FUNCTIONAL ANALYSIS LECTURE NOTES:

WEAK AND WEAK* CONVERGENCE

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1. Weak and Weak* Convergence of Vectors

Definition 1.1. Let X be a normed linear space, and let $x_n,\ x \in X$.

a. We say that x_n *converges*, *converges strongly*, or *converges in norm* to x, and write $x_n \to x$, if

$$\lim_{n\to\infty} \|x - x_n\| \;=\; 0.$$

b. We say that x_n *converges weakly* to x, and write $x_n \xrightarrow{\mathrm{w}} x$, if

$$\forall\, \mu \in X^*, \quad \lim_{n\to\infty} \langle x_n, \mu\rangle \;=\; \langle x, \mu\rangle.$$

Exercise 1.2. a. Show that strong convergence implies weak convergence.

b. Show that weak convergence does not imply strong convergence in general (look for a Hilbert space counterexample).

If our space is itself the dual space of another space, then there is an additional mode of convergence that we can consider, as follows.

Definition 1.3. Let X be a normed linear space, and suppose that $\mu_n,\ \mu \in X^*$. Then we say that μ_n *converges weak** to μ, and write $\mu_n \xrightarrow{\mathrm{w}^*} \mu$, if

$$\forall\, x \in X, \quad \lim_{n\to\infty} \langle x, \mu_n\rangle \;=\; \langle x, \mu\rangle.$$

Note that weak* convergence is just "pointwise convergence" of the operators μ_n!

Remark 1.4. Weak* convergence only makes sense for a sequence that lies in a dual space X^*. However, if we do have a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in X^*, then we can consider three types of convergence of μ_n to μ: strong, weak, and weak*. By definition, these are:

$$\begin{aligned}
\mu_n \to \mu \quad &\Longleftrightarrow \quad \lim_{n\to\infty} \|\mu - \mu_n\| \;=\; 0,\\
\mu_n \xrightarrow{\mathrm{w}} \mu \quad &\Longleftrightarrow \quad \forall\, T \in X^{**}, \quad \lim_{n\to\infty} \langle \mu_n, T\rangle \;=\; \langle \mu, T\rangle,\\
\mu_n \xrightarrow{\mathrm{w}^*} \mu \quad &\Longleftrightarrow \quad \forall\, x \in X, \quad \lim_{n\to\infty} \langle x, \mu_n\rangle \;=\; \langle x, \mu\rangle.
\end{aligned}$$

Exercise 1.5. Given μ_n, $\mu \in X^*$, show that

$$\mu_n \to \mu \quad \implies \quad \mu_n \xrightarrow{\text{w}} \mu \quad \implies \quad \mu_n \xrightarrow{\text{w}^*} \mu. \tag{1.1}$$

If X is reflexive, show that

$$\mu_n \xrightarrow{\text{w}} \mu \quad \iff \quad \mu_n \xrightarrow{\text{w}^*} \mu.$$

In general, however, the implications in (1.1) do not hold in the reverse direction.

Lemma 1.6. a. Weak* limits are unique.

b. Weak limits are unique.

Proof. Suppose that X is a normed linear space, and that we had both $\mu_n \xrightarrow{\text{w}^*} \mu$ and $\mu_n \xrightarrow{\text{w}^*} \nu$ in X^*. Then, by definition,

$$\forall\, x \in X, \quad \langle x, \mu \rangle \;=\; \lim_{n\to\infty} \langle x, \mu_n \rangle \;=\; \langle x, \nu \rangle,$$

so $\mu = \nu$.

b. Suppose that we have both $x_n \xrightarrow{\text{w}} x$ and $x_n \xrightarrow{\text{w}} y$ in X. Then, by definition,

$$\forall\, \mu \in X^*, \quad \langle x, \mu \rangle \;=\; \lim_{n\to\infty} \langle x_n, \mu \rangle \;=\; \langle y, \mu \rangle.$$

Hence, by Hahn–Banach,

$$\|x - y\| \;=\; \sup_{\|\mu\|=1} |\langle x - y, \mu \rangle| \;=\; 0,$$

so $x = y$. □

It is trivial to show that strongly convergent sequences are bounded. However, we need some fairly sophisticated machinery (the Uniform Boundedness Principle) to show that weakly convergent and weak* convergent sequences are likewise bounded.

Exercise 1.7. a. Show that weak* convergent sequences in the dual of a Banach space are bounded.

Give an example of an unbounded but weak* convergence sequence in the dual of an incomplete normed space.

Hint: The dual space of c_{00} under the ℓ^∞ norm is $(c_{00})^* \cong \ell^1$.

b. Show that weakly convergent sequences in a normed space are bounded.

Next, we will show that strong convergence is equivalent to weak convergence in finite-dimensional spaces.

Lemma 1.8. If X is a finite-dimensional vector space, then strong convergence is equivalent to weak convergence.

Proof. Consider first the case that $X = \mathbb{F}^d$ under the Euclidean norm $\|\cdot\|_2$. Suppose that $x_n \xrightarrow{\mathrm{w}} x$ in $\mathbb{F}^d$. Then for each standard basis vector e_k, we have

$$x_n \cdot e_k \to x \cdot e_k, \qquad k = 1, \dots, d.$$

That is, weak convergence implies componentwise convergence. But since there are only finitely many components, this implies norm convergence, since

$$\|x - x_n\|_2^2 \;=\; \sum_{k=1}^{d} |x \cdot e_k - x_n \cdot e_k|^2 \;\to\; 0 \quad \text{as } n \to \infty.$$

For the general case, choose any basis $\mathcal{B} = \{e_1, \dots, e_d\}$ for X, and use the fact that all norms on X are equivalent to define an isomorphism between X and $\mathbb{F}^d$. □

Often, there exists a connection between componentwise or pointwise convergence and weak convergence. This is related to the question of whether "point evaluation" are continuous linear functionals on a given space.

Example 1.9. Fix $1 \le p \le \infty$, and consider the space ℓ^p. As usual, given $x \in \ell^p$ and $y \in \ell^{p'}$, write

$$\langle x, y \rangle \;=\; \sum_{k=1}^{\infty} x_k\, \bar{y}_k.$$

The standard basis vectors e_k belong to every ℓ^p, and hence $e_k \in \ell^{p'} \subseteq (\ell^p)^*$ (with equality if $p < \infty$). Therefore, if we have $x_n = (x_n(k))_{k \in \mathbb{N}}$ and $y = (y(k))_{k \in \mathbb{N}}$ and we know that $x_n \xrightarrow{\mathrm{w}} y$, then we have for each $k \in \mathbb{N}$ that

$$x_n(k) \;=\; \langle x_n, e_k \rangle \;\to\; \langle y, e_k \rangle \;=\; y(k).$$

Thus,

$$\text{weak convergence in } \ell^p \quad\Longrightarrow\quad \text{componentwise convergence in } \ell^p.$$

The converse is not true in general, but the following result gives necessary and sufficient conditions, at least for some p.

Exercise 1.10. Fix $1 < p < \infty$, and let x_n, $y \in \ell^p$ be given. Prove that the following two statements are equivalent.

a. $x_n \xrightarrow{\mathrm{w}} y$.

b. $x_n(k) \to y(k)$ for each k (componentwise convergence) and $\sup \|x_n\|_p < \infty$.

What happens if $p = 1$ or $p = \infty$?

Exercise 1.11. Suppose that x_n, $y \in \ell^1$ are given. Since $\ell^1 \cong c_0^*$, we can consider weak* convergence of x_n to y. Prove that the following two statements are equivalent.

a. $x_n \xrightarrow{\mathrm{w}^*} y$.

b. $x_n(k) \to y(k)$ for each k (componentwise convergence) and $\sup \|x_n\|_1 < \infty$.

Exercise 1.12. Let $f_n,\ f \in C_0(\mathbb{R})$ be given. Since $C_0(\mathbb{R})^* \cong M_b(\mathbb{R})$, we can consider weak convergence of f_n to f. Prove that the following two statements are equivalent.

a. $f_n \xrightarrow{\text{w}} f$.

b. $f_n(x) \to f(x)$ pointwise for each x, and $\sup \|f_n\|_\infty < \infty$.

Exercise 1.13. Let $\mu_n,\ \mu \in M_b(\mathbb{R})$ be given. Show that $\mu_n \xrightarrow{\text{w}^*} \mu$ does not imply $\|\mu_n\| \to \|\mu\|$.

Exercise 1.14. Fix $1 < p < \infty$, and let $f_n \in L^p(\mathbb{R})$ be given. Prove that the following two statements are equivalent.

a. $f_n \xrightarrow{\text{w}} 0$.

b. $\int_E f_n \to 0$ for every $E \subseteq \mathbb{R}$ with $|E| < \infty$, and $\sup \|f_n\|_p < \infty$.

Proof. b $\Rightarrow$ a. Suppose that statement b holds, and let $R = \sup \|f_n\|_p$. Choose any $g \in L^{p'}(\mathbb{R})$. Since the step functions are dense in $L^{p'}(\mathbb{R})$, we can find a function of the norm

$$\varphi \;=\; \sum_{k=1}^{M} c_k \chi_{F_k},$$

with each F_k a measurable subset of $\mathbb{R}$, such that

$$\|g - \varphi\|_{p'} \;<\; \frac{\varepsilon}{4R}.$$

Since $\varphi \in L^{p'}(\mathbb{R})$, there exists a compact set K such that if we set $\psi = \varphi\chi_K$, then we have

$$\|\varphi - \psi\|_{p'} \;<\; \frac{\varepsilon}{4R}.$$

Furthermore, note that ψ is a step function, since

$$\psi \;=\; \sum_{k=1}^{M} c_k \chi_{F_k}\, \chi_K \;=\; \sum_{k=1}^{M} c_k \chi_{E_k},$$

where $E_k = F_k \cap K$.

Set

$$C \;=\; \sum_{k=1}^{M} |c_k|,$$

and assume for now that $C > 0$. Since each E_k has finite measure, by hypothesis we can find an integer N such that

$$n > N \quad\Longrightarrow\quad \left| \int_{E_k} f_n \right| \;<\; \frac{\varepsilon}{2C}.$$

Hence for $n > N$ we have

$$\begin{aligned} |\langle g, f_n\rangle| &\le |\langle g-\varphi, f_n\rangle| + |\langle \varphi-\psi, f_n\rangle| + |\langle \psi, f_n\rangle| \\ &< \|g-\varphi\|\,\|f_n\|| + \|\varphi-\psi\|\,\|f_n\| + \left|\int \sum_{k=1}^{M} c_k \chi_{E_k}\, f_n\right| \\ &< \frac{\varepsilon}{4R}\,R + \frac{\varepsilon}{4R}\,R + \sum_{k=1}^{M} |c_k| \left|\int_{E_k} f_n\right| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \sum_{k=1}^{M} |c_k|\,\frac{\varepsilon}{2C} \\ &= \varepsilon. \end{aligned}$$

If $C = 0$ then we still obtain $|\langle g, f_n\rangle| < \varepsilon$. This shows that

$$0 \le \limsup_{n\to\infty} |\langle g, f_n\rangle| \le \varepsilon.$$

Since this is true for every ε, we conclude that

$$\lim_{n\to\infty} |\langle g, f_n\rangle| = 0.$$

And since this is true for every $g \in L^{p'}(\mathbb{R})$, we have $f_n \xrightarrow{\mathrm{w}} 0$. □

Exercise 1.15. Let H be a Hilbert space. Show that

$$f_n \to f \qquad \Longleftrightarrow \qquad f_n \xrightarrow{\mathrm{w}} f \text{ and } \|f_n\| \to \|f\|.$$

Exercise 1.16. Let H and K be Hilbert spaces, and let $T \in \mathcal{B}(H,K)$ be a compact operator. Show that

$$f_n \xrightarrow{\mathrm{w}} f \implies Tf_n \to Tf.$$

Thus, a compact operator maps weakly convergent sequences to strongly convergent sequences.

2. Convergence of Operators

We can apply similar notions to convergence of operators.

Definition 2.1. Let X, Y be normed linear spaces, and let A_n, $A \in \mathcal{B}(X, Y)$ be given.

a. We say that A_n *converges in operator norm to* A, or that A_n *is uniformly operator convergent to* A, and write $A_n \to A$, if

$$\lim_{n\to\infty} \|A - A_n\| \;=\; 0.$$

Rewriting the definition of operator norm, this is equivalent to

$$\lim_{n\to\infty} \left(\sup_{\|x\|=1} \|Ax - A_n x\| \right) \;=\; 0.$$

b. We say that A_n *converges in the strong operator topology (SOT) to* A, or that A_n *is strongly operator convergent to* A, if

$$\forall\, x \in X, \quad A_n x \to Ax \text{ (strong convergence in } Y).$$

Equivalently, this holds if

$$\forall\, x \in X, \quad \lim_{n\to\infty} \|Ax - A_n x\| \;=\; 0.$$

c. We say that A_n *is weakly operator convergent to* A, if

$$\forall\, x \in X, \quad A_n x \xrightarrow{\text{w}} Ax \text{ (weak convergence in } Y).$$

Equivalently, this holds if

$$\forall\, x \in X, \quad \forall\, \mu \in Y^*, \quad \lim_{n\to\infty} \langle A_n x, \mu\rangle = \langle A_n x, \mu\rangle.$$

Remark 2.2. In particular, consider the case $Y = \mathbb{F}$, i.e., the operators A_n are bounded linear functionals on X. Since $Y = Y^*$, strong and weak convergence in Y are equivalent. Hence for this case, strong operator convergence and weak operator convergence are equivalent, and in fact, they are simply weak* convergence of the operators A_n in X^*. Further, uniform operator convergence is simply operator norm convergence of the operators A_n in X^*.