REAL ANALYSIS LECTURE NOTES:

3.5 FUNCTIONS OF BOUNDED VARIATION

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These lecture notes from 2007 follow and expand on the text "Real Analysis: Modern Techniques and their Applications," 2nd ed., by G. Folland. Additional material is based on the text "Measure and Integral," by R. L. Wheeden and A. Zygmund.

A **far improved and expanded** presentation of bounded variation and related topics can be found in my recent textbook:

C. Heil, "Introduction to Real Analysis," Springer, Cham, 2019.

Links, additional material, and more resources are available at my website for the text:

`http://people.math.gatech.edu/~heil/real`

3.5.1 Definition and Basic Properties of Functions of Bounded Variation

We will expand on the first part of Section 3.5 of Folland's text, which covers functions of bounded variation on the real line and related topics.

We begin with functions defined on finite closed intervals in $\mathbb{R}$ (note that Folland's approach and notation is slightly different, as he begins with functions defined on $\mathbb{R}$ and uses $T_F(x)$ instead of our $V[f;a,b]$).

Definition 1. Let $f\colon [a,b] \to \mathbb{C}$ be given. Given any finite partition

$$\Gamma \;=\; \{a = x_0 < \cdots < x_n = b\}$$

of $[a,b]$, set

$$S_\Gamma \;=\; \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

The *variation* of f over $[a,b]$ is

$$V[f;a,b] \;=\; \sup\bigl\{S_\Gamma : \Gamma \text{ is a partition of } [a,b]\bigr\}.$$

The function f has *bounded variation* on $[a,b]$ if $V[f;a,b] < \infty$. We set

$$\mathrm{BV}[a,b] \;=\; \bigl\{f\colon [a,b] \to \mathbb{C} : f \text{ has bounded variation on } [a,b]\bigr\}.$$

Note that in this definition we are considering f to be defined at all points, and not just to be an equivalence class of functions that are equal a.e.

The idea of the variation of f is that is represents the total *vertical* distance traveled by a particle that moves along the graph of f from $(a, f(a))$ to $(b, f(b))$.

Exercise 2. (a) Show that if $f\colon [a,b] \to \mathbb{C}$, then $V[f;a,b] \ge |f(b) - f(a)|$.

(b) We say that a real-valued function $f\colon [a,b] \to \mathbb{R}$ is *monotone increasing* if $a \le x \le y \le b$ implies $f(x) \le f(y)$. Show that every real-valued, monotone increasing function f on $[a,b]$ has bounded variation and that, in this case, $V[f;a,b] = f(b) - f(a)$.

Exercise 3. (a) Show that the *Dirichlet function* $\chi_{\mathbb{Q}}$ has unbounded variation on any finite interval.

(b) Define $f\colon [-1,1] \to \mathbb{R}$ by

$$f(x) \;=\; \begin{cases} x \sin \frac{1}{x}, & x \ne 0, \\ 0, & x = 0. \end{cases}$$

Show that f is continuous, but has unbounded variation on $[-1,1]$.

(c) Construct a continuous, piecewise linear function on $[0,1]$ that has unbounded variation.

The space $\mathrm{BV}[a,b]$ is sometimes defined to consist of only real-valued functions of bounded variation. However, in that case all the definitions and results extend equally to complex-valued functions. In other words, we could just as well have defined bounded variation for real-valued functions, and then declared a complex-valued function to have bounded variation if its real and imaginary parts have bounded variation.

Exercise 4. Given $f\colon [a,b] \to \mathbb{C}$, write the real and imaginary parts as $f = f_r + i f_i$. Show that $f \in \mathrm{BV}[a,b]$ if and only if $f_r,\ f_i \in \mathrm{BV}[a,b]$.

For functions on the domain $\mathbb{R}$ we make the following definition.

Definition 5. The *variation* of a function $f\colon \mathbb{R} \to \mathbb{C}$ is

$$V[f;\mathbb{R}] \;=\; \sup_{a<b} V[f;a,b].$$

We say that f has *bounded variation* if $V[f;\mathbb{R}] < \infty$, and we define

$$\mathrm{BV}(\mathbb{R}) \;=\; \bigl\{ f\colon \mathbb{R} \to \mathbb{C} : f \text{ has bounded variation on } \mathbb{R} \bigr\}.$$

Lipschitz functions are examples of functions on $[a,b]$ that have bounded variation.

Definition 6. A function $f\colon [a,b] \to \mathbb{C}$ is *Lipschitz* on $[a,b]$ if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \;\le\; C\,|x-y|, \qquad x,y \in [a,b].$$

We define

$$\mathrm{Lip}[a,b] \;=\; \bigl\{f\colon [a,b] \to \mathbb{C} : f \text{ is Lipschitz}\bigr\}.$$

Exercise 7. Prove the following.

(a) If f is Lipschitz on $[a,b]$, then f is uniformly continuous and has bounded variation, with $V[f;a,b] \le C\,(b-a)$.

(b) A Lipschitz function need not be differentiable.

(c) If f is differentiable on $[a,b]$ and f' is bounded on $[a,b]$, then f is Lipschitz, and we can take $C = \|f'\|_\infty$. In particular, if f, f' are both continuous on $[a,b]$, then f is Lipschitz.

(d) Extend the definition of Lipschitz functions to functions on the domain $\mathbb{R}$ instead of $[a,b]$. Show that continuity of f, f' on $\mathbb{R}$ (i.e., $f \in C^1(\mathbb{R})$) need not imply that f is Lipschitz. What further hypotheses need be imposed in order to conclude that f is Lipschitz?

We will need the following basic exercises on variation.

Exercise 8. Let $f\colon [a,b] \to \mathbb{C}$ be given.

(a) Show that if Γ' is a refinement of Γ, then $S_\Gamma \le S_{\Gamma'}$.

(b) Show that if $[a',b'] \subseteq [a,b]$, then $V[f;a',b'] \le V[f;a,b]$.

(c) Show that if $a < c < b$, then $V[f;a,b] = V[f;a,c] + V[f;c,b]$.

3.5.2 The Jordan Decomposition

We will prove that every real-valued function of bounded variation can be written as a difference of two monotone increasing functions. Given a real number x, recall the notations

$$x^+ \;=\; \max\{x,0\} \qquad \text{and} \qquad x^- \;=\; -\min\{x,0\}.$$

Note that $x = x^+ - x^-$, while $|x| = x^+ + x^-$.

Definition 9. Let $f\colon [a,b] \to \mathbb{R}$ be given. Given a partition $\Gamma = \{a = x_0 < \cdots < x_n = b\}$ of $[a,b]$, define

$$S_\Gamma^+ \;=\; \sum_{i=1}^n \bigl[f(x_i) - f(x_{i-1})\bigr]^+ \qquad \text{and} \qquad S_\Gamma^- \;=\; \sum_{i=1}^n \bigl[f(x_i) - f(x_{i-1})\bigr]^-.$$

Thus S_Γ^+ is the sum of the positive terms of S_Γ, and S_Γ^- is the sum of the negative terms.

The *positive variation* of f on $[a,b]$ is

$$V^+[f;a,b] \;=\; \sup\bigl\{S_\Gamma^+ : \Gamma \text{ is a partition of } [a,b]\bigr\},$$

and the *negative variation* is

$$V^-[f;a,b] \;=\; \sup\bigl\{S_\Gamma^- : \Gamma \text{ is a partition of } [a,b]\bigr\}.$$

Exercise 10. Show that for any partition Γ we have

$$S_\Gamma^+ + S_\Gamma^- \;=\; S_\Gamma$$

while

$$S_\Gamma^+ - S_\Gamma^- \;=\; f(b) - f(a).$$

Exercise 11. Show that

$$V(x) \;=\; V[f;a,x], \qquad V^+(x) \;=\; V^+[f;a,x], \qquad V^-(x) \;=\; V^-[f;a,x],$$

are each increasing functions of $x \in [a,b]$. Also observe that $0 \le V^-(x), V^+(x) \le V(x)$ for $x \in [a,b]$.

Lemma 12. For any $f\colon [a,b] \to \mathbb{R}$, we have

$$V^+[f;a,b] + V^-[f;a,b] \;=\; V[f;a,b].$$

Further, if any one of $V[f;a,b]$, $V^+[f;a,b]$, or $V^-[f;a,b]$ is finite, then they are all finite, and in this case we also have

$$V^+[f;a,b] - V^-[f;a,b] \;=\; f(b) - f(a).$$

Proof. Note that, even if these quantities are infinite, we have

$$V^+[f;a,b] \;=\; \sup_\Gamma S_\Gamma^+ \;\le\; \sup_\Gamma S_\Gamma \;=\; V[f;a,b]$$

and similarly $V^-[f;a,b] \le V[f;a,b]$. In particular, if $V[f;a,b]$ is finite, then so are $V^+[f;a,b]$ and $V^-[f;a,b]$.

On the other hand,

$$\begin{aligned} V[f;a,b] \;&=\; \sup_\Gamma S_\Gamma \;=\; \sup_\Gamma\, (S_\Gamma^+ + S_\Gamma^-) \\ &\le\; \sup_\Gamma S_\Gamma^+ \;+\; \sup_\Gamma S_\Gamma^- \\ &=\; V^+[f;a,b] + V^-[f;a,b]. \end{aligned}$$

In particular, if both $V^+[f;a,b]$ and $V^-[f;a,b]$ are finite, then so is $V[f;a,b]$.

Further, for every partition Γ, we have by Exercise 10 that $S_\Gamma^+ = S_\Gamma^- + C$, where C is the fixed and finite constant $C = f(b) - f(a)$. Hence, even if they are infinite,

$$\begin{aligned} V^+[f;a,b] \;&=\; \sup\{S_\Gamma^+ : \text{all partitions } \Gamma\} \\ &=\; \sup\{S_\Gamma^- + C : \text{all partitions } \Gamma\} \;=\; V^-[f;a,b] + C. \end{aligned}$$

In particular, $V^+[f;a,b]$ is finite if and only if $V^-[f;a,b]$ is finite.

The above work establishes that if any one of $V[f;a,b]$, $V^+[f;a,b]$, or $V^-[f;a,b]$ if finite, then so are the other two.

Exercise: Show that we can find partitions Γ_k, where each Γ_{k+1} is a refinement of Γ_k, such that

$$\lim_{k\to\infty} S^-_{\Gamma_k} \;=\; V^-[f;a,b].$$

Then since we have that $S^+_{\Gamma_k} = S^-_{\Gamma_k} + C$ where $C = f(b) - f(a)$ is a finite constant, we have

$$\lim_{k\to\infty} S^+_{\Gamma_k} \;=\; \lim_{k\to\infty} (S^-_{\Gamma_k} + C) \;=\; V^-[f;a,b] + C \;=\; V^+[f;a,b].$$

Further, since $S^-_{\Gamma_k} + S^+_{\Gamma_k} = S_{\Gamma_k}$, we have

$$V^+[f;a,b] + V^-[f;a,b] \;=\; \lim_{k\to\infty} S^+_{\Gamma_k} \;+\; \lim_{k\to\infty} S^-_{\Gamma_k} \;=\; \lim_{k\to\infty} (S^+_{\Gamma_k} + S^-_{\Gamma_k}) \;\le\; V[f;a,b].$$

Finally, in the case that $V[f;a,b]$ is finite, we have finite limits and therefore can write

$$V^+[f;a,b] - V^-[f;a,b] \;=\; \lim_{k\to\infty} S^+_{\Gamma_k} \;-\; \lim_{k\to\infty} S^-_{\Gamma_k} \;=\; \lim_{k\to\infty} (S^+_{\Gamma_k} - S^-_{\Gamma_k}) \;=\; f(b) - f(a). \qquad \square$$

Exercise 13. Show that, when they are finite,

$$V^+[f;a,b] \;=\; \frac{1}{2}\big(V[f;a,b] + f(b) - f(a)\big) \quad \text{and} \quad V^-[f;a,b] \;=\; \frac{1}{2}\big(V[f;a,b] - f(b) + f(a)\big).$$

Now we can give the main result of this part, characterizing real-valued functions of bounded variation as the difference of two monotone increasing functions.

Theorem 14 (Jordan Decomposition)**.** If $f\colon [a,b] \to \mathbb{R}$ is given, then the following statements are equivalent.

(a) $f \in \mathrm{BV}[a,b]$.

(b) There exist monotone increasing functions $f_1,\ f_2\colon [a,b] \to \mathbb{R}$ such that $f = f_1 - f_2$.

Proof. (a) $\Rightarrow$ (b). For $x \in [a,b]$, the functions $V^+[f;a,x]$ and $V^-[f;a,x]$ are monotonically increasing with x. Furthermore, by Lemma 12 we have

$$V^+[f;a,x] - V^-[f;a,x] \;=\; f(x) - f(a).$$

Therefore $f = f_1 - f_2$ where $f_1 = V^+[f;a,x] + f(a)$ and $f_2 = V^-[f;a,x]$ are monotonically increasing functions. $\square$

A complex-valued function $f\colon [a,b] \to \mathbb{C}$ will therefore have bounded variation if we can write $f = (f_1^r - f_2^r) + i(f_1^i - f_2^i)$ where $f_1^r,\ f_2^r,\ f_1^i,\ f_2^i$ are monotone increasing.

Exercise 15. Show that $f \in \mathrm{BV}(\mathbb{R})$ if and only if we can write $f = f_1 - f_2$ where $f_1,\ f_2$ are bounded and monotone increasing functions on $\mathbb{R}$.

3.5.3 Variation for Continuous and Differentiable Functions

Definition 16. Given any finite partition

$$\Gamma \;=\; \{a = x_0 < \cdots < x_n = b\}$$

of $[a,b]$, we declare the *mesh size* of the partition to be

$$|\Gamma| \;=\; \max\{x_k - x_{k-1} : k = 1, \dots, n\}.$$

Theorem 17. If $f\colon [a,b] \to \mathbb{R}$ is continuous, then

$$V[f;a,b] \;=\; \lim_{|\Gamma|\to 0} S_\Gamma,$$

i.e., for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition Γ of $[a,b]$,

$$|\Gamma| < \delta \implies V[f;a,b] - \varepsilon < S_\Gamma \le V[f;a,b].$$

Proof. Choose any $\varepsilon > 0$. Then by definition of $V[f;a,b]$, there must exist a partition $\Gamma' = \{a = x'_0 < \cdots < x'_n = b\}$ such that

$$V[f;a,b] - \frac{\varepsilon}{2} \;\le\; S_{\Gamma'} \;\le\; V[f;a,b].$$

Since f is continuous on the compact domain $[a,b]$, it is uniformly continuous, and therefore we can find an $\eta > 0$ such that for x, $y \in [a,b]$ we have

$$|x-y| < \eta \implies |f(x) - f(y)| < \frac{\varepsilon}{4(n+1)}.$$

Now let

$$\delta \;=\; \min\{\eta,\, x'_1 - x'_0,\, \dots,\, x'_n - x'_{n-1}\}.$$

Suppose that $\Gamma = \{a = x_0 < \cdots < x_m = b\}$ is any partition with $|\Gamma| < \delta$. Let

$$I \;=\; \{k \in \{1,\dots,m\} : (x_{k-1}, x_k) \text{ contains some } x'_j\}$$

and

$$J \;=\; \{k \in \{1,\dots,m\} : (x_{k-1}, x_k) \text{ contains no } x'_j\}.$$

Note that since $|\Gamma| < \delta$, no interval $[x_{k-1}, x_k)$ can contain more than more point x'_j. Therefore, given $k \in I$ we can let $\bar{x}_k$ denote that unique x'_j that is contained in (x_{k-1}, x_k). Further, we conclude that I can contain at most $n+1$ elements, since that is how many x'_j there are.

Now, we have that

$$S_\Gamma \;=\; \sum_{k\in I} |f(x_k) - f(x_{k-1})| \;+\; \sum_{k\in J} |f(x_k) - f(x_{k-1})|.$$

Let $\Gamma_0 = \Gamma \cup \Gamma'$, and observe that Γ_0 is a refinement of both Γ and Γ'. Furthermore, by definition of I and J, we have

$$\begin{aligned} S_{\Gamma_0} \;&=\; \sum_{k\in I} \Big(|f(x_k) - f(\bar{x}_k)| + |f(\bar{x}_k) - f(x_{k-1})|\Big) \;+\; \sum_{k\in J} |f(x_k) - f(x_{k-1})| \\ &=\; \Sigma_I + \Sigma_J. \end{aligned}$$

Now,

$$|x_k - \bar{x}_k| \;<\; |x_k - x_{k-1}| \;<\; \eta$$

and similarly $|\bar{x}_k - x_{k-1}| < \eta$, so

$$|f(x_k) - f(\bar{x}_k)| \;<\; \frac{\varepsilon}{4(n+1)} \qquad \text{and} \qquad |f(\bar{x}_k) - f(x_{k-1})| \;<\; \frac{\varepsilon}{4(n+1)}.$$

Since I can contain at most $n+1$ elements, we therefore have

$$\Sigma_I \;\le\; \sum_{k\in I} \frac{\varepsilon}{2(n+1)} \;\le\; \frac{\varepsilon}{2}.$$

Consequently,

$$\begin{aligned} S_\Gamma \;\ge\; \Sigma_J \;&=\; S_{\Gamma_0} - \Sigma_I \\ &\ge\; S_{\Gamma_0} - \frac{\varepsilon}{2} \\ &\ge\; S_{\Gamma'} - \frac{\varepsilon}{2} \qquad \text{since } \Gamma_0 \text{ is a refinement of } \Gamma' \\ &\ge\; V[f;a,b] - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}, \end{aligned}$$

which completes the proof. □

We will improve on the next theorem later. Specifically, instead of requiring that f be differentiable with a continuous derivative, we will only need f to be *absolutely continuous*. Absolutely continuous functions need only be differentiable almost everywhere (although that alone is not sufficient to imply absolute continuity).

Theorem 18. If $f\colon [a,b] \to \mathbb{R}$ is differentiable and f' is continuous on $[a,b]$, then

$$V[f;a,b] \;=\; \int_a^b |f'|, \qquad V^+[f;a,b] \;=\; \int_a^b (f')^+, \qquad V^-[f;a,b] \;=\; \int_a^b (f')^-.$$

Proof. Given any particular partition $\Gamma = \{a = x_0 < \cdots < x_m = b\}$, by the Mean-Value Theorem we can find points $\xi_k \in (x_{k-1}, x_k)$ such that

$$f(x_k) - f(x_{k-1}) \;=\; f'(\xi_k)\,(x_k - x_{k-1}),$$

so

$$S_\Gamma \;=\; \sum_{k=1}^m |f(x_k) - f(x_{k-1})| \;=\; \sum_{k=1}^m |f'(\xi_k)|\,(x_k - x_{k-1}).$$

Note that this is a Riemann sum for $\int |f'|$. Since f is continuous, we therefore have

$$\lim_{|\Gamma|\to 0} S_\Gamma \;=\; \lim_{|\Gamma|\to 0} \sum_{k=1}^m |f'(\xi_k)|\,(x_k - x_{k-1}) \;=\; \int |f'|.$$

On the other hand, Theorem 17 implies that that

$$\lim_{|\Gamma|\to 0} S_\Gamma \;=\; V[f;a,b],$$

so we conclude that $V[f;a,b] = \int |f'|$.

Exercise: Use Exercise 13 to extend to the positive and negative variations. □

Exercise 19. Give an example that shows that the conclusion of Theorem 17 can fail if f has a single removable discontinuity and is continuous at all other points in $[a,b]$.

3.5.3 Monotonicity and Discontinuities

Definition 20. Let $f\colon [a,b] \to \mathbb{R}$ be given and let $x \in [a,b]$ be fixed. If they exist, we define

$$f(x-) \;=\; \lim_{y \to x^-} f(y) \qquad \text{and} \qquad f(x+) \;=\; \lim_{y \to x^+} f(y).$$

(a) If f is not continuous at x, then we say that f has a *removable discontinuity* at x $f(x-)$ and $f(x+)$ both exist and $f(x-) = f(x+)$.

(b) If f is not continuous at x, then we say that f has a *jump discontinuity* at x if $f(x-)$ and $f(x+)$ exist but are not equal.

Removable and jump discontinuities are often referred to collectively as *discontinuities of the first kind.*

Lemma 21. Suppose $f\colon [a,b] \to \mathbb{R}$ is monotone increasing.

(a) $f(x+)$ exists for all $x \in [a,b)$, and $f(x-)$ exists for all $x \in (a,b]$.

(b) f has at most countably many discontinuities, and they are all jump discontinuities.

Proof. (a) This follows from the fact that f is increasing and bounded.

(b) Since f is increasing and $f(x+)$ and $f(x-)$ exist, any point of discontinuity of f must be a jump discontinuity. Further, since f is bounded and increasing, given any fixed $k \in \mathbb{N}$, the set of x such that

$$f(x+) - f(x-) \;\geq\; \frac{1}{k}$$

must be finite. Since every jump discontinuity must satisfy this inequality for some $k \in \mathbb{N}$, we conclude that there can be at most countably many jump discontinuities. □

Exercise 22. Give an example of a monotone increasing function $f\colon [a,b] \to \mathbb{R}$ that has an infinite number of discontinuities.

Exercise 23. Show that the conclusions of Lemma 21 also hold for monotone increasing functions $f\colon \mathbb{R} \to \mathbb{R}$ that are bounded, and that we further have in this case that $f(-\infty) = \lim_{x\to-\infty} f(x)$ and $f(\infty) = \lim_{x\to\infty} f(x)$ exist.

3.5.4 Monotonicity and Differentiability

In this part we will prove that monotone increasing functions are differentiable a.e. We will formulate the result for monotone increasing functions on $\mathbb{R}$, but it also applies to monotone functions $f\colon [a,b] \to \mathbb{R}$ since any such function can be extended to be a monotone function on $\mathbb{R}$ (e.g., define $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$).

Theorem 24. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is monotone increasing.

(a) The function $g(x) = f(x+)$ is defined everywhere, is monotone increasing, is right-continuous, and is differentiable a.e.

(b) f is differentiable a.e., f' is measurable, $f' \ge 0$ a.e., and $f' = g'$ a.e.

Proof. (a) Exercise: Show that g exists and is monotone increasing and right-continuous.

Consequently, our task is to show that g is differentiable almost everywhere. Since g is monotone increasing and right-continuous, there exists a corresponding Lebesgue–Stieltjes measure μ_g (see Section 1.5). As Lebesgue–Stieltjes measures enjoy many of the same regularity properties as Lebesgue measure, we actually have that μ_g is a regular Borel measure on $\mathbb{R}$.

By a theorem from the end of Section 3.4, since μ_g is regular, if we let $\mu_g = k\,dx + \lambda$ be its Lebesgue–Radon–Nikodym decomposition with respect to Lebesgue measure dx, then we have

$$\lim_{h\to 0^+} \frac{g(x+h)-g(x)}{h} \;=\; \lim_{h\to 0^+} \frac{\mu_g(x,x+h]}{|(x,x+h]|} \;=\; k(x) \text{ a.e.}$$

A similar argument applies for the case $h \to 0^-$, so we conclude that g is differentiable a.e., with $g'(x) = k(x)$ for almost every x.

(b) Let $H = g - f$. Since f has at most countably many discontinuities, we have that $H(x) \ne 0$ for at most countably many x, say $x_1, x_2, \dots$. Further, since we always have $f \le g$, we know that

$$H(x_j) \;=\; g(x_j) - f(x_j) \;=\; f(x_j+) - f(x_j) \;>\; 0$$

for each j. Define

$$\mu \;=\; \sum_j H(x_j)\,\delta_j,$$

so H is a finite or countable sum of point mass measures and hence is a Borel measure on $\mathbb{R}$.

We claim that μ is locally finite, i.e., finite on compact sets. This follows from the fact that

$$\begin{aligned}
\mu(-N,N) \;&=\; \sum_{|x_j|<N} H(x_j) \\
&=\; \sum_{|x_j|<N} \bigl(f(x_j+) - f(x_j)\bigr) \\
&\le\; f(N) - f(-N) \;<\; \infty.
\end{aligned}$$

By a theorem from Section 1.5, every locally finite Borel measure on $\mathbb{R}$ is a Lebesgue–Stieltjes measure. Therefore μ is a regular Borel measure. Further, $\mu \perp dx$ since if we set $E = \{x_j\}$ then we have $|E| = 0 = \mu(E^{\mathrm{C}})$. Therefore the Lebesgue–Radon–Nikodym decomposition of μ with respect to Lebesgue measure is $\mu = 0\,dx + \mu$. Hence,

$$\begin{aligned} \left|\frac{H(x+h)-H(x)}{h}\right| \;\le\; \frac{H(x+h)+H(x)}{|h|} \;&\le\; \frac{\displaystyle\sum_{x-2|h|<|x_j|<x+2|h|} H(x_j)}{|h|} \\ &=\; 4\,\frac{\mu(x-2|h|,x+2|h|]}{|(x-2|h|,x+2|h|]|} \\ &\to\; 0 \quad \text{as } h \to 0, \end{aligned}$$

where the convergence to zero again follows from a theorem at the end of Section 3.4. □

Since every function of bounded variation is a difference of two bounded monotone increasing functions, we obtain the following facts about functions in BV($\mathbb{R}$).

Corollary 25. Suppose that $f \in \mathrm{BV}(\mathbb{R})$.

(a) $f(x+)$ and $f(x-)$ exist for all $x \in \mathbb{R}$, as do $f(-\infty)$ and $f(\infty)$.

(b) f has at most countably many discontinuities, each of which is either a removable discontinuity or a jump discontinuity.

(c) The function $g(x) = f(x+)$ is right-continuous and has bounded variation on $\mathbb{R}$.

(d) f and g are differentiable almost everywhere, and $f' = g'$ a.e.

Further, we also have the following inequality for monotone increasing functions on intervals (a, b). Compare this inequality to the Fundamental Theorem of Calculus.

Theorem 26. If $f\colon [a,b] \to \mathbb{R}$ is monotone increasing, then

$$0 \;\le\; \int_a^b f' \;\le\; f(b) - f(a). \tag{1}$$

Proof. Extend f by setting $f(x) = f(a)$ for $x \le a$ and $f(x) = f(b)$ for $x \ge b$. We have by Theorem 24 that f is differentiable a.e. Therefore, the functions

$$f_k(x) \;=\; \frac{f(x+1/k)-f(x)}{1/k} \;=\; k\left(f(x+\tfrac{1}{k}) - f(x)\right)$$

converge pointwise to $f'(x)$ a.e. as $k \to \infty$. By Fatou's Lemma, we therefore have

$$\begin{aligned} \int_a^b f' \;&\le\; \liminf_{k\to\infty} \int_a^b f_k \\ &=\; k\int_{a+1/k}^{b+1/k} f \;-\; k\int_a^b f \end{aligned}$$

$$\begin{aligned} &= k\int_b^{b+1/k} f \;-\; k\int_a^{a+1/k} f \\ &\le k\int_b^{b+1/k} f(b) \;-\; k\int_a^{a+1/k} f(a) \qquad \text{(since } f \text{ is increasing)} \\ &= f(b) - f(a). \end{aligned}$$

□

Exercise 27. Give an example showing that strict inequality can hold in equation (1).

Exercise 28. Extend Theorem 26 to functions $f \in \mathrm{BV}(\mathbb{R})$.

The next exercise extends some of the preceding facts from monotone increasing functions to functions of bounded variation.

Exercise 29. Show that if $f \in \mathrm{BV}[a,b]$, then $f'(x)$ exists for a.e. x, and furthermore $f' \in L^1[a,b]$. Formulate and prove an analogous result for $f \in \mathrm{BV}(\mathbb{R})$.

3.5.5 The Derivative of the Variation

In this part we will show that if f has bounded variation and we set $V(x) = V[f;a,x]$, then we have $V'(x) = |f'(x)|$ a.e.

First, we need the following useful lemma, due to Fubini.

Lemma 30. For $k \in \mathbb{N}$, let f_k be monotone increasing functions on $[a,b]$. If

$$s(x) \;=\; \sum_{k=1}^{\infty} f_k(x) \;\text{ converges } \forall\, x \in [a,b],$$

then s is differentiable a.e., and

$$s'(x) \;=\; \sum_{k=1}^{\infty} f_k'(x) \quad \text{a.e.}$$

Proof. For each $N \in \mathbb{N}$, define

$$s_N(x) \;=\; \sum_{k=1}^{N} f_k(x) \qquad \text{and} \qquad r_N(x) \;=\; \sum_{k=N+1}^{\infty} f_k(x).$$

By hypothesis, the series defining $r_N(x)$ converges for each x, and we have $s = s_N + r_N$. Further, both s_N and r_N are increasing on $[a,b]$, and hence each is differentiable except possibly on some set $Z_N \subseteq [a,b]$ with measure zero. The set $Z = \cup Z_N$ also has measure zero, so s is differentiable a.e., with

$$s'(x) \;=\; s_N'(x) + r_N'(x), \quad x \notin Z.$$

By Theorem 26, f_k', s_n', $r_n' \geq 0$ a.e. Consequently,

$$0 \;\leq\; \sum_{k=1}^{N} f_k'(x) \;=\; s_N'(x) \;\leq\; s_N'(x) + r_N'(x) \;=\; s'(x) \quad \text{a.e.}$$

Since this is true for every N, we therefore have that

$$\sum_{k=1}^{\infty} f_k'(x) \;\leq\; s'(x) \quad \text{a.e.} \tag{2}$$

Our goal is to show that we actually have equality holding in equation (2) except for a set of measure zero.

Since $s_N(x) \to s(x)$ everywhere, we have that $r_N(x) \to 0$ for every x. Therefore we can choose N_j such that

$$r_{N_j}(a) \;<\; \frac{1}{2^j} \qquad \text{and} \qquad r_{N_j}(b) \;<\; \frac{1}{2^j}, \qquad j \in \mathbb{N}.$$

Combining this with the fact that $r_N(a) \leq r_N(b)$, we have

$$0 \;\leq\; \sum_{j=1}^{\infty} \bigl(r_{N_j}(b) - r_{N_j}(a)\bigr) \;<\; \infty. \tag{3}$$

Hence,

$$\begin{aligned} 0 \;&\leq\; \int_a^b \sum_{j=1}^{\infty} r_{N_j}' \\ &=\; \sum_{j=1}^{\infty} \int_a^b r_{N_j}' \qquad \text{since } r_{N_j}' \geq 0 \text{ a.e.} \\ &\leq\; \sum_{j=1}^{\infty} \bigl(r_{N_j}(b) - r_{N_j}(a)\bigr) \qquad \text{by Theorem 26} \\ &<\; \infty \qquad \text{by equation (3).} \end{aligned}$$

Thus the function $\sum_{j=1}^{\infty} r_{N_j}'$, which is nonnegative a.e., is integrable, and hence must be finite a.e.:

$$0 \;\leq\; \sum_{j=1}^{\infty} r_{N_j}'(x) \;<\; \infty \quad \text{a.e.}$$

Consequently, we have convergence of a sequence of partial sums:

$$\lim_{j\to\infty} \Bigl(s'(x) - \sum_{k=1}^{N_j} f_k'(x)\Bigr) \;=\; \lim_{j\to\infty} \bigl(s'(x) - s_{N_j}'(x)\bigr) \;=\; \lim_{j\to\infty} r_{N_j}'(x) \;=\; 0 \quad \text{a.e..}$$

However, since each term f_k' is nonnegative a.e., this implies that that the full sequence of partial sums converges:

$$\lim_{N\to\infty} \Bigl(s'(x) - \sum_{k=1}^{N} f_k'(x)\Bigr) \;=\; 0 \quad \text{a.e.}$$

This gives us our desired equality. □

Now we can give the main result of this part.

Theorem 31. If $f \in \mathrm{BV}[a,b]$ and $V(x) = V[f;a,x]$, then V is differentiable a.e., and

$$V'(x) \;=\; |f'(x)| \quad \text{a.e. } x \in [a,b].$$

Proof. By definition, $V(b)$ is the supremum of all sums S_Γ over all partitions Γ of $[a,b]$. Therefore, we can choose a sequence of partitions

$$\Gamma_k \;=\; \bigl\{a = x_0^k < x_1^k < \cdots < x_{m_k}^k = b\bigr\}$$

such that

$$0 \;\le\; V(b) - S_{\Gamma_k} \;<\; 2^{-k}, \qquad k \in \mathbb{N},$$

where

$$S_{\Gamma_k} \;=\; \sum_{j=1}^{m_k} |f(x_j^k) - f(x_{j-1}^k)|.$$

Choose scalars c_j^k in such a way that

$$f_k(x) \;=\; \begin{cases} f(x) + c_j^k, & \text{if } x \in [x_{j-1}^k, x_j^k] \text{ and } f(x_j^k) \ge f(x_{j-1}^k), \\ -f(x) + c_j^k, & \text{if } x \in [x_{j-1}^k, x_j^k] \text{ and } f(x_j^k) < f(x_{j-1}^k), \end{cases}$$

is well-defined at each point $x = x_j^k$, and satisfies $f_k(a) = 0$. Then we have for each choice of j and k that

$$f_k(x_j^k) - f_k(x_{j-1}^k) \;=\; |f(x_j^k) - f(x_{j-1}^k)|.$$

Consequently,

$$S_{\Gamma_k} \;=\; \sum_{j=1}^{m_k} \bigl(f_k(x_j^k) - f_k(x_{j-1}^k)\bigr) \;=\; f_k(b) - f_k(a) \;=\; f_k(b).$$

In particular, we have for each $k \in \mathbb{N}$ that

$$0 \;\le\; V(b) - f_k(b) \;\le\; V(b) - S_{\Gamma_k} \;<\; 2^{-k}.$$

We claim now that for each fixed k, the function $V(x) - f_k(x)$ is an increasing function of x. To see this, suppose that $a \le x < y \le b$. If there is a single j such that $x, y \in [x_{j-1}^k, x_j^k]$, then

$$f_k(y) - f_k(x) \;=\; |f(y) - f(x)| \;\le\; V(y) - V(x).$$

On the other hand, if $x \in [x_{j-1}^k, x_j^k]$ and $y \in [x_{\ell-1}^k, x_\ell^k]$ with $j < \ell$, then

$$\begin{aligned} f_k(y) - f_k(x) \;&=\; \bigl(f_k(y) - f_k(x_{\ell-1}^k)\bigr) \;+\; \sum_{i=j+1}^{\ell-1} \bigl(f_k(x_i^k) - f_k(x_{i-1}^k)\bigr) \;+\; \bigl(f_k(x_j^k) - f_k(x)\bigr) \\ &\le\; \bigl(V(y) - V(x_{\ell-1}^k)\bigr) \;+\; \sum_{i=j+1}^{\ell-1} \bigl(V(x_i^k) - V(x_{i-1}^k)\bigr) \;+\; \bigl(V(x_j^k) - V(x)\bigr) \\ &=\; V(y) - V(x). \end{aligned}$$

In any case, we obtain $f_k(y) - f_k(x) \le V(y) - V(x)$, and therefore

$$V(x) - f_k(x) \;\le\; V(y) - f_k(y).$$

Hence $V(x) - f_k(x)$ is indeed increasing with x.

Therefore, for $x \in [a, b]$,

$$0 \;=\; V(a) - f_k(a) \;\le\; V(x) - f_k(x) \;\le\; V(b) - f_k(b) \;<\; 2^{-k}.$$

Consequently,

$$0 \;\le\; \sum_{k=1}^{\infty} \bigl(V(x) - f_k(x)\bigr) \;\le\; \sum_{k=1}^{\infty} 2^{-k} \;<\; \infty.$$

Lemma 30 therefore implies that the series

$$\sum_{k=1}^{\infty} \bigl(V'(x) - f_k'(x)\bigr)$$

converges for almost every x. Hence $f_k'(x) \to V'(x)$ for a.e. x. But since V is increasing we have $V'(x) \ge 0$ a.e., so

$$|f'(x)| \;=\; |f_k'(x)| \;\to\; |V'(x)| \;=\; V'(x) \quad \text{a.e.},$$

and therefore $|f'(x)| = V'(x)$ a.e. □

Additional Problems

Problem 32. If f, $g \in \mathrm{BV}[a,b]$, then $\alpha f + \beta g \in \mathrm{BV}[a,b]$ for all α, $\beta \in \mathbb{C}$ (so $\mathrm{BV}[a,b]$ is a vector space), and $fg \in \mathrm{BV}[a,b]$. If $|g(x)| \ge \varepsilon > 0$ for all $x \in [a,b]$ then $f/g \in \mathrm{BV}[a,b]$.

Problem 33. Set $f(x) = x^2 \sin(1/x)$ and $g(x) = x^2 \sin(1/x^2)$ for $x \ne 0$, and $f(0) = g(0) = 0$. Show that f and g are differentiable everywhere, but $f \in \mathrm{BV}[-1,1]$ while $g \notin \mathrm{BV}[-1,1]$.

Hint: Show that f is differentiable everywhere and f' is bounded, so f is Lipschitz. To show g does not have bounded variation, choose a particular appropriate partition, similar to Exercise 3.

Problem 34. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions of bounded variation on $[a,b]$. If $V[f_k;a,b] \le M < \infty$ for all k and if $f_k \to f$ pointwise on $[a,b]$, show that f is of bounded variation and that $V[f;a,b] \le M$. Give an example of a pointwise convergent sequence of functions of bounded variation whose limit is not of bounded variation.

Problem 35. Let $E \subseteq \mathbb{R}$ be measurable, and suppose that $f\colon E \to \mathbb{R}$ is Lipschitz on E, i.e., $|f(x) - f(y)| \le C\,|x-y|$ for all x, $y \in E$. Prove that if $A \subseteq E$, then

$$|f(A)|_e \;\le\; C\,|A|_e. \tag{4}$$

Note that even if A is measurable, it need not be true that $f(A)$ is measurable, which is why we must use exterior Lebesgue measure in (4).