $\lim_{n\to\infty} h_n(x) = h(x)$ for every $x \in [0,1]$. Thus, convergence with respect to the norm $\|\cdot\|_1$ *does not* imply pointwise convergence in general.

3.5.73. Define $T\colon C[0,1] \to C[0,1]$ by

$$Tf(x) \;=\; x \;+\; \int_0^x t f(t)\,dt, \qquad f \in C[0,1].$$

Prove that T satisfies the hypotheses of the Banach Fixed Point Theorem (see Problem 2.9.20), and the fixed point of T satisfies the differential equation $f'(x) = x f(x) + 1$.

Problems on Lipschitz Functions

3.5.74. Given a, $b > 0$, define

$$f(x) \;=\; \begin{cases} |x|^a \sin |x|^{-b}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove the following statements.

(a) If $a \geq 1 + b$, then $f \in \mathrm{Lip}[-1,1]$.

(b) If $a > 1 + b$, then $f \in C^1[-1,1]$, i.e., f is differentiable everywhere on the interval $[-1,1]$ and both f and f' are bounded on $[-1,1]$.

(c) If $a = 1 + b$, then $f \notin C^1[-1,1]$.

(d) Show that the function f defined in Example 3.50.50 is differentiable at every point, f' is bounded but not continuous on $[-1,1]$, and $f \in \mathrm{Lip}[-1,1]$.

3.5.75. Suppose that $f\colon I \to \mathbb{F}$ is differentiable everywhere on an interval I. Prove that f is Lipschitz if and only if f' is bounded.

3.5.76. Let I be an interval. We say that a function $f\colon I \to \mathbb{F}$ is *Hölder continuous* on I with exponent $\alpha > 0$ if there exists a constant $K > 0$ such that

$$\forall\, x, y \in I, \quad |f(x) - f(y)| \;\le\; K\,|x-y|^{\alpha}.$$

Note that a Lipschitz function is a function that is Hölder continuous with exponent $\alpha = 1$.

(a) Show that if f is Hölder continuous for some exponent $\alpha > 0$, then f is uniformly continuous on I. Further, if $\alpha > 1$, then f is constant.

(b) Show that the function $f(x) = |x|^{1/2}$ is Hölder continuous on $\mathbb{R}$ for all exponents $0 < \alpha \le \frac{1}{2}$ but not for any exponent $\alpha > \frac{1}{2}$, while

$$g(x) \;=\; \begin{cases} 0, & x \le 0, \\ -\frac{1}{\ln x}, & 0 < x < \frac{1}{2}, \\ \frac{1}{\ln 2}, & x \ge \frac{1}{2}, \end{cases}$$

is uniformly continuous on $\mathbb{R}$ but is not Hölder continuous for *any* exponent $\alpha > 0$.

(c) Prove that if $0 < \alpha \le \log_3 2 \approx 0.6309\ldots$, then the Cantor–Lebesgue function φ is Hölder continuous on $[0,1]$ with exponent α, but φ is not Hölder continuous for any exponent $\alpha > \log_3 2$.

3.5.77. Given $0 < \alpha < 1$, let $C^{\alpha}(I)$ be the space of all bounded functions that are Hölder continuous with exponent α on I, i.e.,

$$C^{\alpha}(I) \;=\; \bigl\{f \in C_b(I) : f \text{ is Hölder continuous with exponent } \alpha\bigr\}.$$

Show that $C^{\alpha}(I)$ is a Banach space with respect to the norm

$$\|f\|_{C^{\alpha}} \;=\; \|f\|_{\mathrm{u}} + \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}}, \qquad f \in C^{\alpha}(I).$$

Problems on Convex Functions

3.5.78. This problem introduces *convex functions*, establishes the *Discrete Jensen Inequality*, and uses these to give another proof of equation (3.8).

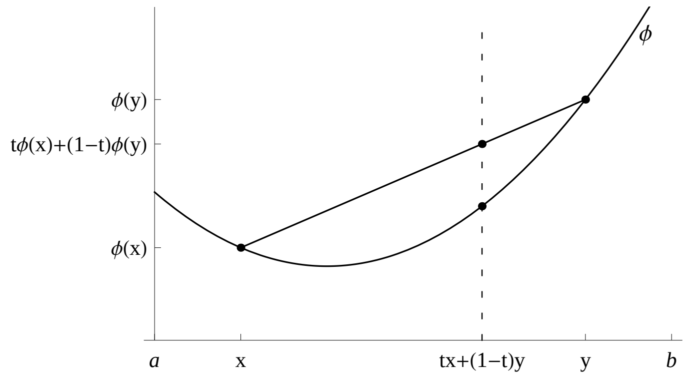

Fig. 3.55 Graph of a convex function ϕ.

Fix $-\infty \le a < b \le \infty$. We say that a function $\phi\colon (a,b) \to \mathbb{R}$ is *convex* on the open interval (a,b) if

$$\forall\, a < x < y < b, \quad \forall\, 0 < t < 1, \quad \phi\bigl(tx + (1-t)y\bigr) \;\le\; t\,\phi(x) + (1-t)\,\phi(y).$$

In other words, as illustrated in Figure 3.55, on any subinterval $[x,y]$ of (a,b), the graph of ϕ lies on or below the line segment that joints the points $(x,\phi(x))$ and $(y,\phi(y))$. An analogous definition is made for *concave functions*.

(a) Assume $\phi\colon (a,b) \to \mathbb{R}$ is a convex function. Prove the *Discrete Jensen Inequality*: If $N \ge 2$, then for any points $x_1,\dots,x_N \in (a,b)$ and positive weights $t_1,\dots,t_N$ that satisfy $t_1 + \cdots + t_N = 1$, we have

$$\phi\biggl(\sum_{j=1}^N t_j x_j\biggr) \;\le\; \sum_{j=1}^N t_j \phi(x_j).$$

(b) Given a convex function ϕ on (a,b) and given $x \in (a,b)$, prove that the function

$$m(y) \;=\; \frac{\phi(y)-\phi(x)}{y-x}, \qquad y \in (a,b),\, y \ne x,$$

is monotone increasing on (a,x) and monotone increasing on (x,b). Also show that if $z < x < y$, then $m(z) \le m(y)$.

(c) Let $\phi\colon (a,b) \to \mathbb{R}$ be given. Prove that ϕ is convex if and only if for all $a < x < y < z < b$ we have

$$\frac{\phi(y)-\phi(x)}{y-x} \;\le\; \frac{\phi(z)-\phi(x)}{z-x}.$$

(d) Show that if $b_1\ b_2 > 0$, then for any $a_1,\, a_2 \in \mathbb{R}$ we have

$$\min\left\{\frac{a_1}{b_1},\, \frac{a_2}{b_2}\right\} \;\le\; \frac{a_1+a_2}{b_1+b_2} \;\le\; \max\left\{\frac{a_1}{b_1},\, \frac{a_2}{b_2}\right\}.$$

(e) Suppose that a function $\phi\colon (a,b) \to \mathbb{R}$ is differentiable at every point of (a,b) and ϕ' is monotone increasing on (a,b). Prove that ϕ is convex.

(f) Prove that the following functions are convex on the given intervals:

$$\begin{aligned} x^p &\quad \text{on } (0,\infty), && \text{where } 1 \le p < \infty,\\ e^{ax} &\quad \text{on } (-\infty,\infty), && \text{where } a \in \mathbb{R},\\ -\ln x &\quad \text{on } (0,\infty). \end{aligned}$$

(g) Use the Discrete Jensen Inequality and the fact that e^x is convex to give another proof of the fact that if $a,\, b \ge 0$ and $1 < p < \infty$ then

$$ab \;\le\; \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

3.5.79. Fix $-\infty \le a < b \le \infty$, and assume that ϕ is a convex function on (a,b). Prove that ϕ has the following properties.

(a) ϕ is right-differentiable on (a,b), i.e., for each point $x \in (a,b)$ the limit

$$\phi'_+(x) \;=\; \lim_{y\to x^+} \frac{\phi(y)-\phi(x)}{y-x}$$

exists and is finite. Similarly, ϕ is left-differentiable at each point of (a,b).

(b) If $a < x < y < b$, then

$$\phi'_+(x) \;\le\; \frac{\phi(y)-\phi(x)}{y-x} \;\le\; \phi'_-(y),$$

and $\phi'_-(x) \le \phi'_+(x)$.

(c) ϕ'_+ and ϕ'_- are monotone increasing on (a,b).

(d) ϕ is continuous on (a,b). (In fact, ϕ is *Lipschitz* on any closed interval $[c,d] \subseteq (a,b)$.)

(e) ϕ is differentiable at all but at most countably many points in (a,b). That is, $\phi'_+(x) = \phi'_-(x)$ except for countably many values of x.

Hints: (a) Apply the Mean-Value Theorem to obtain points $\xi_1 < \xi_2$ such that $\frac{\phi(y)-\phi(x)}{y-x} = \phi'(\xi_1)$ and $\frac{\phi(z)-\phi(y)}{z-y} = \phi'(\xi_2)$. Then apply the inequalities from part (b) to estimate $\min\{\phi'(\xi_1),\, \phi'(\xi_2)\}$.

(b) Part (b) of Exercise 3.5.78 shows that if $z < x < y$, then $m(z) \le m(y)$.

(d) Use part (b) to show that $|\phi(y) - \phi(x)| \le K\,|y-x|$ where $K = \max\bigl\{|\phi'_+(c)|,\, |\phi'_-(d)|\bigr\}$.

(e) Suppose that y is a point of continuity for ϕ'_+. Use part (b) to show that

$$\phi'_-(y) \;=\; \lim_{x\to y^-} \frac{\phi(y)-\phi(x)}{y-x} \;\geq\; \lim_{x\to y^-} \phi'_+(x) \;=\; \phi'_+(y).$$

3.6 Equivalent Norms

As far as convergence in a normed space X is concerned, we can replace the norm on X with any equivalent norm without changing the meaning of convergence in the space, or the topology for the space. To illustrate this, let X be a finite-dimensional vector space. The next exercise constructs a family of norms on X, analogous to the ℓ^p norms on $\mathbb{F}^d$ defined in Theorem 3.2.5, and shows that all of the norms in this family are equivalent. Hence each of these norms determines the same convergence criterion on X, i.e., a sequence converges with respect to one of these norms if and only if it converges with respect to all of the others.

Extra Problems

3.6.50. Suppose that $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on a vector space X, and there exists a constant $C > 0$ such that $\|x\|_a \leq C\,\|x\|_b$ for every $x \in X$. Prove the following statements.

(a) If $x_n,\, x \in X$ and $x_n \to x$ with respect to $\|\cdot\|_b$, then $x_n \to x$ with respect to $\|\cdot\|_a$.

(b) If $U \subseteq X$ and U is open with respect to $\|\cdot\|_a$, then U is open with respect to $\|\cdot\|_b$.

3.6.51. Let X be a vector space. Prove that equivalence of norms is an equivalence relation on the class of all norms on X. In other words, prove that the following three statements hold for all norms $\|\cdot\|_a$, $\|\cdot\|_b$, and $\|\cdot\|_c$ on X (where we write $\|\cdot\|_1 \asymp \|\cdot\|_2$ to mean that the two norms are equivalent).

(a) Reflexivity: $\|\cdot\|_a \asymp \|\cdot\|_a$.

(b) Symmetry: If $\|\cdot\|_a \asymp \|\cdot\|_b$, then $\|\cdot\|_b \asymp \|\cdot\|_a$.

(c) Transitivity: If $\|\cdot\|_a \asymp \|\cdot\|_b$ and $\|\cdot\|_b \asymp \|\cdot\|_c$, then $\|\cdot\|_a \asymp \|\cdot\|_c$.

3.6.52. (Expansion of Problem 3.6.9).

Suppose that $\|\cdot\|_a$ and $\|\cdot\|_b$ are two norms on a vector space X. Let $B^a_r(x)$ and $B^b_r(x)$ denote the open balls of radius r centered at $x \in X$ with respect to the norms $\|\cdot\|_a$ and $\|\cdot\|_b$, respectively. Prove that the following three statements are equivalent.

(a) $\|\cdot\|_a \asymp \|\cdot\|_b$.

(b) There exists some $C > 0$ such that

$$\frac{1}{C}\,\|x\|_a \;\leq\; \|x\|_b \;\leq\; C\,\|x\|_a, \qquad x \in X.$$

(c) There exists some $r > 0$ such that

$$B^a_{1/r}(0) \;\subseteq\; B^b_1(0) \;\subseteq\; B^a_r(0).$$

3.6.53. Given $1 \leq p < q \leq \infty$, prove that $\|\cdot\|_p$ and $\|\cdot\|_q$ are not equivalent norms on ℓ^p.

Remark: $\ell^p \subseteq \ell^q$, so $\|\cdot\|_q$ is a norm on ℓ^p (when restricted from ℓ^q to ℓ^p).

3.6.54. Given $1 \leq p < q \leq \infty$, prove that $\|\cdot\|_p$ and $\|\cdot\|_q$ are not equivalent norms on $C[0,1]$.

3.7 Norms on Finite-Dimensional Spaces

In summary, if X is a finite-dimensional vector space, then every norm on X is equivalent and therefore induces the same topology on X (which we call the *norm topology* on X). In particular, every norm on X is equivalent to the norm $\|\cdot\|_2$ (see Problem 3.6.5). Consequently the norm topology on X can be identified with the usual topology on $\mathbb{F}^d$, in much the same manner as Exercise 2.8.51 showed that the topology of $\mathbb{C}^d$ can be identified with that of $\mathbb{R}^{2d}$. This idea can be used to solve the following exercise, which extends the Heine–Borel Theorem to X.

Exercise 3.7.50. Let X be a finite-dimensional normed space X. Using the notation of Problem 3.6.5, if $\mathcal{B} = \{e_1, \dots, e_d\}$ is a basis for X, then

$$\|x\|_2 \;=\; \bigl(|c_1(x)|^2 + \cdots + |c_d(x)|^2\bigr)^{1/2}, \qquad x \in X,$$

defines a norm on X. Let K be a closed and bounded subset of X, and prove that

$$\widetilde{K} \;=\; \Bigl\{\bigl(c_1(x), \dots, c_d(x)\bigr) \;:\; x \in K\Bigr\}$$

is a closed and bounded set in $\mathbb{F}^d$. Conclude that $\widetilde{K}$ is compact, and use this to prove that K is compact in X. $\diamondsuit$

In contrast, he following two exercises will show that if X is an infinite-dimensional normed space then there exists a subset of X that is closed and bounded but not sequentially compact (and hence not compact by Theorem 2.8.9).

Exercise 3.7.51. Let X be a normed space, let M be a proper, closed subspace of X, and fix $\varepsilon > 0$. This exercise will prove *F. Riesz's Lemma*: There is a unit vector $y \in X$ whose distance from M is at least $1 - \varepsilon$.

Choose $u \in X \setminus M$ and set $a = \operatorname{dist}(u, M) > 0$. Let $\delta > 0$ be small enough that $a/(a + \delta) > 1 - \varepsilon$, and let $v \in M$ satisfy $a \le \|u - v\| < a + \delta$. Set

$$y \;=\; \frac{(u - v)}{\|u - v\|}.$$

Show that $\|y\| = 1$ and

$$\operatorname{dist}(y, M) \;=\; \inf_{x \in M} \|y - x\| \;>\; 1 - \varepsilon. \qquad \diamondsuit$$

The next exercise can be solved by applying Exercise 3.7.51 repeatedly. Note that by Problem 3.7.4, if $x_1, \dots, x_n$ are finitely many elements of X, then $\operatorname{span}\{x_1, \dots, x_n\}$ is closed because it is a finite-dimensional subspace of X.

Exercise 3.7.52. Let X be any infinite-dimensional normed space. Prove that the closed unit disk $D = \{x \in X : \|x\| \le 1\}$ in X is not sequentially compact, and therefore is not compact. $\diamondsuit$

We will study *inner product spaces* and *Hilbert spaces* in Chapter 5. If X happens to be a Hilbert space, then the machinery of orthogonal complements can be used to give a much more direct (and elegant) solution to Exercises 3.7.51 and 3.7.52 (see Problem 5.9.7).

3.50 Extra Section: Spaces of Differentiable Functions

This extra section expands on the material in Section 3.5 where the space $C_b^1(I)$ is introduced and discussed.

Most of Section 3.5 deals with spaces whose elements are continuous functions. We will generalize some of the results obtained there to spaces of m-times differentiable functions. For simplicity of presentation, in this section we we will take the domain of our functions to be $I = \mathbb{R}$. However, only small modifications need to be made if we want to consider functions whose domain is a different type interval (essentially, one-sided instead of two-sided limits need to be used at any boundary point of the interval).

We begin with a technical issue that complicates the issue of how best to define spaces of differentiable functions. The problem is that it is not true that a differentiable function must have a continuous derivative. Here is an example of a function that is differentiable at every point, yet its derivative is not continuous.

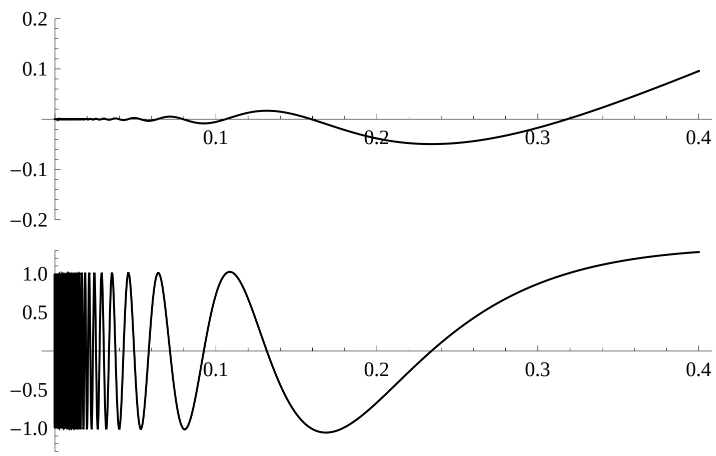

Fig. 3.56 The function f from Example 3.50.50 (top), and its derivative f' (bottom), for $0 < x < 0.4$. Note the differing vertical scales on the two graphs.

Example 3.50.50. Define

$$f(x) \;=\; \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

This function is continuous at every point (why?). We will prove that $f'(x)$ *exists at every point* x, yet f' is *discontinuous at* $x = 0$.

To see that f is differentiable at points other than $x = 0$, note that the function $1/x$ is differentiable on $(0, \infty)$ and $\sin x$ is differentiable on all of $\mathbb{R}$.

Therefore their composition $\sin(1/x)$ is differentiable on $(0, \infty)$. Since x^2 is also differentiable everywhere, it follows that the product $f(x) = x^2 \sin(1/x)$ is differentiable on $(0, \infty)$. Applying the product rule and the chain rule, the derivative of f on this interval is

$$f'(x) \;=\; 2x \, \sin \frac{1}{x} \;+\; x^2 \Bigl(\cos \frac{1}{x}\Bigr) \Bigl(-\frac{1}{x^2}\Bigr) \;=\; 2x \, \sin \frac{1}{x} \;-\; \cos \frac{1}{x}. \tag{3.33}$$

A similar calculation shows that f is differentiable on $(-\infty, 0)$, and $f'(x)$ is given by equation (3.33) for $x \in (-\infty, 0)$.

What about the point $x = 0$? Here we use the definition of the derivative to show that $f'(0)$ exists:

$$f'(0) \;=\; \lim_{t \to 0} \frac{f(0+t) - f(0)}{t - 0} \;=\; \lim_{t \to 0} \frac{t^2 \sin(1/t)}{t} \;=\; \lim_{t \to 0} t \sin(1/t) \;=\; 0.$$

Thus f is differentiable at every point $x \in \mathbb{R}$.

The graphs of f and f' are pictured in Figure 3.56. Considering the figure, it appears that $f'(x)$ cannot converge to any value as $x \to 0$, because $f'(x)$ oscillates more and more as x approaches zero. Indeed, applying equation (3.33), we see that

$$\lim_{x \to 0} f'(x) \;=\; \lim_{x \to 0} \Bigl(2x \, \sin \frac{1}{x} \;-\; \cos \frac{1}{x}\Bigr) \quad \textit{does not exist.}$$

Therefore, even though $f'(0)$ exists, the derivative function f' is not continuous at the point $x = 0$. $\diamondsuit$

For this and other reasons, the space of *all* differentiable functions can be difficult to work with. Usually we impose some restrictions to create a "nicer" space. For example, we can consider the space of all differentiable functions *which have a continuous derivative.* We call this space $C^1(\mathbb{R})$. Extending to higher orders, we let $C^m(\mathbb{R})$ denote the set of all of all m-times differentiable functions on $\mathbb{R}$ such that f and each derivative $f', \dots, f^{(m)}$ is continuous, i.e.,

$$C^m(\mathbb{R}) \;=\; \bigl\{f \in C(\mathbb{R}) \,:\, f, f', \dots, f^{(m)} \in C(\mathbb{R})\bigr\}.$$

On the preceding line, when we write $f, f', \dots, f^{(m)} \in C(\mathbb{R})$ we implicitly mean that $f, f', \dots, f^{(m)}$ all exist (i.e., f is m-times differentiable) and each of these functions belongs to $C(\mathbb{R})$.

The space $C^m(\mathbb{R})$ is still too "large" to define a convenient norm, so we look at the following subspaces of $C^m(\mathbb{R})$:

$$\begin{aligned}
C_b^m(\mathbb{R}) \;&=\; \bigl\{f \in C_b(\mathbb{R}) \,:\, f, f', \dots, f^{(m)} \in C_b(\mathbb{R})\bigr\},\\
C_0^m(\mathbb{R}) \;&=\; \bigl\{f \in C_0(\mathbb{R}) \,:\, f, f', \dots, f^{(m)} \in C_0(\mathbb{R})\bigr\},\\
C_c^m(\mathbb{R}) \;&=\; \bigl\{f \in C_c(\mathbb{R}) \,:\, f, f', \dots, f^{(m)} \in C_c(\mathbb{R})\bigr\}.
\end{aligned}$$

That is, $C_b^m(\mathbb{R})$ is the space of m-times differentiable functions such that f and each derivative $f', \dots, f^{(m)}$ is continuous and bounded, and so forth. According to Problem 3.50.53, these spaces are nested in the following way:

$$C_c^m(\mathbb{R}) \subsetneq C_0^m(\mathbb{R}) \subsetneq C_b^m(\mathbb{R}) \subsetneq C^m(\mathbb{R}). \tag{3.34}$$

The inclusions on the preceding line are proper. For example, the function $f(x) = \sin x$ belongs to $C_b^m(\mathbb{R})$ because it and every derivative is bounded, but $f \notin C_0^m(\mathbb{R})$. Problem 3.50.53 asks for examples that show that the other inclusions in equation (3.34) are also proper.

After giving some examples, we will define the "standard norm" for the subspaces $C_b^m(\mathbb{R})$, $C_0^m(\mathbb{R})$, and $C_c^m(\mathbb{R})$, and determine which of these are complete with respect to that norm. The hardest case is actually $m = 1$; once we understand that case we can extend to higher orders by induction.

Sometimes it is useful to consider analogous spaces of infinitely differentiable functions. Although these are not normed spaces, we make the following definitions:

$$\begin{aligned}
C^\infty(\mathbb{R}) &= \bigl\{f \in C(\mathbb{R}) \ : f, f', \dots \in C(\mathbb{R})\bigr\}, \\
C_b^\infty(\mathbb{R}) &= \bigl\{f \in C_b(\mathbb{R}) : f, f', \dots \in C_b(\mathbb{R})\bigr\}, \\
C_0^\infty(\mathbb{R}) &= \bigl\{f \in C_0(\mathbb{R}) : f, f', \dots \in C_0(\mathbb{R})\bigr\}, \\
C_c^\infty(\mathbb{R}) &= \bigl\{f \in C_c(\mathbb{R}) : f, f', \dots \in C_c(\mathbb{R})\bigr\}.
\end{aligned}$$

It is not obvious that $C_c^\infty(\mathbb{R}) \neq \{0\}$. Some examples of nontrivial elements of $C_c^\infty(\mathbb{R})$ are constructed in Problem 2.10.6.

The Norm for $C_b^1(\mathbb{R})$

There are many ways to define a norm on $C_b^1(\mathbb{R})$. For example, since $C_b^1(\mathbb{R})$ is a subspace of $C_b(\mathbb{R})$, the uniform norm is a norm on $C_b^1(\mathbb{R})$. However, the following exercise shows that $C_b^1(\mathbb{R})$ is *not complete* with respect to the uniform norm.

Exercise 3.50.51. The *hat function* or *tent function* on the interval $[-1, 1]$ is the piecewise linear function W pictured in Figure 3.57. Explicitly,

$$W(x) = \max\bigl\{1 - |x|, 0\bigr\} = \begin{cases} 1 - x, & 0 \le x \le 1, \\ 1 + x, & -1 \le x \le 0, \\ 0, & |x| \ge 1. \end{cases} \tag{3.35}$$

The hat function is continuous, but it is not differentiable at every point. We will construct a differentiable "approximation" to W.

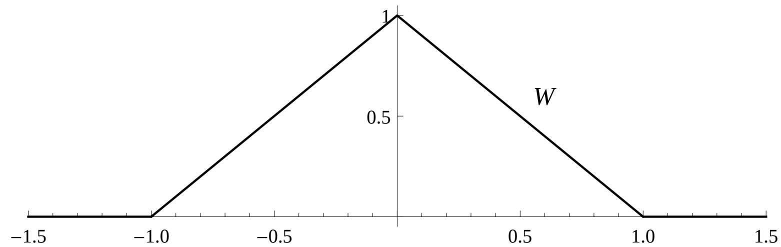

Fig. 3.57 The hat function W from Example 3.50.51.

Omitting the points $-1, 0$, or 1 where W is not differentiable, the derivative of W is

$$W'(x) \;=\; \begin{cases} 1, & -1 < x < 0, \\ -1, & 0 < x < 1, \\ 0, & |x| > 1. \end{cases}$$

Create a continuous approximation to W' by setting

$$f_k(x) \;=\; \begin{cases} 0, & x < 1 - \frac{1}{k}, \\ \text{linear}, & -1 - \frac{1}{k} \le x < -1, \\ 1, & -1 \le x < -\frac{1}{k}, \\ \text{linear}, & -\frac{1}{k} \le x < \frac{1}{k}, \\ -1, & \frac{1}{k} \le x < 1, \\ \text{linear}, & 1 \le x < 1 + \frac{1}{k}, \\ 0, & x \ge 1 + \frac{1}{k}. \end{cases}$$

The function f_5 is pictured in Figure 3.58.

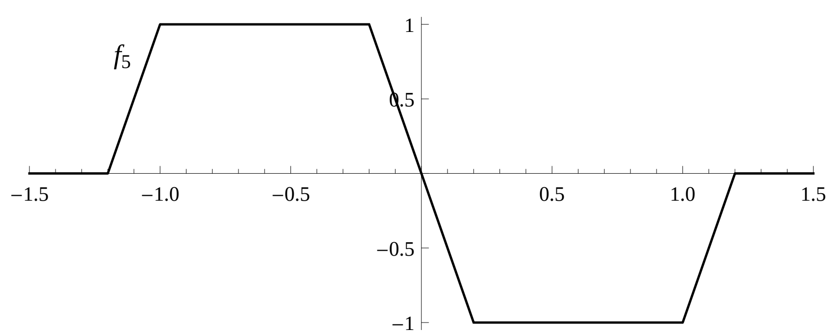

Fig. 3.58 The function f_5 from Example 3.50.51.

Now let g_k be the following indefinite integral of f_k:

$$g_k(x) \;=\; \int_{-2}^{x} f_k(t)\,dt, \qquad x \in \mathbb{R}.$$

The function g_5 is pictured in Figure 3.59. Looking at its graph, it appears that g_5 is a good approximation to W. Prove the following statements.

(a) $g_k \in C_c^1(\mathbb{R})$.

(b) g_k converges uniformly to W as $k \to \infty$.

(c) $\{g_k\}_{k\in\mathbb{N}}$ is a sequence in $C_b^1(\mathbb{R})$ that is Cauchy with respect to the uniform norm, but this sequence does not converge uniformly to an element of $C_b^1(\mathbb{R})$.

(d) $C_b^1(\mathbb{R})$ is not complete with respect to the uniform norm. $\diamondsuit$

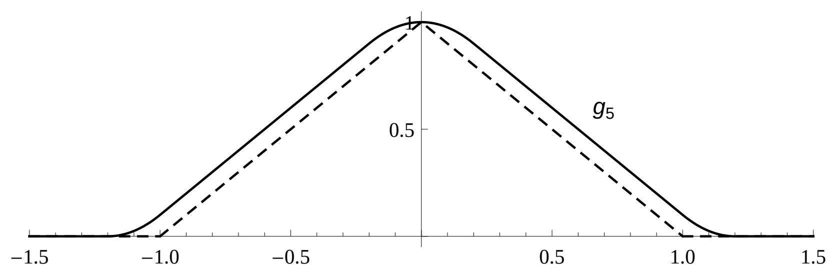

Fig. 3.59 The function g_5 from Example 3.50.51, with the hat function W in dashed lines for comparison.

In summary, $\|\cdot\|_{\mathrm{u}}$ is a norm on $C_b^1(\mathbb{R})$, but $C_b^1(\mathbb{R})$ is not complete with respect to this norm. We need to construct a different norm that incorporates information about the derivative. As a first attempt, let us define

$$|||f||| \;=\; \|f'\|_{\mathrm{u}}, \qquad f \in C_b^1(\mathbb{R}). \tag{3.36}$$

Unfortunately, this does not even define a norm on $C_b^1(\mathbb{R})$. For example, the constant function $f = 1$ satisfies

$$|||f||| \;=\; \|f'\|_{\mathrm{u}} \;=\; 0,$$

even though f is not the zero function. Hence $|||\cdot|||$ does not satisfy the uniqueness requirement of a norm.

The "correct" definition of a norm on $C_b^1(\mathbb{R})$ should incorporate information about *both* f and f'. We can do this by setting

$$\|f\|_{C_b^1} \;=\; \|f\|_{\mathrm{u}} + \|f'\|_{\mathrm{u}}, \qquad f \in C_b^1(\mathbb{R}). \tag{3.37}$$

According to Problem 3.5.25, this is a norm on $C_b^1(\mathbb{R})$, and $C_b^1(\mathbb{R})$ is complete with respect to this norm.

The case $m = 1$ (which is Problem 3.5.25) is actually the hard part. Assuming that problem has been done, we can extend to higher-order derivatives by induction. Given $m \in \mathbb{N}$, the norm on $C_b^m(\mathbb{R})$ is defined by

$$\|f\|_{C_b^m} \;=\; \|f\|_{\mathrm{u}} + \|f'\|_{\mathrm{u}} + \cdots + \|f^{(m)}\|_{\mathrm{u}}, \qquad f \in C_b^m(\mathbb{R}).$$

Exercise 3.50.52. Use induction to prove that $C_b^m(\mathbb{R})$ is a Banach space with respect to the norm $\|\cdot\|_{C_b^m}$. ♢

Problems 3.50.56 and 3.50.57 consider whether the spaces $C_0^m(\mathbb{R})$ and $C_c^m(\mathbb{R})$ are complete with respect to the norm $\|\cdot\|_{C_b^m}$ (yes for $C_0^m(\mathbb{R})$, but no for $C_c^m(\mathbb{R})$).

Extra Problems

3.50.53. Let $m \in \mathbb{N}$ be fixed.

(a) Exhibit a function in $C^m(\mathbb{R})$ that does not belong to $C_b^m(\mathbb{R})$.

(b) Exhibit a function in $C_b^m(\mathbb{R})$ that does not belong to $C_0^m(\mathbb{R})$.

(c) Let $\phi(x) = e^{-x^2}$ be the Gaussian function. Show that for each $n \in \mathbb{N}$ there exists a polynomial of degree n such that $\phi^{(n)}(x) = p_n(x)\, e^{-x^2}$. Prove that $\phi \in C_0^m(\mathbb{R})$, but $\phi \notin C_c^m(\mathbb{R})$.

3.50.54. A function $f \in C^\infty(\mathbb{R})$ is said to be *real analytic* at the origin if its Taylor series converges to $f(x)$ for all x in some neighborhood of 0. In other words, there must exist a $\delta > 0$ such that

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n, \qquad |x| < \delta.$$

Prove that the function γ constructed in Problem 2.10.6 is not real analytic at the origin.

3.50.55. Define

$$\gamma(x) \;=\; e^{-1/x^2}\, \chi_{(0,\infty)}(x) \;=\; \begin{cases} e^{-1/x^2}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Prove the following statements.

(a) For each $n \in \mathbb{N}$, there exists a polynomial p_n of degree $3n$ such that

$$\gamma^{(n)}(x) \;=\; p_n(x^{-1})\, e^{-x^{-2}}\, \chi_{(0,\infty)}(x).$$

(b) $\gamma \in C^\infty(\mathbb{R})$ and $\gamma^{(n)}(0) = 0$ for every $n \ge 0$.

(c) If $a < b$ then the function $f(x) = \gamma(x-a)\,\gamma(b-x)$ belongs to $C_c^\infty(\mathbb{R})$ and satisfies $f(x) > 0$ for $x \in (a,b)$ and $f(x) = 0$ for $x \notin (a,b)$.

3.50.56. Given $m \in \mathbb{N}$, prove that $C_0^m(\mathbb{R})$ is a Banach space with respect to the norm $\|\cdot\|_{C_b^m}$.

3.50.57. Let g be any function in $C_0^m(\mathbb{R})$ such that $g \notin C_c^m(\mathbb{R})$. Let θ be the function constructed in Problem 2.10.7 (so θ is infinitely differentiable, compactly supported, and identically 1 on $[-1,1]$). For each $k \in \mathbb{N}$, set

$$g_k(x) \;=\; g(x)\,\theta(x/k).$$

Prove the following statements.

(a) $g_k \in C_c^m(\mathbb{R})$.

(b) $\lim_{k\to\infty} \|g - g_k\|_{C_b^m} = 0$.

(c) $\{g_k\}_{k\in\mathbb{N}}$ is a sequence in $C_c^m(\mathbb{R})$ that is Cauchy with respect to the norm $\|\cdot\|_{C_b^m}$, but this sequence does not converge to an element of $C_c^m(\mathbb{R})$.

(d) $C_c^m(\mathbb{R})$ is not complete with respect to $\|\cdot\|_{C_b^m}$.

3.51 Extra Section: A Metric for $C(\mathbb{R})$

This extra section expands on Problem 3.5.70.

Since functions in $C(\mathbb{R})$ can be unbounded, the uniform norm $\|\cdot\|_{\mathrm{u}}$ is not a norm on $C(\mathbb{R})$. On the other hand, each function $f \in C(\mathbb{R})$ is bounded on each *bounded closed interval* $[a,b]$. Consequently, if we fix an integer $N \in \mathbb{N}$, then the quantity

$$|||f|||_N \;=\; \sup_{|x|\le N} |f(x)|$$

is finite for every $f \in C(\mathbb{R})$. In fact, considering a single, fixed N, the function $|||\cdot|||_N$ is nonnegative, homogeneous, and satisfies the Triangle Inequality, so it is a *seminorm* on $C(\mathbb{R})$ in the sense of Definition 3.1.1. However, it is not a norm, because there exist functions in $C(\mathbb{R})$ other than the zero function that satisfy $|||f|||_N = 0$.

Remark 3.51.50. Let $\chi_{[-N,N]}$ denote the *characteristic function* of the interval $[-N,N]$, i.e.,

$$\chi_{[-N,N]}(x) \;=\; \begin{cases} 1, & |x| \le N,\\ 0, & |x| > N. \end{cases}$$

Then $f \cdot \chi_{[-N,N]}$ is the function that equals f on $[-N,N]$ and is zero outside of $[-N,N]$. Therefore, an equivalent way to write $|||\cdot|||_N$ is

$$|||f|||_N \;=\; \|f \cdot \chi_{[-N,N]}\|_{\mathrm{u}}. \tag{3.38}$$

That is, the seminorm $|||f|||_N$ is simply the uniform norm of the restriction of f to the interval $[-N,N]$. If f is zero on $[-N,N]$ but nonzero somewhere outside of that interval, then we will have $|||f|||_N = 0$ even though f is not the zero function. ♢

We would like to collect all of these seminorms $|||\cdot|||_N$ together to create a norm or metric on $C(\mathbb{R})$. We could try summing over N, but unfortunately the series

$$\sum_{N=1}^{\infty} |||f|||_N \qquad \text{or even} \qquad \sum_{N=1}^{\infty} 2^{-N}\, |||f|||_N$$

need not converge for every $f \in C(\mathbb{R})$. On the other hand, no matter what $f \in C(\mathbb{R})$ we choose we always have

$$\frac{|||f|||_N}{1+|||f|||_N} \;\le\; 1, \qquad \text{all } N \in \mathbb{N},$$

so we can define

$$\|f\| \;=\; \sum_{N=1}^{\infty} 2^{-N} \frac{|||f|||_N}{1+|||f|||_N}, \qquad f \in C(\mathbb{R}).$$

This series converges for every $f \in C(\mathbb{R})$, but does it define a norm? Unfortunately, the answer is no, because $\|cf\| \neq |c|\,\|f\|$ in general. Even so, Problem 3.5.70 shows that

$$\mathrm{d}(f,g) \;=\; \|f-g\| \;=\; \sum_{N=1}^{\infty} 2^{-N}\,\frac{|||f-g|||_N}{1+|||f-g|||_N}, \qquad f,\,g \in C(\mathbb{R}), \tag{3.39}$$

is a *metric* on $C(\mathbb{R})$.

What kind of *convergence criterion* does the metric d correspond to? That is, what does it mean for f_k to converge to f with respect to the metric d? Instead of uniform convergence, we will see that convergence with respect to d is the type of convergence that is given in the following definition.

Definition 3.51.51 (Uniform Convergence on Compact Sets). Let f_k and f be functions in $C(\mathbb{R})$. If f_k converges uniformly to f on the interval $[-N,N]$ for each $N \in \mathbb{N}$, then we say that f_k *converges uniformly to f on compact sets.* $\diamondsuit$

Example 3.51.52. Let $f = 1$, i.e., f is the constant function that is identically 1. Let f_k be a continuous "cutoff" of f, similar to the cutoff illustrated in Figure 8.41 for a generic continuous function g. e.g.,

$$f_k(x) \;=\; \begin{cases} 1, & |x| \le k, \\ \text{linear}, & \text{on } [-k-1,-k] \text{ and on } [k,k+1], \\ 0, & |x| \ge k+1. \end{cases}$$

If we consider a particular fixed N, then for all $k \ge N$ we have $f_k = f$ on the interval $[-N,N]$. Hence f_k converges uniformly to f on the (fixed) interval $[-N,N]$. This is true for each individual N, so Definition 3.51.51 says that f_k converges uniformly to f on compact sets.

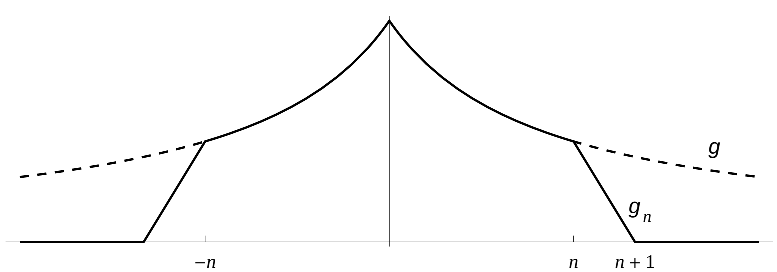

Fig. 3.60 A cutoff g_n for a function g.

On the other hand, f_k does not converge uniformly to f, because for *every* k we have

$$\|f-f_k\|_{\mathrm{u}} \;=\; \sup_{x\in\mathbb{R}} |f(x)-f_k(x)| \;\ge\; |f(k+2)-f_k(k+2)| \;=\; 1-0 \;=\; 1.$$

Thus $\|f-f_k\|_{\mathrm{u}}$ does not converge to zero, so f_k does not converge uniformly to f, even though f_k does converge *uniformly on compact sets* to f. $\diamondsuit$

The following lemma shows that convergence with respect to the metric d is precisely the same as uniform convergence on compact sets.

Lemma 3.51.53. *Given functions $f_k,\ f \in C(\mathbb{R})$, the following two statements are equivalent.*

(a) *f_k converges to f with respect to the metric* d, *i.e.,*

$$\lim_{k\to\infty} \mathrm{d}(f_k,f) \;=\; 0.$$

(b) *f_k converges uniformly to f on compact sets, i.e.,*

$$\forall\, N\in\mathbb{N}, \quad \lim_{k\to\infty} \|(f-f_k)\cdot\chi_{[-N,N]}\|_{\mathrm{u}} \;=\; 0.$$

Proof. (a) $\Rightarrow$ (b). Suppose that $\mathrm{d}(f_k,f)\to 0$ as $k\to\infty$. Given $N\in\mathbb{N}$, we must show that $|||f-f_k|||_N \to 0$ as $k\to\infty$.

Choose $\varepsilon>0$, and set

$$\eta \;=\; \frac{\varepsilon}{2^N(1+\varepsilon)}.$$

Since $\eta>0$ and $\mathrm{d}(f_k,f)\to 0$, there must exist some $K>0$ such that

$$k>K \implies \mathrm{d}(f_k,f)<\eta.$$

Therefore, for $k>K$ we have

$$\begin{aligned}
\frac{|||f-f_k|||_N}{1+|||f-f_k|||_N} &= 2^N 2^{-N}\frac{|||f-f_k|||_N}{1+|||f-f_k|||_N}\\
&\le 2^N \sum_{M=1}^{\infty} 2^{-M}\frac{|||f-f_k|||_M}{1+|||f-f_k|||_M}\\
&= 2^N\,\mathrm{d}(f-f_k)\\
&< 2^N\,\eta\\
&= \frac{\varepsilon}{1+\varepsilon}.
\end{aligned}$$

Since the function $\frac{x}{1+x}$ is monotone increasing on $(0,\infty)$, it follows that for all $k>K$ we have

$$|||f-f_k|||_N \;<\; \varepsilon.$$

This proves that $|||f-f_k|||_N\to 0$ as $k\to\infty$.

(b) ⇒ (a). Suppose that $|||f - f_k|||_N \to 0$ as $k \to \infty$ for every $N \in \mathbb{N}$. Fix $\varepsilon > 0$, and let M be large enough that

$$\sum_{N=M+1}^{\infty} 2^{-N} \;<\; \varepsilon.$$

Then let K be large enough that

$$k > K \quad \Longrightarrow \quad |||f - f_k|||_N \;<\; \varepsilon, \quad N = 1, \dots, M.$$

Then for all $k > K$ we have

$$\begin{aligned} \mathrm{d}(f_k, f) \;&=\; \sum_{N=1}^{M} 2^{-N} \frac{|||f - f_k|||_N}{1 + |||f - f_k|||_N} \;+\; \sum_{N=M+1}^{\infty} 2^{-N} \frac{|||f - f_k|||_N}{1 + |||f - f_k|||_N} \\ &\leq\; \sum_{N=1}^{M} 2^{-N} \frac{\varepsilon}{1+0} \;+\; \sum_{N=M+1}^{\infty} 2^{-N} \cdot 1 \\ &\leq\; \varepsilon \sum_{N=1}^{\infty} 2^{-N} \;+\; \varepsilon \\ &=\; 2\varepsilon. \end{aligned}$$

Hence $\mathrm{d}(f_k, f) \to 0$ as $k \to \infty$. □

Example 3.51.52 shows that a sequence that converges uniformly on compact sets need not converge uniformly. On the other hand, if f_k converges uniformly to f then we have $\|f - f_k\|_{\mathrm{u}} \to 0$ as $n \to \infty$. Therefore, for any particular N,

$$\begin{aligned} \|(f - f_k) \cdot \chi_{[-N,N]}\|_{\mathrm{u}} \;&=\; \sup_{|x| \leq N} |f(x) - f_k(x)| \\ &\leq\; \sup_{x \in \mathbb{R}} |f(x) - f_k(x)| \\ &=\; \|f - f_k\|_{\mathrm{u}} \\ &\to\; 0 \quad \text{as } k \to \infty. \end{aligned}$$

Consequently f_k converges uniformly on compact sets. Thus uniform convergence implies uniform convergence on compact sets, but Example 3.51.52 shows that the converse implication can fail. Here is another example that illustrates the difference between these two types of convergence.

Example 3.51.54 (Triangles Marching to Infinity). Let W be the *hat function* on the interval $[-1, 1]$ that was introduced in equation (3.35). The hat function belongs to $C(\mathbb{R})$; in fact, since it is compact supported, it is an element of $C_c(\mathbb{R})$.

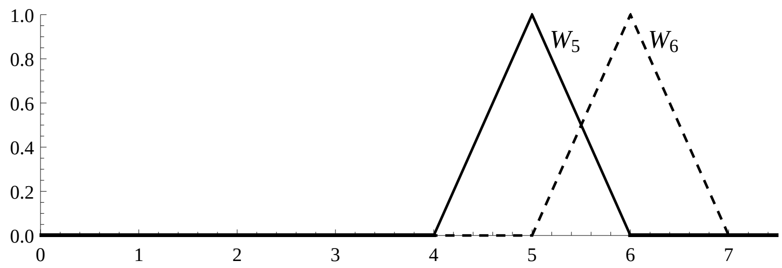

Fig. 3.61 The functions W_k, $k \geq 5$, from Example 3.51.54, vanish on the interval $[-4, 4]$.

The function

$$W_k(x) = W(x-k)$$

is simply the hat function translated to the right by k units. If we fix an integer $N > 0$, then for every $k > N$ we have $W_k = 0$ on the interval $[-N, N]$. Consequently, as shown in Figure 3.61, $\|W_k \cdot \chi_{[-N,N]}\|_{\mathrm{u}} = 0$ for $k > N$. Therefore

$$\forall\, N \in \mathbb{N}, \quad \lim_{k\to\infty} \|W_k \cdot \chi_{[-N,N]}\|_{\mathrm{u}} = 0.$$

So, we conclude that W_k converges uniformly on compact sets to the zero function. However,

$$\|W_k - 0\|_{\mathrm{u}} = \|W_k\|_{\mathrm{u}} = 1 \quad \text{for } \textit{every } k,$$

so W_k does not converge uniformly to the zero function. $\diamondsuit$

In summary, we have created a metric on $C(\mathbb{R})$, and the convergence notion associated with that metric is convergence on compact sets. According to Problem 3.51.56, $C(\mathbb{R})$ is complete with respect to this metric. However, we do not call $C(\mathbb{R})$ a Banach space, because it is not complete with respect to a *norm*. Instead, using the terminology of *topological vector spaces*, $C(\mathbb{R})$ is a *Fréchet space*.

We saw in Problem 3.3.14 that if $p < 1$ then the metric on ℓ^p is not induced from any norm on ℓ^p. It can be shown that the metric d on $C(\mathbb{R})$ is not induced from any norm on $C(\mathbb{R})$. However, in contrast to ℓ^p when $p < 1$, the issue for $C(\mathbb{R})$ is not a lack of convexity.

Extra Problems

3.51.55. Given a function $f \in C(\mathbb{R})$, prove that there exist functions $f_k \in C_c(\mathbb{R})$ such that f_k converges uniformly to f on compact sets.

Remark: Using the terminology of Definition 2.6.3, this says that $C_c(\mathbb{R})$ is *dense* in $C(\mathbb{R})$ with respect to the metric d.

3.51.56. Prove that $C(\mathbb{R})$ is complete with respect to the metric d defined in equation (3.39).

3.51.57. Given integers $n \ge 0$ and $N > 0$, define

$$|||f|||_{N,n} \;=\; \|f^{(n)} \cdot \chi_{[-N,N]}\|_{\mathrm{u}}, \qquad f \in C^\infty(\mathbb{R}).$$

Prove that

$$\mathrm{d}(f,g) \;=\; \sum_{n=0}^{\infty}\sum_{N=1}^{\infty} 2^{-n-N}\,\frac{|||f-g|||_{N,n}}{1+|||f-g|||_{N,n}}, \quad f,\, g \in C^\infty(\mathbb{R}),$$

defines a metric on $C^\infty(\mathbb{R})$.

3.52 Extra Section: Homeomorphisms

We saw in Example 2.9.53 that a continuous function need not map an open set to an open set. We are only guaranteed that the *inverse image* of an open set under a continuous function is open. Now, if $f\colon X \to Y$ is a bijection, then it has an inverse function $f^{-1}\colon Y \to X$. If the inverse function is continuous, then the inverse image of an open set under f^{-1} will be open. As we observe in the following lemma, the inverse image of a set under f^{-1} is simply the direct image of the set under f. So if f and f^{-1} are both continuous, then both direct and inverse images of open sets will be open. Here is the precise statement and proof of this fact.

Lemma 3.52.50. *Let X and Y be metric spaces, and suppose that $f\colon X \to Y$ is a bijection. If f and its inverse function f^{-1} are both continuous, then*

$$U \text{ is open in } X \quad \Longrightarrow \quad f(U) \text{ is open in } Y \tag{3.40}$$

and

$$V \text{ is open in } Y \quad \Longrightarrow \quad f^{-1}(V) \text{ is open in } X. \tag{3.41}$$

Proof. Equation (3.41) is an immediate consequence of the fact that f is continuous. To prove equation (3.40), let U be any open subset of X. Then, since $f^{-1}\colon Y \to X$ is continuous, the inverse image of U under the function f^{-1} must be open. Because f is a bijection, this inverse image is simply

$$(f^{-1})^{-1}(U) \;=\; U.$$

Therefore U is open in X. □

Thus, if f and f^{-1} are both continuous, then f and f^{-1} both preserve topological structure, as they each map open sets to open sets. We have a special name for such functions.

Definition 3.52.51 (Homeomorphism). Let X, Y be metric spaces.

(a) We say that $f\colon X \to Y$ is a *homeomorphism* if f is a bijection and both f and its inverse function f^{-1} are continuous.

(b) We say that X and Y are *homeomorphic metric spaces* if there exists a homeomorphism $f\colon X \to Y$. ♢

If $f\colon X \to Y$ is a homeomorphism, then f maps the open sets in X to the open sets in Y, and f^{-1} does the same in the reverse direction. In essence, homeomorphic spaces have the "same topological structure." The following exercise gives an example.

Exercise 3.52.52. Define $f\colon \mathbb{R}^{2d} \to \mathbb{C}^d$ by

$$f(x_1, y_1, \dots, x_d, y_d) \;=\; (x_1 + iy_1, \dots, x_d + iy_d)$$

for $x = (x_1, y_1, \dots, x_d, y_d) \in \mathbb{R}^{2d}$. Prove that f is a homeomorphism. ♢

Here is a rather different example of a homeomorphism. The domain and codomain for the function T_a in this example are $C_b(\mathbb{R})$, which is the set of all bounded, continuous functions $f\colon \mathbb{R} \to \mathbb{F}$. The standard norm for this space is the uniform norm $\|\cdot\|_{\mathrm{u}}$.

Example 3.52.53. For each fixed real number $a \in \mathbb{R}$ we will define a function $T_a\colon C_b(\mathbb{R}) \to C_b(\mathbb{R})$. That is, T_a will maps elements of $C_b(\mathbb{R})$ to elements of $C_b(\mathbb{R})$. In other words, T_a is a kind of "meta-function" that inputs a function $g \in C_b(\mathbb{R})$ and outputs a new function in $C_b(\mathbb{R})$. Using normal function notation we would denote this output by $T_a(g)$, but in order to avoid multiplicities of parentheses we will simply write $T_a g$ instead of $T_a(g)$.

We need to declare the rule for T_a, i.e., we must define how T_a will take the input g and determine what the corresponding output $T_a g$ is. What we have in mind for T_a is that it will simply *translate* the graph of the input g by a units to give us $T_a g$. That is, given $g \in C_b(\mathbb{R})$, we declare that $T_a g$ is the function whose graph is the identical to the graph of g except that it has been shifted to the right by a units. Explicitly,

$$T_a g(x) \;=\; g(x-a), \qquad x \in \mathbb{R}.$$

For example, if g is the function $g(x) = \sin x$, then $T_a g$ is the function

$$T_a g(x) \;=\; \sin(x-a) \;=\; \sin x \,\cos a \;-\; \cos x \,\sin a,$$

while if $g(x) = e^{-x^2}$ then

$$T_a g(x) \;=\; e^{(x-a)^2} \;=\; e^{x^2-2ax+a^2} \;=\; \frac{e^{a^2} e^{x^2}}{e^{2ax}}.$$

We refer to T_a as a *translation operator* (see the illustration in Figure 3.62).

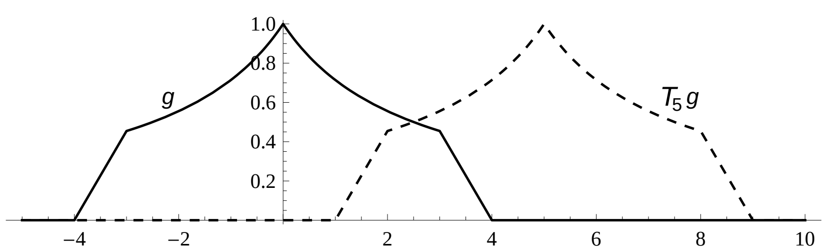

Fig. 3.62 A function g (solid) and its translation $T_5 g$ (dashed).

We will prove that $T_a\colon C_b(\mathbb{R}) \to C_b(\mathbb{R})$ is a homeomorphism. To do this, we must show that T_a is a bijection, and both T_a and its inverse $(T_a)^{-1}$ are continuous.

Proof that T_a is injective. To prove that T_a is injective, we must show that if $T_a f = T_a g$, then $f = g$. So, suppose that functions $f,\, g \in C_b(\mathbb{R})$ are such that $T_a f = T_a g$. This means that $T_a f$ and $T_a g$ are the same function in $C_b(\mathbb{R})$. In other words, $T_a f(t) = T_a g(t)$ for every $t \in \mathbb{R}$. Choose any particular point $x \in \mathbb{R}$, and let $t = x + a$. Then

$$T_a f(t) \;=\; f(t-a) \;=\; f(x)$$

and similarly

$$T_a g(t) \;=\; g(t-a) \;=\; g(x).$$

But $T_a f(t) = T_a g(t)$, so this implies that $f(x) = g(x)$. Since x was an arbitrary real number, we have shown that f and g take the same values at every x, and therefore they are they same function. That is, $f = g$. Therefore we have proved that T_a is injective.

Proof that T_a is surjective. We must prove that the range of T_a is all of $C_b(\mathbb{R})$. So, choose any function $g \in C_b(\mathbb{R})$. We must show that f is in the range of T_a. In other words, we must prove that $g = T_a f$ for some function f, i.e., we want g to be the function f shifted a units to the right. The appropriate function f is therefore g shifted a units to the left. That is, f is the function $f(x) = g(x+a)$. This function f belongs to $C_b(\mathbb{R})$, and for every x we have

$$T_a f(x) \;=\; f(x-a) \;=\; g(x).$$

Note that this discussion also reveals what $(T_a)^{-1}$ is. The inverse of shifting a units to the right is simply shifting a units to the left, so

$$(T_a)^{-1} \;=\; T_{-a}.$$

The inverse of the translation operator T_a is the translation operator T_{-a}.

Proof that T_a is continuous. Before proving continuity, we observe that the uniform norm of a function is not changed by translation, because

$$\|T_a g\|_{\mathrm{u}} \;=\; \sup_{x\in\mathbb{R}} |T_a g(x)| \;=\; \sup_{x\in\mathbb{R}} |g(x-a)| \;=\; \sup_{t\in\mathbb{R}} |g(t)| \;=\; \|g\|_{\mathrm{u}}. \tag{3.42}$$

We will use this *translation-invariance* property of the uniform norm to prove that T_a is continuous.

Exercise 2.9.52 tells us that one way to prove that T_a is continuous is to show that if $g_n \to g$ in $C_b(\mathbb{R})$, then $T_a g_n \to T_a g$ in $C_b(\mathbb{R})$. So, suppose that $g_n \to g$ in $C_b(\mathbb{R})$. This means that g_n converges to g uniformly, i.e.,

$$\lim_{n\to\infty} \|g - g_n\|_{\mathrm{u}} \;=\; 0.$$

But then, by the translation-invariance of the uniform norm and the fact that $T_a(g - g_n) = T_a g - T_a g_n$, we see that

$$\begin{aligned}\lim_{n\to\infty} \|T_a g - T_a g_n\|_{\mathrm{u}} &= \lim_{n\to\infty} \|T_a(g-g_n)\|_{\mathrm{u}} \\ &= \lim_{n\to\infty} \|g-g_n\|_{\mathrm{u}} \\ &= 0.\end{aligned} \tag{3.43}$$

Thus $T_a g_n \to T_a g$ in $C_b(\mathbb{R})$, and therefore T_a is continuous.

Proof that $(T_a)^{-1}$ *is continuous.* Since $(T_a)^{-1} = T_{-a}$, the same argument as given for the continuity of T_a shows that $(T_a)^{-1}$ is continuous. $\diamondsuit$

Thus, T_a is a homeomorphism that maps $C_b(\mathbb{R})$ onto itself, i.e., it is a continuous bijection whose inverse is also continuous. An important property of T_a, which we made use of in equation (3.43), is that T_a is *linear*, which means that

$$T_a(cf+dg) \;=\; c\,T_a f + d\,T_a g, \qquad \text{all } c,d \in \mathbb{F},\ f,g \in C_b(\mathbb{R}).$$

In Section 6.1 we will see much simpler ways to prove that a linear function is continuous.

Remark 3.52.54. We will show in Theorem 6.3.1 that if X and Y are normed spaces and X is *finite-dimensional*, then every linear function $F\colon X \to Y$ is continuous. However, if X is infinite-dimensional, then there exist linear functions $F\colon X \to Y$ that are not continuous! An example of such a discontinuous linear function appears in Example 6.3.4. $\diamondsuit$

Extra Problems

3.52.55. Prove that the property "X is homeomorphic to Y" is an equivalence relation on the class of all metric spaces. In other words, prove that the following three statements hold for all metric spaces X, Y, and Z.

(a) Reflexivity: (X, d_X) is homeomorphic to (X, d_X).

(b) Symmetry: If (X, d_X) is homeomorphic to (Y, d_Y), then (Y, d_Y) is homeomorphic to (X, d_X).

(c) Transitivity: If (X, d_X) is homeomorphic to (Y, d_Y) and (Y, d_Y) is homeomorphic to (Z, d_Z), then (X, d_X) is homeomorphic to (Z, d_Z).

3.52.56. (a) Given $f \in C_b(\mathbb{R})$, define Lf to be the function

$$(Lf)(x) \;=\; f(x)\,\sin x, \qquad x \in \mathbb{R}.$$

Prove that $L\colon C_b(\mathbb{R}) \to C_b(\mathbb{R})$ is continuous and injective, but is not a homeomorphism.

(b) Now define $M\colon C_b(\mathbb{R}) \to C_b(\mathbb{R})$ by

$$(Mf)(x) \;=\; f(x)\,(2+\sin x), \qquad x \in \mathbb{R}.$$

Is M a homeomorphism? Is $\|Mf\|_{\mathrm{u}} = \|f\|_{\mathrm{u}}$ for every function f? Compare this to equation (3.42).

3.52.57. Let X be a normed space, and let z be a point in X.

(a) Define $F_z \colon X \to X$ by

$$F_z(x) \;=\; x+z, \qquad x \in X.$$

Prove that F is a homeomorphism.

(b) Suppose that $X = C_b(\mathbb{R})$. In this case, what is z? Is F_z the same homeomorphism as the function T_a defined in Example 3.52.53, i.e., is $F_z = T_a$ for some a? If not, exhibit a function $f \in C_b(\mathbb{R})$ such that $F_z(f) \neq T_a(f)$.

3.52.58. Given $a > 0$ and $f \in C_b(\mathbb{R})$, define $D_a f$ to be the function

$$D_a f(x) \;=\; f(ax), \qquad x \in \mathbb{R}.$$

We call $D_a f$ the *dilation* of f by a (although if $a > 1$ then it is really a "compression" rather than a "dilation"). Prove that $D_a \colon C_b(\mathbb{R}) \to C_b(\mathbb{R})$ is a homeomorphism. Is D_a uniformly continuous?

3.52.59. Given $f \in C_b^1(\mathbb{R})$, let $Df = f'$ be the derivative of f. By definition of the space $C_b^1(\mathbb{R})$, the derivative f' is continuous and bounded, and therefore belongs to $C_b(\mathbb{R})$. The standard norm for $C_b^1(\mathbb{R})$ is $\|f\|_{C_b^1} = \|f\|_{\mathrm{u}} + \|f'\|_{\mathrm{u}}$; see Section 3.50.

(a) Prove that $D \colon C_b^1(\mathbb{R}) \to C_b(\mathbb{R})$ is continuous, but is neither injective nor surjective, and therefore is not a homeomorphism.

(b) Now we restrict the domain of D to the space $C_0^1(\mathbb{R})$. Prove that $D \colon C_0^1(\mathbb{R}) \to C_0(\mathbb{R})$ is continuous and injective, but is not surjective and therefore is not a homeomorphism.

(c) Finally, replace the standard norm for $C_b^1(\mathbb{R})$ with the uniform norm. Prove that $D \colon (C_b^1(\mathbb{R}), \|\cdot\|_{\mathrm{u}}) \to C_b(\mathbb{R})$ is not continuous.

CHAPTER 4: FURTHER RESULTS ON BANACH SPACES

4.1 Infinite Series in Normed Spaces

Example 4.1.50. Let x be any vector in a normed space X, and set $x_n = 2^{-n}x$. We will show that $x = \sum x_n$. The partial sums of the series $\sum x_n$ are

$$s_N = \sum_{n=1}^{N} x_n = \sum_{n=1}^{N} 2^{-n}x = \frac{2^N - 1}{2^N}x.$$

Since

$$x - s_N = x - \frac{2^N - 1}{2^N}x = 2^{-N}x,$$

we have

$$\lim_{N\to\infty} \|x - s_N\| = \lim_{N\to\infty} \|2^{-N}x\| = \lim_{N\to\infty} 2^{-N}\|x\| = 0.$$

Thus s_N converges to x *with respect to the norm of* X, so Definition 4.1.1 tells us that

$$x = \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} 2^{-n}x. \qquad \diamondsuit$$

Example 4.1.51. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\delta_1$ in $X = \ell^1$ (note that we use δ_n in each term of this series, not δ_n). Is this a convergent series *in the space* ℓ^1? Before answering this, recall the *alternating harmonic series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. We know that this infinite series of scalars converges and equals $-\ln 2 = \ln(1/2)$. This means that its Nth partial sum

$$t_N = \sum_{n=1}^{N} \frac{(-1)^n}{n}$$

converges to the value $-\ln 2$ as N increases.

The Nth partial sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\delta_1$ is

$$s_N = \sum_{n=1}^{N} \frac{(-1)}{n}\delta_1 = \left(\sum_{n=1}^{N} \frac{(-1)^n}{n}, 0, 0, \dots\right) = (t_N, 0, 0, \dots).$$

Therefore, if we let $x = (-\ln 2, 0, 0, \dots)$, then

$$\|x - s_N\|_1 = \big\|(-\ln 2 - t_N, 0, 0, \dots)\big\|_1 = |-\ln 2 - t_N| \to 0 \quad \text{as } N \to \infty.$$

Hence s_N does converge to x in ℓ^1-norm, so $\sum_{n=1}^\infty \frac{(-1)^n}{n}\delta_1$ converges in ℓ^1 and equals x. The reader should check that this series converges in ℓ^p for every index $1 \le p \le \infty$. $\diamondsuit$

Extra Problems

4.1.52. Let X be a nontrivial normed vector space (possibly incomplete).

(a) Show that there exist vectors $x_n \in X$, with $x_n \neq 0$ for every n, such that $\sum x_n$ converges.

(b) Show that there exist vectors $y_n \in X$ such that the series $\sum y_n$ does not converge.

4.2 Absolute Convergence of Series

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of vectors in a normed space X. Then $\{\|x_n\|\}_{n\in\mathbb{N}}$ is a sequence of real scalars, and we can wonder whether there is a connection between the convergence of the series $\sum x_n$ in X and the convergence of the series of real numbers $\sum \|x_n\|$. To address this, let us give names to the partial sums of these two series, say

$$s_N \;=\; \sum_{n=1}^{N} x_n \qquad \text{and} \qquad t_N \;=\; \sum_{n=1}^{N} \|x_n\|.$$

It follows from the Triangle Inequality that $\|s_N\| \le |t_N|$, but this inequality does not give us any useful information about convergence. On the other hand, if $M < N$ then we can use the Triangle Inequality to compute that

$$\|s_N - s_M\| \;=\; \biggl\|\sum_{n=M+1}^{N} x_n\biggr\| \;\le\; \sum_{n=M+1}^{N} \|x_n\| \;=\; |t_N - t_M|.$$

This tells us that *if* $\{t_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence of real scalars, *then* $\{s_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence of vectors in X. Now, all Cauchy sequences in a *complete* space converge, so *if X is complete* then we have the following implications:

$$\sum_{n=1}^{\infty} \|x_n\| \text{ converges} \quad \Longrightarrow \quad \{t_N\}_{N\in\mathbb{N}} \text{ is Cauchy in } \mathbb{R} \tag{4.44}$$

$$\Longrightarrow \quad \{s_N\}_{N\in\mathbb{N}} \text{ is Cauchy in } X \tag{4.45}$$

$$\underset{\text{if } X \text{ is complete}}{\Longrightarrow} \quad \sum_{n=1}^{\infty} x_n \text{ converges.} \tag{4.46}$$

However, if X is not complete then this chain of implication breaks down at the last step. In an incomplete space, even if we know that $\{s_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence, we cannot conclude that that $\sum x_n$ converges! Even so, we see that the issue of whether the series $\sum \|x_n\|$ converges is important.

Suppose that $\sum x_n$ is an absolutely convergent series in X. Definition 4.2.2 does *not* say that the series $\sum x_n$ converges in X! However, the results that we will obtain in Section 3.7 imply that counterexamples only exist in infinite-dimensional normed spaces, so we cannot yet give a good example. On the other hand, we can illustrate the idea by considering $X = \mathbb{Q}$. Technically, this will not give us a true counterexample because $\mathbb{Q}$ is not a vector space over $\mathbb{R}$ or $\mathbb{C}$ (so $\mathbb{Q}$ is not a normed space in the sense of Definition 3.1.1). Still, we will use $\mathbb{Q}$ to illustrate the difference between absolute convergence and convergence of a series.

Example 4.2.50. Define a sequence of rational numbers $(c_n)_{n\in\mathbb{N}}$ by expanding the decimal representation of the irrational number $\pi = 3.14159265\ldots$ in the following way:

$$\begin{aligned} c_1 &= 3, \\ c_2 &= 0.1, \\ c_3 &= 0.04, \\ c_4 &= 0.001, \\ c_5 &= 0.0005, \end{aligned}$$

and so forth. Since $\sum |c_n| < \infty$, the series $\sum c_n$ converges absolutely. However, the series $\sum c_n$ does not converge *in the set* $\mathbb{Q}$. The partial sums $s_N = \sum_{n=1}^{N} c_n$ are rational numbers that converge to the real number π, but the limit π does not belong to $\mathbb{Q}$. ♢

Extra Problems

4.2.51. Give an explicit example of functions $f_n \in C_c(\mathbb{R})$ such that the series $\sum_{n=1}^{\infty} f_n$ converges absolutely in $C_c(\mathbb{R})$ with respect to the uniform norm, but the series does not converge to an element of $C_c(\mathbb{R})$.

4.2.52. Let M be a subspace of a normed space X. Prove that the closure of M equals the set of all absolutely convergent series of elements of M:

$$\overline{M} \;=\; \biggl\{\sum_{n=1}^{\infty} x_n \;:\; x_n \in M,\; \sum_{n=1}^{\infty} \|x_n\| < \infty\biggr\}.$$

Hint: If $x \in \overline{M}$, then there exist vectors $x_n \in M$ such that $x_n \to x$. By Problem 2.2.24, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $\|x_{n_{k+1}} - x_{n_k}\| < 2^{-k}$ for each $k \in \mathbb{N}$.

Question: Why does this problem not imply that equality holds in equation (4.16)?

4.2.53. Let I be an interval in $\mathbb{R}$, and let $\|\cdot\|_{\mathrm{Lip}}$ be the norm on $\mathrm{Lip}(I)$ defined in Problem 4.2.15. Fix a point $x_0 \in I$, and prove that

$$|||f||| \;=\; |f(x_0)| \;+\; \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}$$

is an equivalent norm for $\mathrm{Lip}(I)$.

4.3 Unconditional Convergence of Series

Unconditionally Convergent Series of Scalars

To illustrate conditional and unconditional convergence, we will first consider series of scalars, i.e., infinite series in the Banach space $X = \mathbb{F}$. Here is an example of a convergent series of real numbers that does not converge unconditionally—if we change the ordering of the series, then the new series may converge to a different value, or may not converge at all!

Example 4.3.50. The *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, as its partial sums are unbounded. On the other hand, the *alternating harmonic series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

does converge (in fact, its partial sums converge to $\ln\frac{1}{2}$, the natural logarithm of $\frac{1}{2}$). Although the alternating harmonic series converges, it does not converge absolutely in the sense of Definition 4.2.2.

We will show that the alternating harmonic series does not converge unconditionally either. To do this, set $p_n = \frac{1}{2n-1}$ and $q_n = \frac{1}{2n}$, i.e., the p_n are the positive terms from the alternating series and the q_n are the absolute values of the negative terms. Each series $\sum p_n$ and $\sum q_n$ diverges to infinity.

Hence there must exist an integer $m_1 > 0$ such that

$$p_1 + \cdots + p_{m_1} \ > \ 1.$$

Also, there must exist an integer $m_2 > m_1$ such that

$$p_1 + \cdots + p_{m_1} - q_1 + p_{m_1+1} + \cdots + p_{m_2} \ > \ 2.$$

Continuing in this way, we can create a rearrangement

$$p_1 + \cdots + p_{m_1} - q_1 + p_{m_1+1} + \cdots + p_{m_2} - q_2 + \cdots$$

of $\sum \frac{(-1)^n}{n}$ that diverges to $+\infty$. Likewise, we can construct a rearrangement that diverges to $-\infty$, converges to any given real number r, or simply oscillates without ever converging (see Problem 4.3.6). Hence the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{\sigma(n)}}{\sigma(n)}$$

converges for some bijections $\sigma \colon \mathbb{N} \to \mathbb{N}$ but not for others, which says that the alternating harmonic series converges conditionally. $\diamondsuit$

Unconditional convergence of an infinite series is highly desirable because it says that the particular choice of ordering of the index set is irrelevant—if a series converges unconditionally then it will converge no matter how we choose to order the terms, and each reordering of the series will converge to the same limit. Series that converge *conditionally* (i.e., they converge, but not unconditionally) are surprisingly difficult to deal with, as the following example indicates.

Example 4.3.51. We know that the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges and equals $-\ln 2$. For each $n \in \mathbb{N}$ let

$$p_n \ = \ \frac{1}{2n} \qquad \text{and} \qquad q_n \ = \ \frac{1}{2n-1},$$

so the alternating series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \ = \ -q_1 + p_1 - q_2 + p_2 - \cdots \ = \ -\ln 2 \ = \ -0.693147\ldots.$$

We will show that there is a rearrangement of this series that converges to the real number $\pi = 3.14159\ldots$.

The series $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ both diverge to infinity. In particular, the partial sums of $\sum_{n=1}^{\infty} p_n$ increase without bound, so there is a first integer M_1 such that

$$p_1 \ = \ p_1 + \cdots + p_{M_1} \ \geq \ \pi.$$

As the partial sums of $\sum_{n=1}^\infty q_n$ also increase without bound, there is a first integer N_1 such that

$$p_1 + \cdots + p_{M_1} - q_1 - \cdots - q_{N_1} \ < \ \pi.$$

But $\sum_{n>M_1} p_n$ also diverges, so there is some first integer M_2 such that

$$p_1 + \cdots + p_{M_1} - q_1 - \cdots - q_{N_1} + p_{M_1+1} + \cdots + p_{M_2} \geq \pi.$$

Continuing in this way we obtain a rearrangement of the alternating harmonic series that converges to π:

$$\begin{aligned} p_1 \ + \ \cdots \ + \ p_{M_1} \ - \ q_1 \ - \ \cdots \ - \ q_{N_1} \ + \ p_{M_1+1} \ + \ \cdots \ + \ p_{M_2} \\ - \ q_{N_1+1} \ - \ \cdots \ - \ q_{N_2} \ + \ p_{M_2+1} \ + \ \cdots \ + \ p_{M_3} \ - \ \cdots \ = \ \pi. \end{aligned}$$

There is nothing special about π in this argument—if we replace π by e then we can find a rearrangement of the alternating harmonic series that converges to e, or to any real number that we like! The same is true of any infinite series of real scalars that converges but does not converge unconditionally—given *any* real number x we can find a rearrangement of the series that converges to x. $\diamondsuit$

Example 4.3.52. Recall that the alternating harmonic series $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}$ converges to $\ln 2$. We will show that there is a rearrangement of the alternating harmonic series that converges to a sum different from $\ln 2$.

We do not even need to know exactly what the alternating harmonic series converges to, just let

$$L \ = \ \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}.$$

Since this is an alternating series, we can use the first terms to give bounds for L. We see that $\frac{1}{2} \leq L \leq 1$, so we know that $L \neq 0$.

Multiplying the series by 2, we obtain

$$\begin{aligned} 2L \ = \ 2\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} \ &= \ 2\biggl(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots\biggr) \\ &= \ 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + + \frac{2}{7} - \frac{1}{4} \cdots + \\ &= \ (2-1) - \frac{1}{2} + \biggl(\frac{2}{3} - \frac{1}{3}\biggr) - \frac{1}{4} + \biggl(\frac{2}{5} - \frac{1}{5}\biggr) - \frac{1}{6} + \cdots \\ &= \ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \\ &= \ L. \end{aligned}$$

This is a contradiction since $L \neq 0$. $\diamondsuit$

Problem 4.3.58 asks for a proof that if X is a *finite-dimensional* normed space, then a series in X converges absolutely if and only if it converges unconditionally.

Unconditional Convergence in Banach Spaces

Unfortunately, it is not true that unconditional convergence always implies absolute convergence. For example, we will see later that if $\{e_n\}_{n\in\mathbb{N}}$ is an infinite orthonormal sequence in a Hilbert space H, then the series $\sum \frac{1}{n}\, e_n$ converges unconditionally but not converge absolutely (see Problem 5.7.3).

Although we will not prove it, the following interesting, nontrivial result implies that if X is *any* infinite-dimensional Banach space, then there exists a series in X that converges unconditionally but not absolutely. For a proof of Theorem 4.3.53, see [Heil11, Thm. 3.33].

Theorem 4.3.53 (Dvoretzky–Rogers Theorem). *Let X be an infinite-dimensional Banach space. Given any sequence $(c_n)_{n\in\mathbb{N}} \in \ell^2$, there exist unit vectors $x_n \in X$ such that the infinite series $\sum c_n x_n$ converges unconditionally in X.* ◇

Corollary 4.3.54. *If X is an infinite-dimensional Banach space, then there exists an infinite series that converges unconditionally but not absolutely in X.*

Proof. If we set $c_n = \frac{1}{n}$, then the sequence $(c_n)_{n\in\mathbb{N}}$ is square-summable. Theorem 4.3.53 therefore implies that there exist unit vectors $x_n \in X$ such that the series $\sum c_n x_n$ converges unconditionally. This series does not converge absolutely, because

$$\sum_{n=1}^{\infty} \|c_n x_n\| \;=\; \sum_{n=1}^{\infty} \frac{1}{n} \;=\; \infty. \qquad \diamondsuit$$

Equivalent Characterizations of Unconditional Convergence

We will state a theorem that gives several reformulations of unconditional convergence (see [Heil11, Thm. 3.10] for a proof).

Theorem 4.3.55. *If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in a Banach space X, then the following four statements are equivalent.*

(a) $\sum_{n=1}^{\infty} x_n$ *converges unconditionally.*

(b) *The net* $\{\sum_{n\in F} x_n : F \subseteq \mathbb{N}, F \text{ finite}\}$ *converges in* X. *Explicitly, this means that there exists a vector* $x \in X$ *such that for every* $\varepsilon > 0$, *there is a finite set* $F_0 \subseteq \mathbb{N}$ *such that*

$$F_0 \subseteq F \subseteq \mathbb{N},\ F \text{ finite} \implies \Big\| x - \sum_{n\in F} x_n \Big\| < \varepsilon.$$

(c) *If* $(c_n)_{n\in\mathbb{N}}$ *is any bounded sequence of scalars, then the series* $\sum_{n=1}^\infty c_n x_n$ *converges in* X.

(d) *If* $(\varepsilon_n)_{n\in\mathbb{N}}$ *is any sequence of scalars such that* $\varepsilon_n = \pm 1$ *for every* n, *then the series* $\sum_{n=1}^\infty \varepsilon_n x_n$ *converges in* X. ◇

The equivalence of statements (a) and (c) in Theorem 4.3.55 is quite interesting. It implies that if $\sum x_n$ is a series in X that does not converge unconditionally, then we can find scalars c_n with $|c_n| < 1$ for every n such that $\sum c_n x_n$ does not converge. That is, even though the norm of $c_n x_n$ is *strictly smaller* than the norm of x_n, the series $\sum c_n x_n$ fails to converge compare Problem 4.3.56).

Statement (b) of Theorem 4.3.55 mentions *nets*. Nets are generalizations of ordinary sequences, and are used to characterize convergence in abstract topological spaces.

Extra Problems

4.3.56. Let $x_n = (-1)^n/n$, so $\sum x_n$ is the alternating harmonic series. Although this series converges, show that there exist scalars c_n such that $c_n \to 0$ yet $\sum c_n x_n$ does not converge.

4.3.57. Assume that $\sum c_n$ is a conditionally convergent series of real scalars, i.e., the series converges, but it does not converge unconditionally.

(a) Let (p_n) be the sequence of nonnegative terms of (c_n) in order, and let (q_n) be the sequence of negative terms of (c_n) in order. Show that $\sum p_n$ and $\sum q_n$ must both diverge. Conclude that $\sum c_n$ is not absolutely convergent.

(b) Given $x \in \mathbb{R}$, show there exists a permutation σ of $\mathbb{N}$ such that $\sum c_{\sigma(n)}$ converges and equals x.

(c) Show that there exists a permutation σ of $\mathbb{N}$ such that $\sum c_{\sigma(n)}$ diverges to ∞ (that is, $\lim_{N\to\infty} \sum_{n=1}^N c_{\sigma(n)} = \infty$), and another permutation τ such that $\sum c_{\tau(n)}$ diverges to $-\infty$.

(d) Show that there exists a permutation σ of $\mathbb{N}$ such that $\sum c_{\sigma(n)}$ does not converge and does not diverge to ∞ or $-\infty$.

4.3.58. Let X be a finite-dimensional normed space. Show that a series $\sum x_n$ in X is unconditionally convergent if and only if it is absolutely convergent.

4.3.59. Use Theorem 4.3.55 to show that if a series $\sum_{n=1}^{\infty} x_n$ converges unconditionally in a Banach space X, then there exists a single vector $x \in X$ such that for each bijection $\sigma \colon \mathbb{N} \to \mathbb{N}$ we have $\sum_{n=1}^{\infty} x_{\sigma(n)} = x$.

4.3.60. Let X be a Banach space, and assume that vectors $x_{mn} \in X$, $m, n \in \mathbb{N}$, satisfy $\sum_m \sum_n \|x_{mn}\| < \infty$. Show that if $\sigma \colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection, then the following series all converge and are equal as indicated:

$$\sum_{m=1}^{\infty}\biggl(\sum_{n=1}^{\infty} x_{mn}\biggr) \;=\; \sum_{n=1}^{\infty}\biggl(\sum_{m=1}^{\infty} x_{mn}\biggr) \;=\; \sum_{k=1}^{\infty} x_{\sigma(k)}.$$

4.3.61. Prove that the conclusion of Problem 4.3.60 remains valid if we replace the hypothesis of absolute convergence with unconditional convergence, i.e., we assume that the series $\sum_{(m,n)\in\mathbb{N}^2} c_{mn}$ converges unconditionally in X.

4.5 Independence, Hamel Bases, and Schauder Bases

When we are given a *finite-dimensional* vector space, one of the first things that we are taught to do in undergraduate linear algebra class is to look for a *basis* for that space, i.e., a set that both *spans* and is *linearly independent.* Unfortunately, bases in this ordinary vector space sense are usually not very useful in the infinite-dimensional setting. For one thing, in most infinite-dimensional vector spaces the only way that we can know that a basis exists is by appealing to the Axiom of Choice. For another, any basis for a *complete* infinite-dimensional normed space must be an uncountable set (see Theorem 4.5.3). This makes them too unwieldy for practical use.

When we work with generic vector spaces, we have no choice but to restrict ourselves to *finite* linear combinations of elements. In particular, spanning and linear independence are each defined in terms of *finite linear combinations* of vectors. We cannot form an infinite series in a vector space without having a notion of what it means to converge in that space, because an infinite series is, by definition, the *limit* of the partial sums of the series. If our vector space is lucky enough to have a *norm*, then we do have a convergence criterion and therefore can form infinite series. Consequently, when we work with a normed vector space we can seek “bases” that are defined in terms of “infinite linear combinations” instead of just finite linear combinations of vectors. This is not an issue in finite-dimensional normed spaces, but it can be very important when dealing with infinite-dimensional spaces. As it turns out, defining the most useful “basis” concept in infinite dimensions is not as straightfoward as we might hope, as there are more subtleties in the ways that we can generalize the notions of spanning and independence than might be apparent at first glance.

The Standard Basis Vectors

The standard basis vectors, which were introduced in Notation 2.1.8, will be used to illustrate many of the concepts discussed in this chapter, so we recall their definition here. Given $n \in \mathbb{N}$, the nth *standard basis vector* is the sequence δ_n whose nth component is 1, and all other components are zero. That is,

$$\delta_n \;=\; (0, \dots, 0, 1, 0, 0, \dots),$$

where $n-1$ zeros precede the 1. The *sequence of standard basis vectors*, or simply the *standard basis*, is the collection $\mathcal{E}$ of all of the standard basis vectors. In other words,

$$\mathcal{E} \;=\; \{\delta_n\}_{n\in\mathbb{N}}.$$

Each standard basis vector δ_n has only finite many nonzero components, so δ_n belongs to the space c_{00}. Hence the standard basis $\mathcal{E}$ is a subset of c_{00}, and since c_{00} is contained in the spaces ℓ^p and c_0, the standard basis $\mathcal{E}$ is also a subset of ℓ^p and c_0.

Although we have just called $\mathcal{E} = \{\delta_n\}_{n\in\mathbb{N}}$ a "basis," this should only be viewed as a name for now. As we progress through the chapter, we will discuss in what sense $\mathcal{E}$ is a basis for spaces such as ℓ^p, c_0, or c_{00}.

According to the next exercise, if p is finite then the standard basis is a Schauder basis for ℓ^p. Note that in order to prove that a sequence is a Schauder basis for a given space, we must show that the series in equation (4.17) converge *in the norm of that space* (not just in some other sense like componentwise convergence!).

Exercise 4.5.50. Let $\mathcal{E} = \{\delta_n\}_{n\in\mathbb{N}}$ be the standard basis.

(a) Prove that $\mathcal{E}$ is a Schauder basis for ℓ^p for each index $1 \le p < \infty$ (be sure to show that the partial sums converge *in norm*, not just componentwise).

(b) Prove that $\mathcal{E}$ is not a Schauder basis for ℓ^∞. In particular, exhibit an sequence $x \in \ell^\infty$ such that there are no scalars c_n that satisfy $x = \sum c_n\delta_n$ with convergence of the series in ℓ^∞-norm. $\diamondsuit$

Closed Span

We emphasize that the definition of the closed span does *not* say that

$$\overline{\mathrm{span}}(A) \;=\; \Biggl\{\sum_{n=1}^{\infty} c_n x_n : x_n \in A, c_n \in \mathbb{F}\Biggr\} \qquad \leftarrow \text{This need not hold!}$$

It is *not* true that an arbitrary element of $\overline{\mathrm{span}}(A)$ can always be written as $x = \sum_{n=1}^{\infty} c_n x_n$ for some $x_n \in A$, $c_n \in \mathbb{F}$. Instead, to illustrate the meaning of the closed span, consider a countable set $A = \{x_n\}_{n\in\mathbb{N}}$. For such a set we can write the closed span explicitly as

$$\overline{\mathrm{span}}\{x_n\}_{n\in\mathbb{N}} = \Biggl\{x \in X : \exists\, c_{n,N} \in \mathbb{F} \text{ such that } \sum_{n=1}^{N} c_{n,N} x_n \to x \text{ as } N \to \infty\Biggr\}.$$

That is, x belongs to the closed span of $\{x_n\}_{n\in\mathbb{N}}$ if and only if there exist scalars $c_{n,N} \in \mathbb{F}$ such that

$$\sum_{n=1}^{N} c_{n,N} x_n \to x \quad \text{as } N \to \infty. \tag{4.47}$$

The scalars $c_{n,N}$ in equation (4.47) can depend on N. In contrast, to say that $x = \sum_{n=1}^{\infty} c_n x_n$ means that

$$\sum_{n=1}^{N} c_n x_n \to x \quad \text{as } N \to \infty. \tag{4.48}$$

In order for equation (4.48) to hold, the scalars c_n must be *independent of* N.

Schauder Bases

We will prove that every Schauder basis is a complete sequence and is finitely linearly independent. However, Example 4.5.52 will show that the converse statement need not hold.

Lemma 4.5.51. *If $\mathcal{B} = \{x_n\}_{n\in\mathbb{N}}$ is a Schauder basis for a Banach space X, then $\mathcal{B}$ is both complete and finitely linearly independent in X.*

Proof. By the definition of a Schauder basis, every vector $x \in X$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n(x)\, x_n$. The partial sums $s_N = \sum_{n=1}^{N} \alpha_n(x)\, x_n$ of this infinite series belong to the finite linear span of $\mathcal{B}$. Further, $s_N \to x$, so we conclude that $\mathrm{span}(\mathcal{B})$ is dense in X. Applying equation (4.15), this tells us that the sequence $\mathcal{B}$ is complete.

To show that $\mathcal{B}$ is independent, suppose that some finite linear combination of vectors from $\mathcal{B}$ equals the zero vector. That is, suppose that

$$\sum_{n=1}^{N} c_n x_n \;=\; 0$$

for some integer $N > 0$ and scalars $c_1, \dots, c_N \in \mathbb{F}$. If we define $c_n = 0$ for $n > N$, then $\sum_{n=1}^\infty c_n x_n = 0$, where the series converges with respect to the norm of X. However, we also have $\sum_{n=1}^\infty 0 x_n = 0$, so the uniqueness requirement of a Schauder basis implies that $c_n = 0$ for every n. Hence $\mathcal{B}$ is finitely linearly independent. □

Unfortunately, the converse of Lemma 4.5.51 fails in general, i.e., there are sequences that are both complete and finitely independent but are not Schauder bases. Here is an example in the space $C[a,b]$; also see Problem 5.8.14, which presents some further examples and extensions.

Example 4.5.52. The set of monomials $\mathcal{M} = \{x^k\}_{k=0}^\infty$ is both complete and finitely independent in $C[a,b]$. However, as shown in Problem 4.6.6, there exist functions $f \in C[a,b]$ that cannot be written in the form $f(x) = \sum_{k=0}^\infty c_k x^k$ with convergence of the series in the uniform norm (or even just pointwise). Consequently the set of monomials $\mathcal{M}$ is *not* a Schauder basis for $C[a,b]$. ◇

A complete examination of Schauder bases requires the use of the Hahn–Banach and Uniform Boundedness Theorems, which are part of the field of Functional Analysis. For more information on Schauder bases and related topics, we refer to the following texts:

C. Heil, *A Basis Theory Primer*, Expanded Edition, Birkhäuser, Boston, 2011.

I. Singer, *Bases in Banach Spaces* I, Springer-Verlag, New York, 1970.

J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, I*, Springer-Verlag, New York, 1977.

Separable Banach Spaces

According to Definition 2.6.7, a Banach space X is *separable* if it contains a *countable dense subset*. Now, if $A = \{x_n\}_{n\in\mathbb{N}}$ is a complete sequence in X then $\mathrm{span}(A)$ is dense in X, but $\mathrm{span}(A)$ is not countable since it contains every scalar multiple of each vector x_n (as well as every finite linear combination of these vectors). On the other hand, if we restrict to finite linear combinations that only use rational scalars then we obtain a countable subset of $\mathrm{span}(A)$, and the next exercise shows that this subset is still dense in X. Recall that a *rational scalar* is a complex number whose real and imaginary parts are both rational.

Exercise 4.5.53. Let X be a Banach space. Show that if there exists a countable sequence $\{x_n\}_{n\in\mathbb{N}}$ that is complete in X, then

$$S \;=\; \biggl\{\sum_{n=1}^{N} r_n x_n \,:\, N > 0,\; r_n \text{ is rational}\biggr\}$$

is a countable, dense subset of X, and therefore X is separable. $\diamondsuit$

For example, we can use this exercise to prove that $C[a,b]$ is separable.

Example 4.5.54. The Weierstrauss Approximation Theorem implies that the set of monomials $\{1, x, x^2, \dots\}$ is a countable complete sequence in $C[a,b]$. Applying Exercise 4.5.53, it follows that $C[a,b]$ must be separable. $\diamondsuit$

The following corollary is a special case of Exercise 4.5.53.

Corollary 4.5.55. *If a Banach space X has a Schauder basis $\mathcal{B} = \{x_n\}_{n\in\mathbb{N}}$, then X is separable.*

Proof. By Lemma 4.5.51, if $\mathcal{B} = \{x_n\}_{n\in\mathbb{N}}$ is a Schauder basis for a Banach space X, then $\mathcal{B}$ is both countable and complete. It therefore follows from Exercise 4.5.53 that X must be separable. $\square$

Thus, a Banach space that has a Schauder basis must be separable. The question of whether every separable Banach space has a Schauder basis was a longstanding open problem known as the *Basis Problem*. It was finally shown by Enflo [Enf73] that there exist separable Banach spaces that have no Schauder bases! For constructions of Schauder bases in some specific Banach spaces such as $C[0,1]$, we refer to the text [Heil11].

The HRT Conjecture

The problem given in Problem 4.5.12 is one particular open problem that is part of the *Linear Independence of Time-Frequency Translates Conjecture*, also known as the *HRT Conjecture*. Many special cases are known. For example, if $g \in C_c(\mathbb{R})$ then it is easy to prove that the set of four functions given in equation (4.21) is independent. More special cases can be found in the survey paper [Heil06]. The original conjecture is in the paper:

C. Heil, J. Ramanathan, and P. Topiwala, **Linear independence of time-frequency translates**, Proc. Amer. Math. Soc., **124** (1996), pp. 2787–2795.

The actual statement of the conjecture is as follows.

Conjecture 4.5.56 (HRT Conjecture). If g in $L^2(\mathbb{R})$ is not the zero function and $\Lambda = \{(a_k, b_k) : k = 1, \dots, N\}$ is a set of finitely many distinct points in $\mathbb{R}^2$, then

$$\mathcal{G}(g, \Lambda) \;=\; \{e^{2\pi i b_k x} g(x - a_k) : k = 1, \dots, N\}$$

is linearly independent. $\diamondsuit$

Despite the simplicity of the *statement* of the conjecture, it appears to be a very difficult problem, still open as of 2018. The conjecture is open if $L^2(\mathbb{R})$ is replaced by $C_0(\mathbb{R})$, and the real-valued version obtained by replacing $e^{2\pi i b x}$ with $\cos bx$ is likewise open. If we replace $L^2(\mathbb{R})$ or $C_0(\mathbb{R})$ with $C_b(\mathbb{R})$ then it is easy to find counterexamples (e.g., any periodic function).

Please contact me with any news, progress, or questions regarding this conjecture.

Extra Problems

4.5.57. Let c_{00} and c_0 be the spaces of sequences introduced in Chapter 2, and let c denote the set of all convergent sequences of scalars, i.e.,

$$c \;=\; \Big\{x = (x_k)_{k\in\mathbb{N}} \,:\, \lim_{k\to\infty} x_k \text{ exists}\Big\}.$$

Since

$$c_{00} \;\subsetneq\; c_0 \;\subsetneq\; c \;\subsetneq\; \ell^\infty,$$

the sup-norm is a norm of each of these four spaces. Further, by Problem 2.2.19, ℓ^∞ is complete with respect to this norm. Prove the following statements.

(a) Prove that c is a closed subspace of ℓ^∞, and therefore is a Banach space with respect to the sup-norm.

(b) c_0 is a closed subspace of c, and therefore is a Banach space with respect to the sup-norm.

(c) c_{00} is not a closed subspace of c_0, and therefore is not complete with respect to the sup-norm.

4.5.58. This problem adds to Problem 4.6.6.

Let $(c_k)_{k\geq 0}$ be a fixed sequence of scalars, and suppose that the series $\sum_{k=0}^\infty c_k x^k$ converges for some number $x \in \mathbb{R}$. Set $r = |x|$, and prove the following statements.

(a) The series $f(t) = \sum_{k=0}^\infty c_k t^k$ converges absolutely for all $t \in (-r, r)$, i.e.,

$$\sum_{k=0}^\infty |c_k|\,|t|^k \;<\; \infty, \qquad \text{all } |t| < r.$$

(b) f is infinitely differentiable on the interval $(-r, r)$.

(c) f is real analytic at the origin in the sense defined in Problem 3.50.54. Specifically,

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, t^k, \qquad |t| < r.$$

4.5.59. Prove the following statements.

(a) c_0 is separable.

(b) ℓ^p is separable for $1 \le p < \infty$.

(c) $C_0(\mathbb{R})$ is separable.

Hint: Let θ_M be 1 on $[-M, M]$, zero outside $[-M-1, M+1]$, and linear on each of the two intervals $[-M-1, -M]$ and $[M, M+1]$. Use the Weierstrass Approximation Theorem to show that the countable set

$$S = \left\{ \sum_{k=0}^{N} r_k x^k \theta_M(x) : M > 0,\ N \ge 0,\ r_k \text{ rational} \right\}$$

is dense in $C_0(\mathbb{R})$.

4.6 The Stone–Weierstrass Theorem

Extra Problems

4.6.50. For this problem we take $\mathbb{F} = \mathbb{R}$. Assume that $f \in C[a, b]$ is a strictly positive function that is also injective. Prove that $\mathcal{A} = \operatorname{span}\{f(x)^n\}_{n \ge 0}$ is dense in $C[0, 1]$.

For example, we can take f to be x^α (if we choose a and b correctly) or e^x.

4.50 Extra Section: Hamel Bases and the Axiom of Choice

If X is a finite-dimensional vector space, then it is usually not too difficult to construct some Hamel basis for X, and we can use basic linear algebra techniques to prove that any two Hamel bases for X contain exactly the same number of vectors. That is, any two Hamel bases for X have the same cardinality, and we call this number the *dimension* of X.

Bases do exist for infinite-dimensional spaces, but we cannot always explicitly display them. The *Axiom of Choice* is one of the axioms of the standard form of set theory most commonly accepted in mathematics (Zermelo–Fraenkel set theory with the Axiom of Choice, or ZFC). One consequence of the Axiom of Choice is that

every vector space has a Hamel basis.

We will show belows how to use one of the equivalent forms of the Axiom of Choice known as *Zorn's Lemma* to prove the existence of Hamel bases. Even so, while the Axiom of Choice implies that X has a Hamel basis, if X is infinite dimensional then there often is no way to *construct* such a basis. Although we know Hamel bases exist, we may not be able to exhibit any specific examples (just try to construct a Hamel basis for ℓ^1; you won't succeed!). This is one reason why Hamel bases are not very useful in infinite-dimensional spaces.

The *Axiom of Choice* is one of the axioms of the standard form of set theory most commonly accepted in mathematics (Zermelo–Fraenkel set theory with the Axiom of Choice, or ZFC). Here is the formal statement of this axiom.

Axiom 4.50.50 (Axiom of Choice). Let S be a nonempty set S, and let $\mathcal{P}$ be the family of all nonempty subsets of S. Then there exists a function $f\colon \mathcal{P} \to S$ such that $f(A) \in A$ for each set $A \in \mathcal{P}$. $\diamondsuit$

There are many statements that are equivalent to the Axiom of Choice. In the remainder of this appendix, we will discuss another equivalent form of the Axiom of Choice, known as Zorn's Lemma or the Kuratowski–Zorn Lemma. First we need to introduce the concept of a partial order on a set.

Definition 4.50.51 (Partial Order). Let S be a set, and let $\leq$ be a relation on $S \times S$.

(a) The relation $\leq$ is a *partial order* on S if the following conditions are satisfied for all elements $A, B, C \in S$:

   i. Reflexivity: $A \leq A$,
   ii. Symmetry: $A \leq B$ and $B \leq A$ implies $A = B$, and
   iii. Transitivity: $A \leq B$ and $B \leq C$ implies $A \leq C$.

(b) Two elements $A, B \in S$ are *comparable* if either $A \leq B$ or $B \leq A$.

(c) A partial ordering $\le$ on S such that every pair A, B of elements of S are comparable is called a *linear ordering* or a *total ordering* of S.

(d) A nonempty subset of S that is linearly ordered by $\le$ is called a *chain* in S.

(e) An element $A \in S$ is *maximal* in S if for every $B \in S$ we have that

$$B \text{ is comparable to } A \implies B \le A.$$

(f) An element $A \in S$ is an *upper bound* for $U \subseteq S$ if $B \le A$ for every $B \in U$. $\diamondsuit$

Note that a maximal element need not be comparable to all elements of S, and it need not be unique.

Now we state Zorn's Lemma (although it is more precise to refer to this result as the Kuratowski–Zorn Lemma, as it is due independently to Kazimierz Kuratowski and Max Zorn). A proof of that Zorn's Lemma is equivalent to the Axiom of Choice can be found in Theorem A.4 of:

J. J. Rotman, *Advanced Modern Algebra*, Prentice Hall, Upper Saddle River, NJ, 2002.

An exposition of the history of the Kuratowski–Zorn Lemma appears in:

P. J. Campbell, The origin of "Zorn's Lemma," *Historia Math.*, **5** (1978), pp. 77–89.

Axiom 4.50.52 (Kuratowski–Zorn Lemma). Let $\le$ be a partial order on a set S. If every chain in S has an upper bound in S, then S contains a maximal element. $\diamondsuit$

To illustrate the use of the Kuratowski–Zorn Lemma, we will prove that every vector space X (other than the trivial space $\{0\}$) possesses a Hamel basis, i.e., a subset that is both finitely linearly independent and whose finite linear span is X (see Definition 4.5.2).

Theorem 4.50.53. *If X is a nontrivial vector space, then there exists a subset B that is a Hamel basis for X.*

Proof. Let

$$S = \big\{A \subseteq X : A \text{ is finitely linearly independent}\big\}.$$

Note that S is nonempty, since if $x \ne 0$ then the singleton $\{x\}$ is a linearly independent set. The inclusion relation $\subseteq$ is a partial order on S.

Suppose that C is a chain in S, say $C = \{A_i\}_{i \in I}$ where I is some index set. By definition, each set A_i is finitely independent, and we claim that $A = \cup A_i$ is also finitely independent. To see this, choose finitely many distinct vectors

$x_1, \dots, x_n \in A$. Then for each $k = 1, \dots, n$ we have $x_k \in A_{i_k}$ for some index $i_k \in I$. Since C is a chain, it is linearly ordered by inclusion. Therefore there is a largest set A_{i_k}, i.e., there is a j such that $A_{i_k} \subseteq A_{i_j}$ for $k = 1, \dots, n$. Hence $x_1, \dots, x_n$ all belong to A_{i_j} and therefore $\{x_1, \dots, x_n\}$ is a linearly independent set. Thus every finite subset of S is independent, so $A \in S$. Since we have $A_i \subseteq A$ for each $i \in I$, it follows that A is an upper bound for the chain C.

Applying the Kuratowski–Zorn Lemma, we conclude that S contains a maximal element B. By definition, B is finitely independent, so if its finite span is X then it is a Hamel basis for X. Suppose that there exists a vector $x \in X$ that does not belong to $\mathrm{span}(B)$. Then $B' = B \cup \{x\}$ is finitely independent and hence belongs to S. However, $B \subsetneq B'$, which implies that B is not a maximal element in S. This is a contradiction, so we must have $X = \mathrm{span}(B)$. □

In summary, we have shown that Theorem 4.50.53 is a consequence of the Axiom of Choice. The converse implication is also true, i.e., the statement "every vector space has a Hamel basis" is another equivalent form of the Axiom of Choice and the Kuratowski–Zorn Lemma; see the following reference:

A. Blass, Existence of bases implies the axiom of choice, in: *Axiomatic Set Theory* (Boulder, Colo., 1983), J. E. Baumgartner, D. A. Martin, and S. Shelah, Eds., Contemp. Math., Vol. 31, Amer. Math. Soc., Providence, RI, 1984, pp. 31–33.

CHAPTER 5: INNER PRODUCTS AND HILBERT SPACES

5.7 Orthogonal and Orthonormal Sequences

Extra Problems

5.7.50. This problem will give a constructive solution to Exercises 3.7.51 and 3.7.52 in the setting of Hilbert spaces.

(a) Let M be a proper, closed subspace of a Hilbert space H. Given a vector $f \notin M$, let p be the orthogonal projection of f onto M and set $e = f - p$. Show that the vector $g = e/\|e\|$ satisfies

$$\|g\| = 1, \qquad g \in M^{\perp}, \qquad \mathrm{dist}(g, M) = \inf_{m \in M} \|g - m\| = 1.$$

(b) Let H be an infinite-dimensional Hilbert space. Prove directly that H must contain an infinite orthonormal sequence $\{e_n\}_{n\in\mathbb{N}}$. Show that no subsequence of $\{e_n\}_{n\in\mathbb{N}}$ is Cauchy, and conclude that the closed unit disk $D = \bigl\{f \in H : \|f\| \le 1\bigr\}$ is not a compact subset of H.

5.8 Orthonormal Bases

Remark 5.8.50. If $\{f_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence, then the scalars $\langle f, f_n\rangle$ are sometimes called the *generalized Fourier coefficients* of f with respect to $\{f_n\}_{n\in\mathbb{N}}$. ◇

Remark 5.8.51. In summary, if $\{e_n\}$ is a complete orthonormal sequence in a Hilbert space H, then every vector $f \in H$ has an explicit, unique representation as

$$f = \sum_n \langle f, e_n\rangle\, e_n, \qquad f \in H, \tag{5.49}$$

and therefore $\{e_n\}$ is a Schauder basis for H. This fact is very useful. Not every complete sequence $\{f_n\}$ is a Schauder basis, and even if it is, it can be a highly nontrivial task to find an explicit formula for the scalars $c_n(f)$ that satisfy $f = \sum c_n f_n$. Remarkably, if $\{e_n\}$ is both *complete* and *orthonormal*, then it is a automatically a Schauder basis with explicitly known representations. Furthermore, such an orthonormal basis has a number of important "stability" properties. First, it follows from Theorem 5.7.1 that the series $f = \sum \langle f, e_n\rangle\, e_n$ converges *unconditionally.* That is, the series converges no matter how we order the index set (and other important "stability" features of unconditional convergence appear in Theorem 4.3.55). Second, the

Plancherel Formula tells us that the norm of a vector f is precisely equal to the ℓ^2-norm of the sequence of coefficients $(\langle f, e_n \rangle)$. Hence a small change in f yields a small change in these coefficients, and conversely a small change in the coefficients results in a small change to f. ♢

The results of Theorems 5.7.1 and 5.8.1 can be generalized to uncountable orthonormal sets (see Problems 5.9.10 and 5.9.11). Specifically, Problem 5.9.11 shows that if a Hilbert space H contains a complete, uncountable orthonormal set, then we obtain basis-like representations analogous to those given in equation (5.11). Sometimes such a set is called an "orthonormal basis," but in keeping with Definition 4.5.4 we will restrict the use of the word "basis" to countable sequences.

Still, this raises a question: Does every Hilbert space contain a complete orthonormal set? We could try to create such a set by an iterative process. For example, suppose that S is any orthonormal set in H, and let $M = \overline{\mathrm{span}}(S)$ be the closed span of S. By definition, S is complete if and only if $M = H$. If S is not complete then M is a proper closed subspace of H, so its orthogonal complement $M^\perp$ is not trivial. If we choose any nonzero vector $g \in M^\perp$ and divide g by its length, we obtain a unit vector that is orthogonal to S. Hence $S \cup \{g\}$ is a larger orthonormal set, and if this new orthonormal set is not complete then we can keep going and add another orthonormal vector to it. Unfortunately, if H is infinite-dimensional then this process might never end. However, the same type of Zorn's Lemma argument that is used in Theorem 4.50.53 to show that every vector space has a Hamel basis can be adapted to show that H must contain a complete orthonormal set. Of course, since this proof depends on the Axiom of Choice it is nonconstructive. On the other hand, Problem 5.8.54 shows how to give a constructive proof for *separable* Hilbert spaces.

Exercise 5.8.52. Use the Axiom of Choice in the form of Zorn's Lemma to show that every Hilbert space contains a complete, orthonormal set. ♢

Thus every Hilbert space contains subsets that are both complete and orthonormal. Given this, the following lemma shows that H contains a countable complete orthonormal set (i.e., an orthonormal basis) if and only if H is separable.

Lemma 5.8.53. *A Hilbert space H is separable if and only if there exists a countable sequence $\{e_n\}$ that is a orthonormal basis for H.*

Proof. $\Leftarrow$. Exercise 4.5.53 tells us that if a Banach space X contains a countable complete sequence, then X is separable.

$\Rightarrow$. Let H be a separable Hilbert space. By Exercise 5.8.52, there is a subset S that is both complete and orthonormal. Because of this orthonormality, any two distinct vectors in S are a distance $\sqrt{2}$ apart:

$$f \neq g \in S \quad \Longrightarrow \quad \|f - g\| = \sqrt{2}. \tag{5.50}$$

If S was uncountable, then by applying Problem 5.4.6 we would find that X is not separable. Since we know that H is separable, S must be countable, and therefore S is an orthonormal basis for H.

For an alternative (and constructive) proof that every separable Hilbert space contains an orthonormal basis, see Problem 5.8.54. □

In summary, if H is a finite-dimensional Hilbert space, then it contains a finite orthonormal basis $\{e_1, \dots, e_d\}$ where d is the dimension of H, while if H is separable, infinite-dimensional Hilbert space, then it contains a countably infinite orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. If H is nonseparable, then it does contain a complete orthonormal set S, but any such set must be uncountable.

Extra Problems

5.8.54. (a) Let M be a closed subspace of a Hilbert space H, and suppose that $f \notin M$. Let p be the orthogonal projection of f onto M and set $e = f - p$. Prove that

$$M \oplus \operatorname{span}\{e\} \;=\; \{m + ce : m \in M,\, c \in \mathbb{F}\}$$

is a closed subspace of H, and

$$M \oplus \operatorname{span}\{e\} \;=\; M + \operatorname{span}\{f\} \;=\; \{m + cf : c \in \mathbb{F}\}.$$

(b) Let $\{f_n\}_{n\in\mathbb{N}}$ be a linearly independent sequence in a Hilbert space H. Show that there exists an orthogonal sequence $\{e_n\}_{n\in\mathbb{N}}$ in H that satisfies

$$\operatorname{span}\{e_1, \dots, e_N\} \;=\; \operatorname{span}\{f_1, \dots, f_N\} \qquad \text{for each } N \in \mathbb{N}.$$

This is called the *Gram–Schmidt orthogonalization procedure*.

(c) How should the Gram–Schmidt orthogonalization procedure be modified if $\{f_n\}_{n\in\mathbb{N}}$ is not linearly independent?

(d) If H is a separable Hilbert space then, by definition, H contains a countable dense subset $\{f_n\}_{n\in\mathbb{N}}$. By applying the Gram–Schmidt orthogonalization procedure to this set, prove that H contains a countable complete orthonormal subset $\{e_n\}_{n\in\mathbb{N}}$ (which, by Theorem 5.8.1, is therefore an orthonormal basis for H).

5.11 The Complex Trigonometric System

The trigonometric system is a countable sequence indexed by the set of integers $\mathbb{Z}$. In contrast, the majority of the sequences we have encountered before have been indexed by the natural numbers $\mathbb{N}$. Even so, all of the results of this chapter on orthonormal sequences carry over with only minor notational changes to sequences indexed by $\mathbb{Z}$ or any other countably infinite index set. In particular, Theorems 5.7.1 and 5.8.1 both carry over to orthonormal sequences that are indexed by any countable index set. Essentially, this is a consequence of the unconditional convergence of the series that appear in those theorems. An unconditionally convergent series converges regardless of what ordering we impose on the index set, and hence we can use any countably infinite index set that we like without changing the meaning of the series.

For the index set $\mathbb{Z}$, we typically use the "standard ordering"

$$0,\,1,\,-1,\,2,\,-2,\,3\,-3\,\ldots.$$

Unless we specify otherwise, this will be the ordering that we have in mind. This means that we consider a bi-infinite series $\sum_{n\in\mathbb{Z}} x_n$ to be ordered as

$$\sum_{n\in\mathbb{Z}} x_n \;=\; x_0 + x_1 + x_{-1} + x_2 + x_{-2} + \cdots.$$

The partial sums of this series are

$$\begin{aligned} s_1 &= x_0,\\ s_2 &= x_0 + x_1,\\ s_3 &= x_0 + x_1 + x_{-1},\\ s_4 &= x_0 + x_1 + x_{-1} + x_{-2}, \end{aligned}$$

and so forth. Therefore, using this ordering, an infinite series $\sum_{n\in\mathbb{Z}} x_n$ converges if and only if these partial sums s_N converge as $N\to\infty$. If the series converges unconditionally (the typicaly situation when working with orthonormal vector), then it will converge no matter what ordering of the series that we choose. When dealing with bi-infinite series it is often convenient to consider the partial sum $s_{2N-1} = x_{-N} + \cdots + x_N$. We call this the Nth *symmetric partial sum* and often denote it by

$$S_N \;=\; s_{2N-1} \;=\; \sum_{n=-N}^{N} x_n \;=\; x_0 + x_1 + x_{-1} + \cdots + x_N + x_{-N}.$$

Extra Problems

5.11.50. Let $f(x) = x$ for $x \in [0,1]$. Compute the Fourier coefficients of f, and use this to give another proof of Euler's formula.

CHAPTER 6: OPERATOR THEORY

6.1 Linear Operators on Normed Spaces

Functionals

If the codomain of an operator is the field of scalars $\mathbb{F}$, then we say that the operator is a *functional*. In other words, a functional is simply a function of the form $f\colon X \to \mathbb{F}$. We often denote functionals by Greek letters such as λ, μ, or ρ, or sometimes by lowercase Roman letters such as d or q. The following exercise gives two examples of nonlinear functionals. We will see some linear functionals in Example 6.3.2.

Exercise 6.1.50. (a) Let X be a nontrivial normed space. Define a functional $\rho\colon X \to \mathbb{F}$ by

$$\rho(x) \;=\; \|x\|, \qquad x \in X.$$

Prove that ρ is not linear, not surjective, and not injective, even though $\ker(\rho) = \{0\}$.

(b) Fix $n \geq 2$ and let X be the set of all $n \times n$ matrices with scalar entries. Let $\det(A)$ denote the determinant of a matrix A, and define $d\colon X \to \mathbb{F}$ by $d(A) = \det(A)$. Show that the functional d is surjective, but it is not linear and not injective. ◊

By definition, an $n \times n$ matrix A is *singular* if its determinant is zero. Hence the kernel of the functional d given in part (b) of Exercise 6.1.50 is precisely the set of all $n \times n$ singular matrices.

6.4 Equivalence of Bounded and Continuous Linear Operators

Recall from Exercise 2.9.52 that if X and Y are normed spaces, then a function $A\colon X \to Y$ is *continuous at a point* $x \in X$ if $x_n \to x$ in X implies $Ax_n \to Ax$ in Y, and A is *continuous* if it is continuous at every point.

Here is an expanded version of Theorem 6.4.1.

Theorem 6.4.50 (Boundedness Equals Continuity). *Let X and Y be normed vector spaces. If $A\colon X \to Y$ is a linear operator, then the following four statements are equivalent.*

(a) *A is continuous at some point $x \in X$.*

(b) *A is continuous at $x = 0$.*

(c) *A is continuous.*

(d) *A is bounded.*

Proof. (a) ⇒ (b), (c). Suppose that A is continuous at x, and fix any vector $y \in X$. If $y_n \to y$ then $y - y_n + x \to x$, so by linearity and the definition of continuity at f we have

$$Ay - Ay_n + Ax \;=\; A(y - y_n + x) \;\to\; Ax \quad \text{as } n \to \infty.$$

Rearranging, we see that $Ay_n \to Ay$, so A is continuous at the point y. This is true for every vector y (including $y = 0$ in particular), so A is continuous.

(c) ⇒ (d). Suppose that A is continuous but unbounded. Then $\|A\| = \infty$, so for each $n \in \mathbb{N}$ there must exist a vector $x_n \in X$ such that $\|x_n\| = 1$ but $\|Ax_n\| \geq n$. Setting $y_n = x_n/n$, we compute that

$$\|y_n - 0\| \;=\; \|y_n\| \;=\; \frac{\|x_n\|}{n} \;\to\; 0 \quad \text{as } n \to \infty.$$

Thus $y_n \to 0$. Since A is continuous and linear, it follows that $Ay_n \to A0 = 0$. By the continuity of the norm (Lemma 3.3.3), this implies that

$$\lim_{n\to\infty} \|Ay_n\| \;=\; \|0\| \;=\; 0.$$

However, for each $n \in \mathbb{N}$ we have

$$\|Ay_n\| \;=\; \frac{1}{n}\,\|Ax_n\| \;\geq\; \frac{1}{n} \cdot n \;=\; 1,$$

which is a contradiction. Therefore A must be bounded.

(d) ⇒ (a). Suppose that A is bounded, and choose any vector $x \in X$. Suppose that $x_n \to x$ in X. Applying linearity and Lemma 6.2.4, we see that

$$\|Ax_n - Ax\| \;=\; \|A(x_n - x)\| \;\leq\; \|A\|\,\|x_n - x\| \;\to\; 0.$$

Thus $Ax_n \to Ax$, so A is continuous at f. □

6.5 The Space $\boldsymbol{\mathcal{B}(X, Y)}$

Using abstract language, Lemma 6.5.5 shows that $\mathcal{B}(X)$ is a *Banach algebra*. This Banach algebra is noncommutative since $AB \neq BA$ in general, but on the other hand it does have an identity element (the identity operator I).

Extra Problems

6.5.50. Let X, Y be normed spaces. Fix $A \in \mathcal{B}(X,Y)$ and $x_n \in X$. Prove the following statements.

(a) If a series $\sum_{n=1}^\infty x_n$ converges in X, then $\sum_{n=1}^\infty Ax_n$ converges in Y.

(b) If $\sum_{n=1}^\infty x_n$ converges unconditionally in X, then $\sum_{n=1}^\infty Ax_n$ converges unconditionally in Y.

(c) If $\sum_{n=1}^\infty x_n$ converges absolutely in X, then $\sum_{n=1}^\infty Ax_n$ converges absolutely in Y.

6.5.51. (a) What is the difference between $C_b(\mathbb{R})$ and $\mathcal{B}(\mathbb{R},\mathbb{F})$? Is one a subset of the other?

(b) Give a complete characterization of the elements of $\mathcal{B}(\mathbb{R},\mathbb{F})$.

6.5.52. This is an extension of Problem 6.3.6.

Let $A = [a_{ij}]$ be an $m \times n$ matrix, which we identify with the linear operator $A\colon \mathbb{F}^n \to \mathbb{F}^m$ that maps x to Ax. Since $\mathbb{F}^n$ is finite dimensional, Theorem 6.3.1 implies that A is bounded with respect to any norm on $\mathbb{F}^n$. However, the exact value of the operator norm of A does depend on which norms we choose to place on $\mathbb{F}^n$ and $\mathbb{F}^m$.

(a) Prove that if the norm on both $\mathbb{F}^n$ and $\mathbb{F}^m$ is $\|\cdot\|_1$ then the operator norm of A is

$$\|A\|_{\ell^1\to\ell^1} \;=\; \max_{j=1,\dots,n}\biggl\{\sum_{i=1}^m |a_{ij}|\biggr\}.$$

Show further that if $m = n$ then this operator norm is submultiplicative, i.e,. for all $n \times n$ matrices A and B we have

$$\|AB\|_{\ell^1\to\ell^1} \;\le\; \|A\|_{\ell^1\to\ell^1}\,\|B\|_{\ell^1\to\ell^1}.$$

(b) Prove that if the norm on $\mathbb{F}^n$ and $\mathbb{F}^m$ is $\|\cdot\|_\infty$ then the operator norm of A is

$$\|A\|_{\ell^\infty\to\ell^\infty} \;=\; \max_{i=1,\dots,m}\biggl\{\sum_{j=1}^n |a_{ij}|\biggr\}.$$

Show that this operator norm is also submultiplicative when $m = n$.

(c) Let $\|A\|_\infty$ be the sup-norm of the entries of A, i.e.,

$$\|A\|_\infty \;=\; \max_{i,j} |a_{ij}|.$$

Prove that this is a norm on the set of all $m \times n$ matrices, but if $m = n \ge 2$ then the sup-norm is *not* submultiplicative.

6.6 Isometries and Isometric Isomorphisms

The translation operator $T_a\colon C_b(\mathbb{R}) \to C_b(\mathbb{R})$ defined in Example 3.52.53 is an isometric isomorphism because it is a bijection and equation (3.42) tells us that $\|T_a g\|_{\mathrm{u}} = \|g\|_{\mathrm{u}}$ for every vector f in the domain $C_b(\mathbb{R})$.

Extra Problems

6.6.50. (a) Prove that the dilation operator $D_a\colon C_b(\mathbb{R}) \to C_b(\mathbb{R})$ defined in Problem 3.52.58 is a bounded linear bijection, and its operator norm is $\|D_a\| = 1$. Is it an isometry?

(b) Prove that the operators L and M defined in Problem 3.52.56 are bounded and linear, and find their operator norms. Is either one an isometry?

6.6.51. (a) We say that a function $f\colon \mathbb{R} \to \mathbb{F}$ is 1-*periodic* if $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Prove that

$$C_b^{\mathrm{per}}(\mathbb{R}) \;=\; \bigl\{f \in C_b(\mathbb{R}) : f \text{ is 1-periodic}\bigr\}$$

is a closed subspace of $C_b(\mathbb{R})$ with respect to the uniform norm on $C_b(\mathbb{R})$.

(b) Prove that

$$C^{\mathrm{per}}[0,1] \;=\; \bigl\{f \in C[0,1] : f(0) = f(1)\bigr\}$$

is a closed subspace of $C[0,1]$ with respect to the uniform norm on $C[0,1]$.

(c) Prove that

$$C_b^{\mathrm{per}}(\mathbb{R}) \;\cong\; C^{\mathrm{per}}[0,1],$$

i.e., there exists an isometric isomorphism $A\colon C_b^{\mathrm{per}}(\mathbb{R}) \to C^{\mathrm{per}}[0,1]$.

6.6.52. Suppose that X and Y are normed spaces, and $\{x_n\}_{n\in\mathbb{N}}$ is a Schauder basis for X. Show that if $A\colon X \to Y$ is a topological isomorphism, then $\{Ax_n\}_{n\in\mathbb{N}}$ is a Schauder basis for Y.

6.7 Infinite Matrices

Expanded Discussion of Convolution

In this section we will consider bi-infinite sequences of scalars, i.e., sequences of the form

$$x \;=\; (x_k)_{k\in\mathbb{Z}} \;=\; \bigl(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\bigr),$$

where each component x_k is a scalar. As discussed in Remark 3.2.50, if p is finite then such a sequence x is p-summable if $\sum_{k\in\mathbb{Z}} |x_k|^p < \infty$. We let $\ell^p(\mathbb{Z})$ be the space of all p-summable bi-infinite sequences, and $\ell^\infty(\mathbb{Z})$ be the space of all bounded bi-infinite sequences.

All of the results of Section 3.2 carry over from ℓ^p to $\ell^p(\mathbb{Z})$. Thus, if $1 \le p \le \infty$ then $\ell^p(\mathbb{Z})$ is a Banach space with respect to the norm

$$\|x\|_p = \begin{cases} \left(\displaystyle\sum_{k=-\infty}^{\infty} |x_k|^p \right)^{1/p}, & 1 \le p < \infty, \\ \displaystyle\sup_{k\in\mathbb{Z}} |x_k|, & p = \infty. \end{cases}$$

In particular, $\ell^p(\mathbb{Z})$ is closed under the operations of addition of sequences and multiplication of a sequence by a scalar. We will introduce another operation on sequences, called *convolution*, and we will prove that the particular space $\ell^1(\mathbb{Z})$ is closed with respect to this new operation. This gives us an example of a Banach space with an extra operation that makes it into a *Banach algebra*.

We motivate convolution by looking at multiplication of polynomials. If $p(x) = a_0 + a_1 x + \cdots a_m x^m$ and $q(x) = b_0 + b_1 x + \cdots b_n x^n$ are two polynomials, then their product is the polynomial

$$p(x)q(x) = c_0 + c_1 x + \cdots + c_{m+n} x^{m+n}$$

whose coefficients c_k are given by the formula

$$c_k = \sum_{j=0}^{k} a_j\, b_{k-j}, \qquad k = 0, \dots, m+n, \tag{6.51}$$

where we take $a_k = 0$ for $k > m$ and $b_k = 0$ for $k > n$. We can think of equation (6.51) as a rule that tells us how to combine two finite sequences $a = (a_0, \dots, a_m)$ and $b = (b_0, \dots, b_n)$ to obtain a new sequence $c = (c_0, \dots, c_{m+n})$. Extending this idea to infinite sequences gives us the definition of convolution.

Definition 6.7.50 (Convolution of Sequences). Let $x = (x_k)_{k\in\mathbb{Z}}$ and $y = (y_k)_{k\in\mathbb{Z}}$ be bi-infinite sequences of scalars. If the series

$$c_k = \sum_{j=-\infty}^{\infty} x_j\, y_{k-j}, \tag{6.52}$$

converges for each $k \in \mathbb{Z}$, then the *convolution of x with y* is the bi-infinite sequence $x * y$ whose components are c_k, i.e.,

$$x * y = (c_k)_{k\in\mathbb{Z}} = \left(\sum_{j\in\mathbb{Z}} x_j\, y_{k-j} \right)_{k\in\mathbb{Z}}.$$

If the series in equation (6.52) does not converge for some k, then the convolution of x with y is undefined. ♢

The convolution of two arbitrary sequences need not be defined. For example, if

$$x = \big(2^k\big)_{k\in\mathbb{Z}} \qquad \text{and} \qquad y = \big(2^{-k}\big)_{k\in\mathbb{Z}},$$

then the series given in equation (6.52) does not converge for *any* k, so the convolution of these two particular sequences does not exist. If the convolution $x * y$ does exist, then we usually denote the kth component of $x * y$ by $(x * y)_k$ instead of c_k. That is, if $x * y$ exists then its components are

$$(x * y)_k = \sum_{j=-\infty}^{\infty} x_j\, y_{k-j}, \qquad k \in \mathbb{Z}.$$

The following theorem shows that the convolution of two *summable* sequences always exists, and it also gives us a bound for the ℓ^1-norm of a convolution. In the proof, we use the fact that we can reindex convergent series. In particular, if k is fixed then by setting $i = k - j$ we see that

$$\sum_{j=-\infty}^{\infty} y_{k-j} = \sum_{i=-\infty}^{\infty} y_i.$$

Note that it is important here that the summation is over all of $\mathbb{Z}$, otherwise the range of summation would change when we reindex. This is one reason why we use bi-infinite sequences when we define convolution. The proof of the following result is Problem 6.7.9.

Theorem 6.7.51. *If $x,\, y \in \ell^1(\mathbb{Z})$, then $x * y \in \ell^1(\mathbb{Z})$. Further, the following* submultiplicative norm inequality *holds:*

$$\|x * y\|_1 \le \|x\|_1\, \|y\|_1, \qquad x, y \in \ell^1(\mathbb{Z}). \qquad ♢ \tag{6.53}$$

Thus $\ell^1(\mathbb{Z})$ is closed under convolution. Problem 6.7.10 gives some further properties of convolution.

Extra Material on Banach Algebras

In summary, not only is $\ell^1(\mathbb{Z})$ a Banach space, but it is also closed under the operation of convolution. Indeed, convolution is a "multiplication-like" operation on $\ell^1(\mathbb{Z})$ in the sense that it has properties similar to ordinary multiplication of scalars. We have a special name for normed spaces that have such a "multiplicative" operation.

Definition 6.7.52 (Normed Algebras and Banach Algebras). Let $\mathcal{A}$ be a normed vector space. We say that $\mathcal{A}$ is a *normed algebra* if for each choice of $x, y \in \mathcal{A}$ there exists a unique product $xy \in \mathcal{A}$ that satisfies the following conditions for all $x, y, z \in \mathcal{A}$ and all scalars $c \in \mathbb{F}$.

(a) Associativity: $(xy)z = x(yz)$.

(b) Distributive Law: $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$.

(c) Interaction with scalar multiplication: $c(xy) = (cx)y = x(cy)$.

(d) Submulticativity: $\|xy\| \le \|x\|\,\|y\|$ for all $x, y \in \mathcal{A}$.

If $xg = gx$ for all $x, g \in \mathcal{A}$ then we say that $\mathcal{A}$ is *commutative.*

If there exists an element $e \in \mathcal{A}$ such that $ex = xe = x$ for every $x \in \mathcal{A}$ then we say that $\mathcal{A}$ is a normed algebra *with identity.*

If $\mathcal{A}$ is compete, then we say that $\mathcal{A}$ is a *Banach algebra.* In other words, a Banach algebra is a Banach space that is also a normed algebra. ♢

Thus $\ell^1(\mathbb{Z})$ is a commutative Banach algebra with identity with respect to the operation of convolution.

Example 6.7.53. (a) $C_b(\mathbb{R})$ is a commutative Banach algebra with identity under the operation of pointwise products of functions, $(fg)(x) = f(x)g(x)$.

(b) $C_0(\mathbb{R})$ is a commutative Banach algebra without identity with respect to pointwise products.

(c) Since the operator norm is submultiplicative, if X is a Banach space then the space $\mathcal{B}(X)$ of all bounded linear operators mapping X into itself is a noncommutative Banach algebra with identity with respect to composition of operators. ♢

Some normed algebras also have an additional operation that has properties similar to that of conjugation of complex numbers.

Definition 6.7.54 (Involution). Let $\mathcal{A}$ be a normed algebra. We say that $\sim$ is an *involution* on $\mathcal{A}$ (with respect to the product on $\mathcal{A}$) if for each $x \in \mathcal{A}$ there is an element $\widetilde{x} \in A$ such that the following conditions are satisfied for all $x, y \in \mathcal{A}$ and all scalars $c \in \mathbb{F}$:

(a) $\widetilde{\widetilde{x}} = x$,

(b) $\widetilde{xy} = \widetilde{y}\,\widetilde{x}$,

(c) $(x+y)^{\sim} = \widetilde{x} + \widetilde{y}$, and

(d) $\widetilde{cx} = \bar{c}\,\widetilde{x}$. ♢

If $\mathbb{F} = \mathbb{R}$, then the complex conjugate that appears on the scalar c in part (d) of the preceding definition is superfluous.

The following exercise shows that $\ell^1(\mathbb{Z})$ has an involution.

Exercise 6.7.55. Given $x = (x_k)_{k\in\mathbb{Z}}$ in $\ell^1(\mathbb{Z})$, define

$$\widetilde{x} \;=\; \bigl(\overline{x_{-k}}\bigr)_{k\in\mathbb{Z}} \;=\; \bigl(\dots,\, \overline{x_2},\, \overline{x_1},\, \overline{x_0},\, \overline{x_{-1}},\, \overline{x_{-2}},\, \dots\bigr).$$

Prove that $\sim$ is an involution on $\ell^1(\mathbb{Z})$ with respect to convolution. ♢

Extra Problems

6.7.56. (a) Let $c_{00}(\mathbb{Z})$ be the set of all bi-infinite sequences $x = (x_k)_{k\in\mathbb{Z}}$ such that $x_k \neq 0$ for at most finitely many k. Prove that $c_{00}(\mathbb{Z})$ is closed under convolution, i.e., if $x, y \in c_{00}(\mathbb{Z})$ then $x * y$ exists and belongs to c_{00}.

(b) Let $c_0(\mathbb{Z})$ be the set of all bi-infinite sequences $x = (x_k)_{k\in\mathbb{Z}}$ such that $x_k \to 0$ as $k \to \pm\infty$. Exhibit sequences $x, y \in c_0(\mathbb{Z})$ such that the convolution $x * y$ does not exist.

6.7.57. Prove the following statements.

(a) If $1 \le p \le \infty$, $x \in \ell^p(\mathbb{Z})$, and $y \in \ell^{p'}(\mathbb{Z})$, then $x * y \in \ell^\infty(\mathbb{Z})$ and

$$\|x * y\|_\infty \;\le\; \|x\|_p\, \|y\|_{p'}.$$

(b) If $1 < p < \infty$, $x \in \ell^p(\mathbb{Z})$, and $y \in \ell^{p'}(\mathbb{Z})$, then $x * y \in c_0(\mathbb{Z})$.

(c) There exist sequences $x \in \ell^1(\mathbb{Z})$ and $y \in \ell^\infty(\mathbb{Z})$ such that $x*y \notin c_0(\mathbb{Z})$.

(d) If $x \in \ell^1(\mathbb{Z})$, and $y \in c_0(\mathbb{Z})$, then $x * y \in c_0(\mathbb{Z})$.

6.7.58. Fix $1 < p < \infty$, and suppose that $x \in \ell^p(\mathbb{Z})$ and $y \in \ell^1(\mathbb{Z})$.

(a) Show that

$$|(x*y)_k| \;\le\; \sum_{j=-\infty}^{\infty} \Bigl(|x_j|\, \bigl|y_{k-j}\bigr|^{1/p}\Bigr)\, \bigl|y_{k-j}\bigr|^{1/p'}\, dy. \tag{6.54}$$

Apply Hölder's Inequality with exponents p and p' to the two factors that appear on the right-hand side of equation (6.54) to show that

$$|(x*y)_k| \;\le\; \|y\|_1^{1/p'} \left(\sum_{j=-\infty}^{\infty} |x_j|^p\, |y_{k-j}|\right)^{1/p}.$$

(b) Use part (a) and Tonelli's Theorem to directly prove *Young's Inequality*:

$$\|x*y\|_p \;\le\; \|x\|_p\, \|g\|_1. \tag{6.55}$$

Remark: Equation (6.55) also holds for $p = 1$ (see Problem 6.7.9) and for $p = \infty$ (see Problem 6.7.11).

CHAPTER 7: OPERATORS ON HILBERT SPACES

7.8 The Spectral Theorem for Compact Self-Adjoint Operators

Extra Problems

7.8.50. Let $\mathbb{F} = \mathbb{C}$ for this problem. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of an $n \times n$ matrix A, ordered so that $|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_n|$. Let $\|x\|$ be the Euclidean norm on $\mathbb{C}^n$, and let $\|A\|$ be the operator norm of A, defined by $\|A\| = \max\{\|Ax\| : \|x\| = 1\}$.

(a) Show that $\|A^k\|^{1/k} \ge |\lambda_1|$ for any positive integer k.

(b) Suppose that A is diagonalizable, i.e., there exists an invertible matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$ Show that $\|\Lambda\| = |\lambda_1|$. Use this to show that

$$\lim_{k\to\infty} \|A^k\|^{1/k} \;\le\; |\lambda_1|.$$

(c) Combine parts (a) and (b) to deduce that

$$\lim_{k\to\infty} \|A^k\|^{1/k} \;=\; |\lambda_1|.$$

This limit is called the *spectral radius* of A.