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A Brief Review of Lebesgue Measure and Integration

February 3, 2018

Chapter 1
Lebesgue Measure and Integration

We will give a brief review, without proofs and without motivation or discussion, of Lebesgue measure and integration on subsets of $\mathbb{R}^d$. Details and proofs can be found in texts on real analysis, such as [Fol99], [WZ77], [SS05], or [Heil19]. While some proofs are quite easy, other proofs are surprisingly difficult, and there are many counterintuitive issues related to Lebesgue measure and integration.

1.1 Exterior Lebesgue Measure

For compactness of notation, we will refer to rectangular parallelepipeds in $\mathbb{R}^d$ whose sides are parallel to the coordinate axes simply as "boxes."

Definition 1.1.1. (a) A *box* in $\mathbb{R}^d$ is a set of the form

$$Q \;=\; [a_1,b_1]\times\cdots\times[a_d,b_d] \;=\; \prod_{i=1}^{d}[a_i,b_i].$$

The *volume* of this box is

$$\mathrm{vol}(Q) \;=\; (b_1-a_1)\cdots(b_d-a_d) \;=\; \prod_{i=1}^{d}(b_i-a_i).$$

(b) The *exterior Lebesgue measure* or *outer Lebesgue measure* of a set $E \subseteq \mathbb{R}^d$ is

$$|E|_e \;=\; \inf\Big\{\sum_k \mathrm{vol}(Q_k)\Big\},$$

where the infimum is taken over all *finite or countable* collections of boxes Q_k such that $E \subseteq \bigcup_k Q_k$. ♢

Thus, every subset of $\mathbb{R}^d$ has a uniquely defined exterior measure that lies in the range $0 \le |E|_e \le \infty$. Here are some of the basic properties of exterior measure (not all of the proofs are trivial, especially the proof of statement (a) of Theorem 1.1.2).

Theorem 1.1.2. (a) *If Q is a box in $\mathbb{R}^d$, then $|Q|_e = \mathrm{vol}(Q)$.*

(b) Monotonicity: *If $E \subseteq F \subseteq \mathbb{R}^d$, then $|E|_e \le |F|_e$.*

(c) Countable subadditivity: *If $E_k \subseteq \mathbb{R}^d$ for $k \in \mathbb{N}$, then*

$$\Big|\bigcup_{k=1}^{\infty} E_k\Big|_e \;\le\; \sum_{k=1}^{\infty} |E_k|_e.$$

(d) Translation invariance: *If $E \subseteq \mathbb{R}^d$ and $h \in \mathbb{R}^d$, then $|E+h|_e = |E|_e$, where $E+h = \{t+h : t \in E\}$.*

(e) Regularity: *If $E \subseteq \mathbb{R}^d$ and $\varepsilon > 0$, then there exists an open set $U \supseteq E$ such that $|U|_e \le |E|_e + \varepsilon$, and hence*

$$|E|_e \;=\; \inf\bigl\{|U|_e : U \text{ open, } U \supseteq E\bigr\}. \qquad \diamondsuit$$

1.2 Lebesgue Measure

Definition 1.2.1. A set $E \subseteq \mathbb{R}^d$ is *Lebesgue measurable*, or simply *measurable* for short, if

$$\forall\, \varepsilon > 0, \quad \exists \text{ open } U \supseteq E \text{ such that } |U \backslash E|_e \le \varepsilon. \tag{1.1}$$

If E is Lebesgue measurable, then its *Lebesgue measure* is its exterior Lebesgue measure and is denoted by $|E| = |E|_e$. $\diamondsuit$

Note that equation (1.1) does not follow from Theorem 1.1.2(e). One consequence of the Axiom of Choice is that there exist subsets of $\mathbb{R}^d$ that are not measurable (see [Heil19] for details).

The following result summarizes some of the properties of measurable sets.

Theorem 1.2.2. (a) *The class of measurable subsets of $\mathbb{R}^d$ is a σ-algebra, meaning that:*

- *$\varnothing$ and $\mathbb{R}^d$ are measurable,*
- *if $E_1, E_2, \dots$ are measurable, then $\cup E_k$ is measurable,*
- *if E is measurable, then $\mathbb{R}^d \backslash E$ is measurable.*

(b) *Every open and every closed subset of $\mathbb{R}^d$ is measurable.*

(c) *Every subset E of $\mathbb{R}^d$ with $|E|_e = 0$ is measurable.* $\diamondsuit$

Since measurability is preserved under complements and countable unions, it is also preserved under countable intersections.

We will give some equivalent formulations of measurability. First we make the following definitions.

Definition 1.2.3. (a) A set $H \subseteq \mathbb{R}^d$ is a *G_δ-set* if there exist finitely or countably many open sets U_k such that $H = \cap U_k$.

(b) A set $H \subseteq \mathbb{R}^d$ is an *F_σ-set* if there exist finitely or countably many closed sets F_k such that $H = \cup F_k$. $\diamondsuit$

Theorem 1.2.4. *Let $E \subseteq \mathbb{R}^d$ be given. Then the following statements are equivalent.*

(a) *E is measurable.*

(b) *For every $\varepsilon > 0$, there exists a closed set $F \subseteq E$ such that $|E \backslash F|_e \le \varepsilon$.*

(c) *$E = H \backslash Z$ where H is a G_δ-set and $|Z| = 0$.*

(d) *$E = H \cup Z$ where H is an F_σ-set and $|Z| = 0$.* $\diamondsuit$

Next we list some properties of Lebesgue measure.

Theorem 1.2.5. *Let E and E_k for $k \in \mathbb{N}$ be measurable subsets of $\mathbb{R}^d$.*

(a) Countable additivity: *If $E_1, E_2, \dots$ are disjoint measurable subsets of $\mathbb{R}^d$, then*

$$\Big|\bigcup_{k=1}^{\infty} E_k\Big| \;=\; \sum_{k=1}^{\infty} |E_k|.$$

(b) *If $E_1 \subseteq E_2$ and $|E_1| < \infty$, then $|E_2 \backslash E_1| = |E_2| - |E_1|$.*

(c) Continuity from below: *If $E_1 \subseteq E_2 \subseteq \cdots$, then $\big|\cup E_k\big| = \lim_{k\to\infty} |E_k|$.*

(d) Continuity from above: *If $E_1 \supseteq E_2 \supseteq \cdots$ and $|E_1| < \infty$, then $\big|\cap E_k\big| = \lim_{k\to\infty} |E_k|$.*

(e) Translation invariance: *If $h \in \mathbb{R}^d$, then $|E + h| = |E|$, where $E + h = \{x + h : x \in E\}$.*

(f) Linear changes of variable: *If $T\colon \mathbb{R}^d \to \mathbb{R}^d$ is linear, then $T(E)$ is measurable and $|T(E)| = |\det(T)|\,|E|$.*

(g) Cartesian products: *If $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ are measurable, then $E \times F \subseteq \mathbb{R}^{m+n}$ is measurable and $|E \times F| = |E|\,|F|$ (using the convention that $0 \cdot \infty = 0$).* $\diamondsuit$

We end this section with some terminology.

Definition 1.2.6. A property that holds except possibly on a set of measure zero is said to hold *almost everywhere*, abbreviated a.e. $\diamondsuit$

For example, if C is the classical Cantor middle-thirds set, then $|C| = 0$. Hence, the characteristic function χ_C of C satisfies $\chi_C(t) = 0$ except for those t that belong to the zero measure set C. Therefore we say that $\chi_C(t) = 0$ for almost every t, or $\chi_C = 0$ a.e. for short.

The essential supremum of a function is an example of a quantity that is defined in terms of a property that holds almost everywhere.

Definition 1.2.7 (Essential Supremum). The *essential supremum* of a function $f\colon E \to \mathbb{R}$ is

$$\operatorname*{ess\,sup}_{t\in E} f(t) \;=\; \inf\{M : f \le M \text{ a.e.}\}.$$

We say that f is *essentially bounded* if $\operatorname{ess\,sup}_{t\in E} |f(t)| < \infty$. $\diamondsuit$

1.3 Measurable Functions

Now we define the class of measurable functions on subsets of $\mathbb{R}^d$. For simplicity we will only consider real-valued and complex-valued functions. However, it is often important to consider *extended real-valued functions*, which are functions that can take real values or the value $\pm\infty$. For details on the extension to extended real-valued functions, we refer to [Heil19].

Definition 1.3.1 (Real-Valued Measurable Functions). Fix a measurable set $E \subseteq \mathbb{R}^d$, and let $f\colon E \to \mathbb{R}$ be given. Then f is a *Lebesgue measurable function*, or simply a *measurable function*, if $f^{-1}(\alpha,\infty) = \{t \in E : f(t) > \alpha\}$ is a measurable subset of $\mathbb{R}^d$ for each $\alpha \in \mathbb{R}$. $\diamondsuit$

In particular, every continuous function $f\colon \mathbb{R}^d \to \mathbb{R}$ is measurable, because the inverse image of an open set under a continuous function is open, and all open sets are measurable. However, a measurable function need not be continuous.

Measurability is preserved under most of the usual operations, including addition, multiplication, and limits. Some care does need to be taken with compositions, but if we compose a measurable function with a continuous function in the correct order, then measurability will be assured.

Theorem 1.3.2. *Let $E \subseteq \mathbb{R}^d$ be measurable.*

(a) *If $f\colon E \to \mathbb{R}$ is measurable and $g = f$ a.e., then g is measurable.*

(b) *If $f,\, g\colon E \to \mathbb{R}$ are measurable, then so is $f + g$.*

(c) *If $f\colon E \to \mathbb{R}$ is measurable and $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous, then $\varphi \circ f$ is measurable. Consequently, $|f|$, f^2, f^+, f^-, and $|f|^p$ for $p > 0$ are all measurable.*

(d) *If* $f, g\colon E \to \mathbb{R}$ *are measurable, then so is* fg.

(e) *If functions* $f_n\colon E \to \mathbb{R}$ *are measurable for* $n \in \mathbb{N}$, *then so are* $\sup f_n$, $\inf f_n$, $\limsup f_n$, *and* $\liminf f_n$.

(f) *If functions* $f_n\colon E \to \mathbb{R}$ *are measurable for* $n \in \mathbb{N}$ *and* $f(t) = \lim_{n\to\infty} f_n(t)$ *exists for a.e.* t, *then* f *is measurable.* $\diamondsuit$

Definition 1.3.3 (Complex-Valued Measurable Functions). Given a measurable domain $E \subseteq \mathbb{R}^d$ and a complex-valued function $f\colon E \to \mathbb{C}$, write f in real and imaginary parts as $f = f_r + if_i$. Then we say that f is *measurable* if both f_r and f_i are measurable. $\diamondsuit$

Egoroff's Theorem says that pointwise convergence of measurable functions is uniform convergence on "most" of the set if the set has finite measure.

Theorem 1.3.4 (Egoroff's Theorem). *Let* $E \subseteq \mathbb{R}^d$ *be measurable with* $|E| < \infty$. *If* $f_n, f\colon E \to \mathbb{C}$ *are measurable functions and* $f_n(t) \to f(t)$ *for a.e.* $t \in E$, *then for every* $\varepsilon > 0$ *there exists a measurable set* $A \subseteq E$ *such that* $|A| < \varepsilon$ *and* f_n *converges uniformly to* f *on* $E\backslash A$, *i.e.,*

$$\lim_{n\to\infty} \left(\sup_{t\notin A} |f(t) - f_n(t)| \right) \;=\; 0. \qquad \diamondsuit$$

1.4 The Lebesgue Integral

To define the Lebesgue integral of a measurable function, we first begin with "simple functions" and then extend to nonnegative functions, real-valued functions, and complex-valued functions.

Definition 1.4.1. Let $E \subseteq \mathbb{R}^d$ be measurable.

(a) A *simple function* on E is a measurable function $\phi\colon E \to \mathbb{C}$ that takes only finitely many distinct values. That is, $\phi\colon E \to \mathbb{C}$ is simple if

$$\phi \;=\; \sum_{k=1}^{N} a_k \, \chi_{E_k}, \tag{1.2}$$

where $N > 0$, $a_k \in \mathbb{C}$, and the E_k are measurable subsets of E.

(b) If $a_1, \dots, a_N \in \mathbb{C}$ are the distinct values assumed by a simple function ϕ and we set $E_k = \{t \in E : \phi(t) = a_k\}$, then ϕ has the form given in equation (1.2) and the sets $E_1, \dots, E_N$ form a partition of E. We call this the *standard representation* of ϕ.

(c) If ϕ is a nonnegative simple function on E with standard representation $\phi = \sum_{k=1}^{N} a_k\, \chi_{E_k}$, then the *Lebesgue integral of ϕ over E* is

$$\int_E \phi \;=\; \int_E \phi(t)\,dt \;=\; \sum_{k=1}^{N} a_k\,|E_k|.$$

(d) If $f\colon E \to [0,\infty)$ is a measurable function, then the *Lebesgue integral of f over E* is

$$\int_E f \;=\; \int_E f(t)\,dt \;=\; \sup\biggl\{\int_E \phi : 0 \le \phi \le f,\ \phi \text{ simple}\biggr\}.$$

If A is a measurable subset of E, then we write $\int_A f = \int_E f\,\chi_A$. $\diamondsuit$

Following are some of the basic properties of integrals of nonnegative functions.

Theorem 1.4.2. *Let $E \subseteq \mathbb{R}^d$ and assume that $f,\, g\colon E \to [0,\infty)$ are measurable.*

(a) *If ϕ is a simple function on E, then the integrals of ϕ given in parts* (c) *and* (d) *of Definition* 1.4.1 *coincide.*

(b) *If $f \le g$ then $\int_E f \le \int_E g$.*

(c) Tchebyshev's Inequality: *If $\alpha > 0$, then $|\{t \in E : f(t) > \alpha\}| \le \frac{1}{\alpha}\int_E f$.*

(d) *$\int_E f = 0$ if and only if $f = 0$ a.e.* $\diamondsuit$

The definition of $\int_E f$ given in Definition 1.4.1 is often cumbersome to implement. One application of the next result (which is also known as the *Beppo Levi Theorem*) is that the integral of f can be obtained as a limit instead of a supremum of integrals of a sequence of monotone increasing simple functions. We say that a sequence of real-valued functions $\{f_n\}_{n\in\mathbb{N}}$ is *monotone increasing* if $f_1(t) \le f_2(t) \le \cdots$ for all t. We write $f_n \nearrow f$ to mean that $\{f_n\}_{n\in\mathbb{N}}$ is monotone increasing and $f_n(t) \to f(t)$ pointwise.

Theorem 1.4.3 (Monotone Convergence Theorem). *Let $E \subseteq \mathbb{R}^d$ be measurable, and assume $\{f_n\}_{n\in\mathbb{N}}$ are nonnegative measurable functions on E such that $f_n \nearrow f$. Then*

$$\lim_{n\to\infty} \int_E f_n \;=\; \int_E f. \qquad \diamondsuit$$

Theorem 1.4.4. *If $E \subseteq \mathbb{R}^d$ and $f\colon E \to [0,\infty)$ are measurable then there exist simple functions ϕ_n such that $\phi_n \nearrow f$, and consequently $\int_E \phi_n \nearrow \int_E f$.* $\diamondsuit$

Corollary 1.4.5. *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of measurable, nonnegative functions on a measurable set* $E \subseteq \mathbb{R}^d$. *Then*

$$\int_E \biggl(\sum_{n=1}^{\infty} f_n\biggr) \;=\; \sum_{n=1}^{\infty} \int_E f_n.$$

In particular, if $f\colon E \to [0,\infty)$ *is measurable,* $A_1, A_2, \dots$ *are disjoint and measurable, and* $A = \cup A_k$, *then*

$$\int_A f \;=\; \sum_k \int_{A_k} f. \qquad \diamondsuit$$

If we have functions f_n that are not monotone increasing, then we may not be able to interchange a limit with an integral. The following result states that as long as the f_n are all nonnegative, we do at least have an inequality.

Theorem 1.4.6 (Fatou's Lemma). *If* $\{f_n\}_{n\in\mathbb{N}}$ *is a sequence of measurable, nonnegative functions on a measurable set* $E \subseteq \mathbb{R}^d$, *then*

$$\int_E \Bigl(\liminf_{n\to\infty} f_n\Bigr) \;\le\; \liminf_{n\to\infty} \int_E f_n. \qquad \diamondsuit$$

We define the integral of a general real-valued function by writing it as a difference of two nonnegative functions, and that of a complex-valued function by splitting it into real and imaginary parts.

Definition 1.4.7. Let $E \subseteq \mathbb{R}^d$ be measurable.

(a) Given a measurable function $f\colon E \to \mathbb{R}$, define

$$f^+(t) \;=\; \max\bigl\{f(t),\, 0\bigr\} \qquad \text{and} \qquad f^-(t) \;=\; \max\bigl\{-f(t),\, 0\bigr\}.$$

Then $f^+,\, f^- \ge 0$, and we have $f = f^+ - f^-$ and $|f| = f^+ + f^-$. The *Lebesgue integral of* f *on* E is

$$\int_E f \;=\; \int_E f^+ \;-\; \int_E f^-,$$

as long as this does not have the form $\infty - \infty$ (in that case, the integral is undefined).

(b) Given a measurable function $f\colon E \to \mathbb{C}$, write the real and imaginary parts of f as $f = f_r + i f_i$. If $\int_E f_r$ and $\int_E f_i$ both exist and are finite, then the *Lebesgue integral of* f *on* E is

$$\int_E f \;=\; \int_E f_r \;+\; i\int_E f_i.$$

Otherwise, the integral is undefined. $\diamondsuit$

Theorem 1.4.8. *Let f be a measurable function on a measurable set $E \subseteq \mathbb{R}^d$. Then $\int_E f$ exists and is a finite scalar if and only if $\int_E |f| < \infty$, and in this case $\bigl|\int_E f\,\bigr| \le \int_E |f|$.* $\diamondsuit$

1.5 L^p Spaces and Convergence

Let E be a measurable subset of $\mathbb{R}^d$. Given $1 \le p < \infty$, for each measurable function $f\colon E \to \mathbb{C}$ we define the L^p-norm of f to be

$$\|f\|_p \;=\; \biggl(\int_E |f(t)|^p\,dt\biggr)^{1/p}.$$

$L^p(E)$ is the space of all functions for which $\|f\|_p$ is finite. Technically, $\|\cdot\|_p$ is only a seminorm on $L^p(E)$ because any function f satisfying $f = 0$ a.e. will have $\|f\|_p = 0$. However, if we identify functions that are equal almost everywhere, i.e., we consider them as defining the same element of $L^p(E)$, then $\|\cdot\|_p$ is a norm on $L^p(E)$. Further, it can be shown that $L^p(E)$ is complete with respect to this norm (i.e., every Cauchy sequence converges), and hence it is a Banach space.

For $p = \infty$ we define the L^∞-norm of f to be

$$\|f\|_\infty \;=\; \operatorname*{ess\,sup}_{t\in E} |f(t)| \;=\; \inf\bigl\{M \ge 0 : |f(t)| \le M \text{ a.e.}\bigr\}.$$

Then $L^\infty(E)$ is a Banach space with respect to this norm if we again identify functions that are equal almost everywhere.

Remark 1.5.1. (a) Technically, an element of $L^p(E)$ is an equivalence class of functions that are equal almost everywhere rather than a single function. We can usually safely ignore the distinction between a function and the equivalence class of functions that are equal to it a.e., but on occasion some care needs to be taken. One such situation arises when dealing with continuous functions. Let $C_b(\mathbb{R})$ denote the set of all continuous, bounded functions $f\colon \mathbb{R} \to \mathbb{R}$. We often write $C_b(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$, but in doing so we are really identifying $C_b(\mathbb{R})$ with its image in $L^\infty(\mathbb{R})$ under the equivalence relation of equality almost everywhere. That is, if $f \in C_b(\mathbb{R})$ then it determines an equivalence class $\widetilde{f}$ of functions that are equal to it almost everywhere, and it is this equivalence class $\widetilde{f}$ that belongs to $L^\infty(\mathbb{R})$. Conversely, if we are given $f \in L^\infty(\mathbb{R})$ (really an equivalence class $\widetilde{f}$ of functions) and there is a representative of this equivalence class that belongs to $C_b(\mathbb{R})$, then we write $f \in C_b(\mathbb{R})$, meaning that there is a representative of f that belongs to $C_b(\mathbb{R})$.

(b) The two statements "f is continuous a.e." and "f equals a continuous function a.e." are distinct. The first means that $\lim_{y\to x} f(y) = f(x)$ for almost every x, while the second means that there exists a continuous function g such that $f(x) = g(x)$ for almost every x. Only in the latter case can we say that there is a representative of f that is a continuous function. The function $\chi_{[0,1]}$ is an example of a function that is continuous a.e. but does not equal any continuous function a.e. $\diamondsuit$

Convergence in L^p-norm is not equivalent to pointwise convergence of functions, but we do have the following important fact.

Theorem 1.5.2. *Let $E \subseteq \mathbb{R}^d$ be measurable and fix $1 \le p \le \infty$. If f_n, $f \in L^p(E)$ and $f_n \to f$ in L^p-norm, then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k}(t) \to f(t)$ for almost every $t \in E$.* $\diamondsuit$

The Dominated Convergence Theorem is one of the most important convergence theorems for integrals.

Theorem 1.5.3 (Lebesgue Dominated Convergence Theorem). *If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions on a measurable set $E \subseteq \mathbb{R}^d$ that satisfies:*

(a) *$f(t) = \lim_{n\to\infty} f_n(t)$ exists for a.e. $t \in E$, and*

(b) *there exists a single function $g \in L^1(E)$ such that $|f_n(t)| \le g(t)$ a.e. for every n,*

then f_n converges to f in L^1-norm, i.e.,

$$\lim_{n\to\infty} \|f - f_n\|_{L^1} \;=\; \lim_{n\to\infty} \int_E |f - f_n| \;=\; 0.$$

Consequently,

$$\lim_{n\to\infty} \int_E f_n = \int_E f. \qquad \diamondsuit$$

There is also a series version of the Dominated Convergence Theorem.

Theorem 1.5.4 (Dominated Convergence Theorem for Series). *If $(a_{mn})_{m,n\in\mathbb{N}}$ is a sequence of complex scalars that satisfies:*

(a) *$a_m = \lim_{n\to\infty} a_{mn}$ exists for all $m \in \mathbb{N}$, and*

(b) *there exists a sequence $b = (b_m)_{m\in\mathbb{N}} \in \ell^1$ such that $|a_{mn}| \le b_m$ for every m and n,*

then

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} a_{mn} = \sum_{m=1}^{\infty} a_m. \qquad \diamondsuit$$

It is often useful to know that we can approximate a given L^p function by functions that have some special properties. For example, combining the Lebesgue Dominated Convergence Theorem with Theorem 1.4.4 shows that the set of L^p simple functions is dense in $L^p(E)$. When p is finite, we can restrict further to simple functions with compact support.

Theorem 1.5.5. *Let $E \subseteq \mathbb{R}^d$ be Lebesgue measurable. Then the set S consisting of all compactly supported simple functions is dense in $L^p(E)$ for each $1 \le p < \infty$.* $\diamondsuit$

Here are some other examples of dense subspaces of L^p. Recall that a subset S of L^p is *complete* (or *total* or *fundamental*) in L^p if $\operatorname{span}(S)$, the finite linear span of S, is dense. Also, we let $C_c(\mathbb{R}^d)$ be the set of all functions with *compact support*, which means that the function is identically zero outside of some ball of finite radius, and if K is closed then we let $C(K)$ denote the set of continuous functions that are identically zero outside of K.

Lemma 1.5.6. *$C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for each $1 \le p < \infty$. If $K \subseteq \mathbb{R}^d$ is compact, then $C(K)$ is dense in $L^p(K)$ for each $1 \le p < \infty$.* $\diamondsuit$

Lemma 1.5.7. *$\{\chi_{[a,b]} : -\infty < a < b < \infty\}$ is complete in $L^p(\mathbb{R})$ for each $1 \le p < \infty$.* $\diamondsuit$

Lemma 1.5.8. *$\{\chi_{E\times F} : E, F \subseteq \mathbb{R}\}$ is complete in $L^p(\mathbb{R}^2)$ for each index $1 \le p < \infty$.* $\diamondsuit$

An important property of integrable functions is given in the next theorem.

Theorem 1.5.9 (Lebesgue Differentiation Theorem). *Fix $f \in L^1[a,b]$. Then for almost every $x \in (a,b)$,*

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(y)\,dy \;=\; \lim_{h\to 0}\frac{1}{2h}\int_{x-h}^{x+h} f(y)\,dy \;=\; f(x).$$

Consequently, the indefinite integral of f,

$$F(x) \;=\; \int_a^x f(y)\,dy,$$

is differentiable a.e., and $F' = f$ a.e. $\diamondsuit$

In fact, the intervals $[x, x+h]$ or $[x-h, x+h]$ can be replaced by any collection of sets $\{S_h\}_{h>0}$ that *shrink regularly* to x, which means that $\operatorname{diam}(S_h) \to 0$, and there exists a constant $C > 0$ such that if Q_h is the smallest interval centered at x that contains S_h, then $|Q_h| \le C\,|S_h|$. The Lebesgue Differentiation Theorem can also be generalized to higher dimensions.

1.6 Repeated Integration

Let $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ be measurable. If f is a measurable function on $E \times F$ then there are three natural integrals of f over $E \times F$. First, there is the integral of f over the set $E \times F \subseteq \mathbb{R}^{m+n}$, which we write as the *double integral*

$$\iint_{E\times F} f \;=\; \iint_{E\times F} f(x,y)\,(dx\,dy).$$

Second, for each fixed y we can integrate $f(x,y)$ as a function of x, and then integrate the result in y, obtaining the *iterated integral*

$$\int_F \left(\int_E f(x,y)\,dx\right) dy.$$

Third, we also have the iterated integral

$$\int_E \left(\int_F f(x,y)\,dy\right) dx.$$

In general these three integrals need not be equal, even if they all exist. The theorems of Fubini and Tonelli give sufficient conditions under which we can exchange the order of integration. We begin with Tonelli's Theorem, which states that interchange is allowed if f is nonnegative.

Theorem 1.6.1 (Tonelli's Theorem). *Let E be a measurable subset of $\mathbb{R}^m$ and F a measurable subset of $\mathbb{R}^n$. If $f\colon E\times F \to [0,\infty)$ is measurable, then the following statements hold.*

(a) $f_x(y) = f(x,y)$ *is measurable on F for each $x \in E$.*

(b) $f^y(x) = f(x,y)$ *is measurable on E for each $y \in F$.*

(c) $g(x) = \int_F f_x(y)\,dy$ *is a measurable function on E.*

(d) $h(y) = \int_E f^y(x)\,dx$ *is a measurable function on F.*

(e) *We have*

$$\begin{aligned}\iint_{E\times F} f(x,y)\,(dx\,dy) \;&=\; \int_F \left(\int_E f(x,y)\,dx\right) dy\\ &=\; \int_E \left(\int_F f(x,y)\,dy\right) dx,\end{aligned}$$

in the sense that either all three of the quantities above are finite and equal, or all are infinite. $\diamondsuit$

As a corollary, we obtain the useful fact that to test whether a given function belongs to $L^1(E\times F)$ we can simply show that any one of three possible integrals is finite.

Corollary 1.6.2. *Let E be a measurable subset of $\mathbb{R}^m$ and F a measurable subset of $\mathbb{R}^n$. If f is a measurable function on $E\times F$, then (as nonnegative real numbers or as infinity):*

$$\iint_{E\times F} |f(x,y)|\,(dx\,dy) = \int_F \left(\int_E |f(x,y)|\,dx\right) dy = \int_E \left(\int_F |f(x,y)|\,dy\right) dx.$$

Consequently, if any one of these three integrals is finite, then f belongs to $L^1(E \times F)$. $\diamondsuit$

Fubini's Theorem allows the interchange of integrals if f is integrable.

Theorem 1.6.3 (Fubini's Theorem). *Let E be a measurable subset of $\mathbb{R}^m$ and F a measurable subset of $\mathbb{R}^n$. If $f \in L^1(E \times F)$, then the following statements hold.*

(a) $f_x(y) = f(x,y)$ *is measurable and integrable on F for almost every $x \in E$.*

(b) $f^y(x) = f(x,y)$ *is measurable and integrable on E for almost every $y \in F$.*

(c) $g(x) = \int_F f_x(y)\,dy$ *is a measurable and integrable function on E.*

(d) $h(y) = \int_E f^y(x)\,dx$ *is a measurable and integrable function on F.*

(e) *We have*

$$\iint_{E\times F} f(x,y)(dx\,dy) = \int_F \left(\int_E f(x,y)\,dx\right) dy = \int_E \left(\int_F f(x,y)\,dy\right) dx,$$

where each of these quantities is a finite scalar. $\diamondsuit$

There are also corresponding discrete versions of Fubini's Theorem and Tonelli's Theorem for series.

Theorem 1.6.4. *Let $(a_{mn})_{m,n\in\mathbb{N}}$ be a sequence of real or comples numbers.*

(a) *If $a_{mn} \geq 0$ for every m and n, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn},$$

in the sense that either both are finite and equal, or both are infinite.

(b) *If $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{mn}| < \infty$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}. \qquad \diamondsuit$$

An entirely similar result holds for interchanging an integral with a series.

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