

C.16 Weak and Weak* Convergence

In this section we will discuss weak and weak* convergence of sequences. A detailed discussion of the weak and weak* topologies and the corresponding convergence criterion for nets is given in Section E.6. For now, we take the following as our definition of weak or weak* convergence of sequences.

Definition C.115. Let X be a normed linear space.

(a) A sequence $\{f_n\}_{n \in \mathbb{N}}$ in X converges weakly to $f \in X$, denoted $f_n \xrightarrow{w} f$, if

$$\forall \mu \in X^*, \quad \lim_{n \rightarrow \infty} \langle f_n, \mu \rangle = \langle f, \mu \rangle.$$

(b) A sequence $\{\mu_n\}_{n \in \mathbb{N}}$ in X^* converges weak* to $\mu \in X^*$, denoted $\mu_n \xrightarrow{w^*} \mu$, if

$$\forall f \in X, \quad \lim_{n \rightarrow \infty} \langle f, \mu_n \rangle = \langle f, \mu \rangle.$$

Thus, weak* convergence is just “pointwise convergence” of the operators μ_n . Weak* convergence is only defined for sequences that lie in a dual space X^* . Given a sequence $\{\mu_n\}_{n \in \mathbb{N}}$ in X^* , we can consider three types of convergence: strong (norm), weak, and weak* convergence. By definition, these are:

$$\begin{aligned} \mu_n \rightarrow \mu &\iff \lim_{n \rightarrow \infty} \|\mu - \mu_n\| = 0, \\ \mu_n \xrightarrow{w} \mu &\iff \forall T \in X^{**}, \quad \lim_{n \rightarrow \infty} \langle \mu_n, T \rangle = \langle \mu, T \rangle, \\ \mu_n \xrightarrow{w^*} \mu &\iff \forall x \in X, \quad \lim_{n \rightarrow \infty} \langle x, \mu_n \rangle = \langle x, \mu \rangle. \end{aligned}$$

Exercise C.116. Let X be a normed space and suppose that $f_n, f \in X$ and $\mu_n, \mu \in X^*$. Show that

$$f_n \rightarrow f \implies f_n \xrightarrow{w} f$$

and

$$\mu_n \rightarrow \mu \implies \mu_n \xrightarrow{w} \mu \implies \mu_n \xrightarrow{w^*} \mu.$$

If X is reflexive, then weak and weak* convergence coincide. In particular, this is the case for Hilbert spaces. In any finite-dimensional space, strong and weak convergence coincide, but they are distinct in any infinite-dimensional space. For example, if $\{e_n\}_{n \in \mathbb{N}}$ is an orthonormal sequence in a Hilbert space H , then it follows from Bessel's Inequality that $\langle e_n, f \rangle \rightarrow 0$ for each $f \in H$ and hence $e_n \xrightarrow{w} 0$, but $\|e_n - 0\| = 1 \not\rightarrow 0$.

Here are some properties of weak and weak* convergent sequences. While the fact that norm-convergent sequences are bounded is trivial, the corresponding fact for weak and weak* convergent sequences is much more subtle and is most easily proved by applying the Uniform Boundedness Principle.

Exercise C.117. Let X be a Banach space. Show that if $\mu_n \xrightarrow{w^*} \mu$ in X^* , then μ is unique and $\sup \|\mu_n\| < \infty$.

In order to conclude boundedness of a weak* convergent sequence, the hypothesis in Exercise C.117 that X be complete is necessary. For example, if we take $X = c_{00}$ then we have $\ell^1 \cong (c_{00})^*$ (compare Problem D.28) and the sequence $\{n\delta_n\}_{n \in \mathbb{N}}$ in ℓ^1 converges weak* to 0, but is unbounded. In general, given any normed linear space X there is always a unique Banach space \tilde{X} , called the *completion* of X , that contains X as a dense subspace. Consequently $X^* = (\tilde{X})^*$, and we should always use \tilde{X} instead of X when discussing weak* convergence. For the space c_{00} , the completion is c_0 .

On the other hand, the analogous uniqueness and boundedness results for weakly convergent sequence hold for any normed space, because of the fact that X^* is complete.

Exercise C.118. Let X be a normed space. Show that if $f_n \xrightarrow{w} f$ in X , then f is unique and $\sup \|f_n\| < \infty$.

Additional Problems

C.41. (a) Given $1 < p < \infty$ and $x_n, y \in \ell^p$, show that $x_n \xrightarrow{w} y$ in ℓ^p if and only if $x_n(k) \rightarrow y(k)$ for each k (componentwise convergence) and $\sup \|x_n\|_p < \infty$. Does either implication remain valid if $p = 1$?

(b) Problem D.28 shows that $\ell^1 \cong c_0^*$. Assuming this, given $x_n, y \in \ell^1$, show that $x_n \xrightarrow{w^*} y$ in ℓ^1 if and only if $x_n(k) \rightarrow y(k)$ for each k and $\sup \|x_n\|_1 < \infty$.

(c) Recall that $\ell^\infty \cong \ell^1$. Given $x_n, y \in \ell^\infty$, show that $x_n \xrightarrow{w^*} y$ in ℓ^∞ if and only if $x_n(k) \rightarrow y(k)$ for each k and $\sup \|x_n\|_\infty < \infty$.

D.27. Given $f_n, f \in C_0(\mathbb{R})$, show that $f_n \xrightarrow{w} f$ if and only if $f_n(x) \rightarrow f(x)$ pointwise for each $x \in \mathbb{R}$ and $\sup \|f_n\|_\infty < \infty$ (weak convergence of sequences is discussed in Section C.16).

Definition

We let $M[0,1]$ denote the space of complex Radon measures on $[0,1]$ (= space of complex Borel measures on $[0,1]$). By the Riesz Representation Theorem,

$$M[0,1] = C[0,1]^*$$

The dual space of the continuous functions on $[0,1]$.

Exercise

Define

$$D = \{ \mu \in M[0,1] : \|\mu\| \leq 1 \}$$

$$d(\mu, \nu) = \sum_{n=0}^{\infty} 2^{-n} \left| \int_0^1 x^n d\mu - \int_0^1 x^n d\nu \right|$$

Show that d is a metric on $M[0,1]$ that induces the weak* topology on D but not on $M[0,1]$.

Hint: Weierstrass Approximation Theorem.

Generalize the preceding exercise to other spaces that have countable complete sequences.

The Relation Between the Weak & Norm Topologies.

If X is a normed space then for any net $\{x_i\}_{i \in I}$ in X ,

$$x_i \rightarrow x \implies x_i \xrightarrow{w} x.$$

Thus it is "easier" to converge weakly than to converge in norm. Hence the weak topology on X has fewer open sets than the norm topology, so the weak topology is weaker than the norm topology:

$$\text{Weak Topology} \subseteq \text{Norm Topology}.$$

Moreover, if X is ∞ -dimensional then these two topologies are not equal.

Exercise

Let X be an ∞ -dimensional normed space, and let

$$S = \{x \in X : \|x\| = 1\}$$

be the unit sphere in X . This set is closed in the norm topology. Show that the weak closure of S is the closed unit disk in X :

$$\bar{S}^{(wk^*)} = \{x \in X : \|x\| \leq 1\}.$$

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Even so, the space l^1 has the interesting property that weak convergence of sequences in l^1 implies norm convergence.

Theorem

Given a sequence $\{x_n\}_{n \in \mathbb{N}}$ in l^1 ,

$$x_n \xrightarrow{w} x \iff x_n \rightarrow x.$$

Proof.

\Leftarrow This direction is trivial.

\Rightarrow . Recall that $(l^1)^* = l^\infty$. Since l^1 is separable, Alaoglu's Theorem implies that

D closed unit ball

$$D = \{x \in l^1 : \|x\|_1 \leq 1\} \quad (\text{norm-closed!})$$

is sequentially weak* compact, and the

weak* topology $\sigma = \sigma(l^\infty, l^1)$ ~~restricted to D~~

restricted to D is metrizable.

Suppose $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in ~~(D, weak^*)~~
 (D, weak^*) . Then since D is sequentially compact

in this topology, \exists subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ and $x \in D$ such that $x_{n_k} \xrightarrow{w^*} x$. Since D is metric and $\{x_n\}$ is Cauchy, it has a subsequence that converges to x , we must have $x_n \xrightarrow{w^*} x$. Thus (D, weak^*) is actually a complete metric space.

We will apply the Baire Category Theorem to this space. Suppose that ~~that~~ $x_n \xrightarrow{w} 0$ in l^1 , which means that

$$\forall y \in l^\infty, \quad \langle x_n, y \rangle \rightarrow 0.$$

Fix $\epsilon > 0$, and for each $m \in \mathbb{N}$ define

$$F_m = \{ y \in D : |\langle x_n, y \rangle| \leq \epsilon \text{ for } n \geq m \}.$$

Since $x_n \xrightarrow{w} 0$ we have $\cup F_m = D$. We claim that F_m is weak* closed in D .

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To see \mathcal{D} is, suppose $\{y_i\}_{i \in \mathbb{I}}$ is a net in F_m and $y_i \xrightarrow{w^*} y \in \mathcal{D}$. We have $|\langle x_n, y_i \rangle| \geq \varepsilon$ for all $n \geq m$, but also $\langle x_n, y_i \rangle \rightarrow \langle x_n, y \rangle$ with respect to \mathbb{I} , so we must have $|\langle x_n, y \rangle| \geq \varepsilon$ for $n \geq m$. Thus $y \in F_m$, so F_m is weak* closed.

Thus the complete metric space $(\mathcal{D}, \text{weak}^*)$ is a countable union of the weak*-closed subsets F_m .

The Baire Category Theorem therefore implies that some F_m has a nonempty weak* interior, i.e., it contains some nonempty weak*-open set.

Now, since $\mathcal{D} \subseteq \ell^\infty$, we can define

$$d(x, y) = \sum_{j=1}^{\infty} \frac{|x(j) - y(j)|}{2^j}, \quad x, y \in \mathcal{D}$$

Exercise: This is a metric on \mathcal{D} .

We claim that d is an equivalent metric

for (D, weak^*) . We have already seen that this is a complete metric space, but rather than trying to deal with the two metrics directly, we will show that the topology induced on D by the metric d is the same as the weak^* topology on D .

Let $\sigma = \sigma(l^\infty, l^1)$ denote the weak^* topology on D , and let τ be the topology induced by the metric d .

Not the same x_n as before!
 →

Suppose that $x_n \xrightarrow{\text{weak}^*} x$ for some $x_n, x \in D$.

~~Fix $\epsilon > 0$, and let~~ Fix $\epsilon > 0$, and let

M be large enough that

$$\sum_{j=M+1}^{\infty} \frac{2}{2^j} < \epsilon.$$

Since weak^* convergence in l^∞ implies

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Componentwise convergence, $\exists N > 0$ such that

$$n > N \implies |x(j) - x_n(j)| < \varepsilon, \quad j=1, \dots, M.$$

Then for $n > N$ we have

$$d(x, x_n) \leq \sum_{j=1}^M \frac{|x(j) - x_n(j)|}{2^j} + \sum_{j=M+1}^{\infty} \frac{\|x - x_n\|_{\infty}}{2^j}$$

$$\leq \sum_{j=1}^M \frac{\varepsilon}{2^j} + \sum_{j=M+1}^{\infty} \frac{2}{2^j}$$

$$< \varepsilon + \varepsilon = 2\varepsilon.$$

Thus $x_n \rightarrow x$ with respect to the metric d .

That is:

$$\text{weak* convergence} \implies \text{convergence w.r.t. } d.$$

Hence the d -topology is weaker, so

$$T \subseteq \sigma.$$

~~Therefore, T is a Hausdorff topology because it is induced from a metric, and we know that D is~~

~~Further, T is a~~

Hausdorff topology because it is induced from

a metric, and we know that D is

Alternative approach to this part; Use the exercise that weak* convergence is equivalent to componentwise convergence + boundedness.

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compact in \mathcal{D} σ (weak*) topology.

An earlier lemma therefore implies that

$T = \sigma$. Hence \mathcal{D} metric d induces
 \mathcal{D} weak* topology on D .

Now we return to \mathcal{D} set F_m . Since this
set has a nonempty weak* interior and \mathcal{D}
metric d induces \mathcal{D} weak* topology, there
exists some d -open ball

$$B_r(y) = \{z \in D : d(y, z) < r\} \subseteq F_m$$

for some $y \in F_m$ & $r > 0$. Let M be large
enough that

$$\sum_{j=M+1}^{\infty} \frac{2}{2^j} < r.$$

Since $x_n \xrightarrow{\omega} 0$ and weak convergence implies
componentwise convergence, $\exists N > 0$ such that

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$$n \geq N \Rightarrow \sum_{j=1}^M |x_n(j)| < \varepsilon.$$

Now fix any $n \geq m, N$. Let $|\alpha_j| = 1$ be such that $\overline{\alpha_j} x_n(j) = |x_n(j)|$. Define $z = (z(j))_{j \in \mathbb{N}}$ by

$$z(j) = \begin{cases} y(j), & j=1, \dots, M \\ \alpha_j, & j > M \end{cases}$$

Then $\|z\|_\infty \leq 1$ so $z \in D$, and furthermore

$$d(y, z) = \sum_{j=1}^{\infty} \frac{|y(j) - z(j)|}{2^j} \leq \sum_{j=M+1}^{\infty} \frac{2}{2^j} < \varepsilon,$$

so $z \in B_\varepsilon(y) \subseteq F_m$. Since $n \geq m$, this implies $|\langle x_n, z \rangle| \leq \varepsilon$. Consequently;

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$$\|x_n\|_1 = \sum_{n=1}^M |x_n(j)| + \sum_{n=M+1}^{\infty} |x_n(j)|$$

$$< \varepsilon + \sum_{n=M+1}^{\infty} x_n(j) \cdot \bar{\alpha}_j$$

$$= \varepsilon + \langle x_n, z \rangle - \sum_{j=1}^M x_n(j) y(j)$$

$$\leq \varepsilon + |\langle x_n, z \rangle| + \sum_{j=1}^M |x_n(j)| \cdot 1$$

$$< \varepsilon + \varepsilon + \varepsilon$$

$$= 3\varepsilon.$$

This is valid for all $n \geq m, N$, and

both m & N depend only on ε . Hence

$\|x_n\|_1 \rightarrow 0$, which says that $x_n \rightarrow 0$ in norm. Q.E.D.

Corollary

The ~~weak~~ weak topology on

$$D = \{x \in \ell^1 : \|x\|_1 \leq 1\}$$

is not metrizable.