FUNCTIONAL ANALYSIS LECTURE NOTES:

EGOROFF AND LUSIN'S THEOREMS

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1. Egoroff's Theorem

Egoroff's Theorem is a useful fact that applies to general bounded positive measures.

Theorem 1 (Egoroff's Theorem)**.** Suppose that μ is a finite measure on a measure space X, and $f_n,\ f\colon X \to \mathbb{C}$ are measurable. If $f_n \to f$ pointwise a.e., then for every $\varepsilon > 0$ there exists a measurable $E \subseteq X$ such that

(a) $\mu(E) < \varepsilon$, and

(b) f_n converges uniformly to f on $E^{\mathrm{C}} = X \backslash E$, i.e.,

$$\lim_{n\to\infty} \sup_{x \notin E} |f(x) - f_n(x)| \;=\; 0.$$

Proof. Let Z be the set of measure zero where $f_n(x)$ does not converge to $f(x)$. For $k, n \in \mathbb{N}$, define the measurable sets

$$E_n(k) \;=\; \bigcup_{m=n}^{\infty} \Big\{|f - f_m| \ge \frac{1}{k}\Big\} \qquad \text{and} \qquad Z_k \;=\; \bigcap_{n=1}^{\infty} E_n(k).$$

Now, if $x \in Z_k$, then $x \in E_n(k)$ for every n. Hence, for each n there must exist an $m \ge n$ such that $|f(x) - f_m(x)| > \frac{1}{k}$. Therefore $f_n(x)$ does not converge to $f(x)$, so $x \in Z$. Thus

$$Z_k \;\subseteq\; Z,$$

and therefore $\mu(Z_k) = 0$ by monotonicity. Since $E_1(k) \supseteq E_2(k) \supseteq \cdots$, we therefore have by continuity from above that

$$\lim_{n\to\infty} \mu(E_n(k)) \;=\; \mu(Z_k) \;=\; 0.$$

Choose now any $\varepsilon > 0$. Then for each k, we can find an n_k such that

$$\mu(E_{n_k}(k)) \;<\; \frac{\varepsilon}{2^k}.$$

Define

$$E \;=\; \bigcup_{k=1}^{\infty} E_{n_k}(k),$$

then we have by subadditivity that $\mu(E) \leq \varepsilon$. And if $x \notin E$, then $x \notin E_{n_k}(k)$ for every k, and therefore $|f(x) - f_m(x)| < \frac{1}{k}$ for all $m \geq n_k$. Thus, we have shown that for each $k \in \mathbb{N}$, there exists an $n_k > 0$ such that for all $m \geq n_k$ we have

$$\sup_{x \notin E} |f(x) - f_m(x)| \;\leq\; \frac{1}{k}.$$

This implies that f_n converges uniformly to f on E^{C}. □

2. Lusin's Theorem

In this section we will prove some useful facts about Radon measures on the real line. Most of these arguments extend to more general locally compact Hausdorff (LCH) domains without much change.

We begin with the following application of Urysohn's Lemma.

Theorem 2. If μ is a Radon measure then $C_c(\mathbb{R})$ is dense in $L^1(\mu)$.

Proof. First consider the function $f = \chi_E$ where $\mu(E) < \infty$. Since μ is both inner and outer regular, we can find $K \subseteq E \subseteq U$ with K compact and U open such that $\mu(U \backslash E) < \varepsilon$ and $\mu(E \backslash K) < \varepsilon$. By Urysohn's Lemma, we can find a function $\theta \in C_c(\mathbb{R})$ such that $0 \leq \theta \leq 1$ everywhere, $\theta = 1$ on K, and $\theta = 0$ on $\mathbb{R} \backslash U$. Then

$$\|\chi_E - \theta\|_1 \;=\; \int |\chi_E - \theta| \;=\; \int_{U \backslash K} |\chi_E - \theta| \;\leq\; |U \backslash K| \;<\; 2\varepsilon.$$

Hence χ_E can be approximated arbitrarily closely in L^1-norm by elements of $C_c(\mathbb{R})$.

Now let $f \in L^1(\mu)$ be arbitrary. Then there exists a simple function $\phi = \sum_{k=1}^{N} c_k \chi_{E_k}$ with all $\mu(E_k) < \infty$ such that $\|f - \phi\|_1 < \varepsilon$. Without loss of generality we may assume $c_k \neq 0$ for each k. Then by the work above, for each k there exists a function $g_k \in C_c(\mathbb{R})$ such that

$$\left\|\chi_{E_k} - g_k\right\|_1 \;<\; \frac{\varepsilon}{N\,|c_k|}, \qquad k = 1, \ldots, N.$$

Hence $g = \sum_{k=1}^{N} c_k g_k \in C_c(\mathbb{R})$ satisfies

$$\|f - g\|_1 \;\leq\; \|f - \phi\|_1 + \left\| \sum_{k=1}^{N} c_k \left(\chi_{E_k} - g_k\right) \right\|_1 \;\leq\; \varepsilon + \sum_{k=1}^{N} |c_k| \, \frac{\varepsilon}{N\,|c_k|} \;=\; 2\varepsilon.$$

Therefore $C_c(\mathbb{R})$ is dense in $L^1(\mu)$. □

Now we will prove Lusin's Theorem.

Theorem 3 (Lusin's Theorem)**.** Let μ be a Radon measure on $\mathbb{R}$. If $f\colon \mathbb{R} \to \mathbb{C}$ is Borel measurable and $\mu(\{f \neq\}) < \infty$, then for every $\varepsilon > 0$, there exists $\varphi \in C_c(\mathbb{R})$ such that

$$\mu\bigl(\{f \neq \varphi\}\bigr) \;<\; \varepsilon.$$

Further, if f is bounded, then φ can be constructed so that

$$\|\varphi\|_\infty \;\leq\; \|f\|_\infty.$$

Proof. Let $E = \{f \neq 0\}$, which by hypothesis has finite measure.

Suppose first that f is bounded. Then $f \in L^1(\mu)$ since $\mu(E) < \infty$. By Theorem 2, there exist functions $g_k \in C_c(\mathbb{R})$ such that $g_n \to f$ in $L^1(\mu)$. Since L^1-convergence implies convergence in measure, there exists a subsequence $\{h_k\}_{k\in\mathbb{N}}$ of $\{g_k\}_{k\in\mathbb{N}}$ such that $h_k \to f$ pointwise μ-a.e. By Egoroff's Theorem, given $\varepsilon > 0$ there exists some subset $A \subseteq E$ such that $\mu(E\backslash A) < \varepsilon$ and h_k converges uniformly to f on A. Since each h_k is continuous and the uniform limit of continuous functions is continuous, we conclude that f is continuous when restricted to the set A. We need to find a continuous function on $\mathbb{R}$ that agrees with f on A and has the required additional properties.

Now, since Radon measures are inner and outer regular, we can find a compact set $K \subseteq A$ and an open set $U \supseteq E$ such that $\mu(A\backslash K) < \varepsilon$ and $\mu(U\backslash E) < \varepsilon$. Note that f is continuous on the compact set K. The *Tietze Extension Theorem* (closely related to Urysohn's Lemma) implies that there exists a continuous function h such that $h = f$ on K and $\mathrm{supp}(h) \subseteq U$.

If we don't care whether $\|\varphi\|_\infty \leq \|f\|_\infty$ holds then we can simply take $\varphi = h$ and stop here. Otherwise, we define

$$\varphi(x) \;=\; \begin{cases} h(x), & |h(x)| \leq \|f\|_\infty, \\ \|f\|_\infty \, e^{i\arg(h(x))}, & |h(x)| > \|f\|_\infty, \end{cases}$$

where $\arg(z) = \theta$ if $z = re^{i\theta}$. The function φ is continuous because $\varphi = \beta \circ h$ where $\beta\colon \mathbb{C} \to \mathbb{C}$ is the continuous function

$$\beta(z) \;=\; \begin{cases} z, & |z| \leq \|f\|_\infty, \\ \|f\|_\infty \, e^{i\arg(z)}, & |z| > \|f\|_\infty. \end{cases}$$

Further, φ is compactly supported because $\varphi(x) = 0$ whenever $h(x) = 0$. Finally, $\varphi(x) = h(x) = f(x)$ for all $x \in K$, so

$$\mu\big(\{\varphi \neq f\}\big) \;\leq\; \mu(U\backslash K) \;\leq\; \mu(U\backslash E) + \mu(E\backslash A) + \mu(A\backslash K) \;<\; 3\varepsilon.$$

Now suppose that f is unbounded. Set $A_n = \{0 < |f| \leq n\}$. Since the A_n are nested and increase to $E = \{f \neq 0\}$, we have $\mu(A_n) \to \mu(E) < \infty$. Hence there exists some n such that $\mu(E\backslash A_n) < \varepsilon$. The function $f\chi_{A_n}$ is bounded, so by the above work there exists some function $\varphi \in C_c(\mathbb{R})$ such that $\varphi = f\chi_{A_n}$ except on a set of measure ε. Therefore $\varphi = f$ except on a set of measure 2ε. $\square$