Banach Limits

Theorem (Banach Limits for $\mathbb{F}=\mathbb{R}$).

$\exists \mu \in (\ell^\infty)^*$ such that

(a) $\|\mu\| = 1$,

(b) $x \in c \implies \langle x, \mu \rangle = \lim_{n\to\infty} x_n$

(c) $x \in \ell^\infty,\ x \geq 0 \implies \langle x, \mu \rangle \geq 0$

(d) $\mu \circ L = \mu$ where $Lx = (x_2, x_3, \dots)$ is the left-shift operator.

Such a functional μ is called a Banach limit.

Proof (Taking $\mathbb{F}=\mathbb{R}$).

Set

$$M = (I-L)(\ell^\infty) = \{x - Lx : x \in \ell^\infty\}.$$

This is a subspace of ℓ^∞.

Step 1. We will show that $\mathrm{dist}(M, \delta_0) = 1$, where $\delta_0 = (1, 1, 1, \dots)$.

Since $0 \in M$ and $\|\delta_0 - 0\|_\infty = 1$, we have

$$\operatorname{dist}(M, \delta_0) = \inf\{\|y - \delta_0\|_\infty : y \in M\} \leq 1.$$

To show the converse inequality, fix any vector $y = x - Lx \in M$. If $y_n < 0$ for any n, then

$$\|\delta_0 - y\|_\infty \geq |1 - y_n| > 1.$$

On the other hand, if every component of $y = x - Lx = (x_1 - x_2, x_2 - x_3, \ldots)$ is nonnegative, then $x_n - x_{n+1} \geq 0$ for every n. Hence (x_n) is a decreasing sequence of real numbers that is bounded below (since $x \in \ell^\infty$), so $x_0 = \lim_{n\to\infty} x_n$ exists. Consequently $x_n - x_{n+1} \to 0$ as $n \to \infty$, so

$$\|\delta_0 - y\|_\infty \geq \lim_{n\to\infty} 1 - (x_n - x_{n+1}) = 1.$$

Thus in any case we have $\|\delta_0 - y\|_\infty \geq 1$, so $\operatorname{dist}(M, \delta_0) \geq 1$.

Step 2. We construct μ.

Since M is a subspace of ℓ^∞ and $\operatorname{dist}(M, \delta_0) = 1$, the Hahn-Banach Theorem implies that $\exists\, \mu \in (\ell^\infty)$ that satisfies

$$\|\mu\| = 1, \qquad \langle \delta_0, \mu \rangle = 1, \qquad \mu|_M = 0.$$

Hence property (a) is satisfied. Also, if $x \in \ell^\infty$ then $y = x - Lx \in M$, so $\langle y, \mu \rangle = 0$. But then $\langle x, \mu \rangle - \langle Lx, \mu \rangle = 0$. Since $\langle Lx, \mu \rangle = \mu(Lx) = (\mu \circ L)(x)$, we see that $\mu - \mu \circ L = 0$, so property (d) is satisfied.

Step 3 We show $c_0 \subseteq \ker(\mu)$ & conclude that (b) holds.

By Step 2, $\mu = \mu \circ L^k$ for every $k \in \mathbb{N}$. Given $x \in c_0$, we therefore have

$$|\langle x,\mu\rangle| = |\langle L^k x,\mu\rangle|$$
$$\leq \|L^k x\|_\infty \|\mu\|$$
$$\to 0 \quad \text{since } x \in c_0.$$

Now if x is any element of c, then $x_0 = \lim_{n\to\infty} x_n$ exists, and $x - x_0\delta_0 \in M$.

Hence $\langle x - x_0\delta_0, \mu\rangle = 0$, so

$$\langle x,\mu\rangle = \langle x_0\delta_0,\mu\rangle = x_0\langle\delta_0,\mu\rangle = x_0 = \lim_{n\to\infty} x_n.$$

Thus property (b) holds.

Step 4 We show property (c) holds.

Suppose $x \in \ell^\infty$ & $x \geq 0$, i.e., $x_n \geq 0$ $\forall n$.

Replacing x by $\frac{x}{\|x\|_\infty}$, we can assume $\|x\|_\infty = 1$.

Then $0 \leq x_n \leq 1$ for every n, so $0 \leq 1 - x_n \leq 1$ $\forall n$.

Hence $0 \leq \|\delta_0 - x\|_\infty \leq 1$, so

$$|\langle \delta_0 - x, \mu\rangle| \leq \|\delta_0 - x\|_\infty \|\mu\| \leq 1.$$

But

$$\langle \delta_0 - x, \mu \rangle = \langle \delta_0, \mu \rangle - \langle x, \mu \rangle$$
$$= 1 - \langle x, \mu \rangle.$$

Since $|\langle \delta_0 - x, \mu \rangle| \leq 1$, we must have

$\langle x, \mu \rangle \geq 0$. Thus property (c) holds. ∎

In summary a Banach limit is a bounded linear functional on ℓ^∞ that

- respects limits of components:
  $\langle x, \mu \rangle = \lim_{n \to \infty} x_n$ whenever this limit exists
- respects positivity:
  $\langle x, \mu \rangle \geq 0$ when $x \geq 0$
- is invariant under left-shifts.

<u>Exercise</u>
Show that a Banach limit cannot respect products. That is, if we set $xy = (x_1 y_1, x_2 y_2, \ldots)$ then we cannot have

$$\langle xy, \mu \rangle = \langle x, \mu \rangle \langle y, \mu \rangle \quad \text{for all } x, y \in \ell^\infty.$$

Extend to $\mathbb{F}=\mathbb{C}$.

Now let $\mathbb{F}=\mathbb{C}$. Note that the space ℓ^∞ is the space of all bounded functions on $\mathbb{N}$. By "rounding down," just as we do for functions on $\mathbb{R}$ or a generic measure space, it follows that the set of simple functions in dense in ℓ^∞

Exercise

Let us say that a sequence x is simple if it only takes finitely many distinct values.

(a) Show that the set $S=\{x\in\ell^p : x \text{ is simple}\}$ is dense in ℓ^p for each $1\le p\le\infty$.

(b) Show that if we restrict to simple functions with compact support, we have the same result except for $p=\infty$:

$$\{x\in c_{00} : x \text{ is simple}\} \text{ is dense in } \begin{cases} \ell^p, & 1\le p<\infty \\ c_0, & p=\infty. \end{cases}$$

<u>Theorem</u> (Banach limits for $\mathbb{F} = \mathbb{C}$).

$\exists\, \nu \in (\ell^\infty)^*$ such that

(a) $\|\nu\| = 1$,

(b) $x \in c \Rightarrow \langle x, \nu \rangle = \lim_{n \to \infty} x_n$

(c) $x \in \ell^\infty,\ x \geq 0 \Rightarrow \langle x, \nu \rangle \geq 0$

(d) $\nu \circ L = \nu$, where L is the left-shift

<u>Proof:</u>

Let $\ell^\infty_{\mathbb{R}}$ be the space of bounded real-valued sequences. By the preceding theorem, $\exists$ a Banach limit $\mu \in (\ell^\infty_{\mathbb{R}})^*$.

Given $x \in \ell^\infty$, write $x = a + ib$ where $a, b \in \ell^\infty_{\mathbb{R}}$. Define

$$\langle x, \nu \rangle = \langle a, \mu \rangle + i \langle b, \mu \rangle.$$

Since a, b are uniquely determined by x, ν is well-defined.

Exercise: ν is continuous on ℓ^∞.

Exercise: Use the fact that μ is $\mathbb{R}$-linear to show that ν is $\mathbb{C}$-linear.

Exercise: Show that properties (b), (c), (d) hold for ν.

So, it only remains to show that $\|\nu\| = 1$.

Since $\delta_0 = (1,1,1,\dots)$ is real-valued, we have

$$\langle \delta_0, \nu \rangle = \langle \delta_0, \mu \rangle + i \langle 0, \mu \rangle = 1$$

Hence $\|\nu\| \geq 1$.

Suppose that $x \in \ell^\infty$ is simple and $\|x\|_\infty \leq 1$. Then we can write $x = \sum_{k=1}^{N} c_k \chi_{E_k}$ where the E_k are disjoint subsets of $\mathbb{N}$, and we must have $|c_k| \leq 1$ for each k. Since $\chi_{E_k} \geq 0$ we have

$\langle \chi_{E_k}, \nu \rangle = \langle \chi_{E_k}, \mu \rangle \geq 0$. Also,

$\sum_{k=1}^{N} \chi_{E_k} = \chi_E$ where $E = \bigcup E_k$. Hence

$$
\begin{aligned}
|\langle x, \nu\rangle| &= \left|\sum_{k=1}^{N} c_k \langle \chi_{E_k}, \nu\rangle\right| \\
&\le \sum_{k=1}^{N} |c_k|\, |\langle \chi_{E_k}, \mu\rangle| \\
&\le \sum_{k=1}^{N} \langle \chi_{E_k}, \mu\rangle \\
&= \langle \chi_E, \mu\rangle \\
&\le \|\chi_E\|_\infty \|\mu\| \\
&= 1 \cdot 1 = 1.
\end{aligned}
$$

Hence

$$\sup_{\substack{x \text{ simple}, \\ \|x\|_\infty \le 1}} |\langle x, \nu\rangle| \le 1.$$

Since ν is continuous & the simple functions are dense, we conclude that $\|\nu\| \le 1$. ■

Example

If $x \in \ell^\infty$ & μ is a Banach limit, then $\langle x, \mu \rangle$ need not be an accumulation point of $x = (x_n)_{n \in \mathbb{N}}$.

To see this, consider

$$X = (1, -1, 1, -1, \ldots).$$

We have $Lx = (-1, 1, -1, \ldots)$, so $Lx = -X$. Therefore

$$\langle X, \mu \rangle = \langle Lx, \mu \rangle = -\langle x, \mu \rangle,$$

So $\langle x, \mu \rangle = 0$.

Moral: Banach limits & free ultrafilters on $\mathbb{N}$ are similar, but not the same.

Example 12. Let $\mathcal{E} = \{e_i\}_{i\in\mathbf{Z}}$ be an orthonormal basis for H, and let $a\colon \mathbf{Z} \to \mathbf{Z}$ be the identity map. Fix $\frac{1}{2} < c_0 < 1$, and for $i \neq 0$ choose $c_i > 0$ in such a way that

$$\sum_{i\in\mathbf{Z}} c_i^2 = 1 \qquad \text{and} \qquad \sum_{i\in\mathbf{Z}} c_i = \infty.$$

Define

$$f_0 = \sum_{i\in\mathbf{Z}} c_i e_i \qquad \text{and} \qquad f_i = e_i \text{ for } i \neq 0.$$

If we set $T(e_i) = f_i$, then T extends to a bounded mapping on H. Further, if $f = \sum_i \langle f, e_i\rangle\, e_i \in H$, then

$$\begin{aligned}\|(\mathbf{1} - T)f\|^2 &= \big\|\langle f, e_0\rangle\,(e_0 - f_0)\big\|^2 \\ &= |\langle f, e_0\rangle|^2 \Big(|1 - c_0|^2 + \sum_{i\neq 0} c_i^2\Big) \;\le\; (2 - 2c_0)\,\|f\|^2,\end{aligned}$$

so $\|\mathbf{1} - T\| \le 2 - 2c_0 < 1$. Hence T is a continuous bijection of H onto itself, so $\mathcal{F} = \{f_i\}_{i\in\mathbf{Z}}$ is a Riesz basis for H. Therefore $\langle f_i, \tilde{f}_j\rangle = \delta_{ij}$, so $(\mathcal{F}, a)$ is ℓ^1-localized with respect to its dual frame. However, $\langle f_0, f_j\rangle = c_j$, so $(\mathcal{F}, a)$ is not ℓ^1-self-localized.

Appendix

A. Ultrafilters

In this appendix we provide a brief review of ultrafilters and their basic properties. For additional information, we refer to [46, Chapter 3]. Filters were introduced by H. Cartan [9, 10] in order to characterize continuous functions on general topological spaces. Soon after, it was realized that the set of ultrafilters endowed with the proper topology is the Stone-Čech compactification of a discrete (or more generally, a completely regular) topological space. In the following we will restrict our attention to ultrafilters over the natural numbers $\mathbf{N}$.

Definition A.1. A collection p of subsets of $\mathbf{N}$ is a *filter* if:

(a) $\emptyset \notin p$,

(b) if $A, B \in p$ then $A \cap B \in p$,

(c) if $A \in p$ and $A \subset B \subset \mathbf{N}$, then $B \in p$.

A filter p is an *ultrafilter* if it is maximal in the sense that:

(d) if p' is a filter on $\mathbf{N}$ such that $p \subset p'$, then $p' = p$,

or, equivalently, if

(d') for any $A \subset \mathbf{N}$, either $A \in p$ or $\mathbf{N} \setminus A \in p$ (but not both, because of properties a and b).

The set of ultrafilters is denoted by $\beta\mathbf{N}$.

Definition A.2. Given any $n \in \mathbf{N}$, $e_n = \{A \subset \mathbf{N} : n \in A\}$ is an ultrafilter, called a *principal ultrafilter*. It is straightforward to show that any ultrafilter p that contains a finite set must be one of these principal ultrafilters. An ultrafilter which contains no finite sets is called *free*. The set of free ultrafilters is denoted by $\mathbf{N}^*$.

Our main use for ultrafilters is that they provide a notion of convergence for arbitrary sequences.

Definition A.3. Let $p \in \beta\mathbf{N}$ be an ultrafilter. Then we say that a sequence $\{c_k\}_{k\in\mathbf{N}}$ of complex numbers *converges to* $c \in \mathbf{C}$ *with respect to* p if for every $\varepsilon > 0$ there exists a set $A \in p$ such that $|c_k - c| < \varepsilon$ for all $k \in A$. In this case we write $c = p\text{-}\lim_{k\in\mathbf{N}} c_k$ or simply $c = p\text{-}\lim c_k$.

The following proposition summarizes the basic properties of convergence with respect to an ultrafilter.

Proposition A.1. *Let $p \in \beta\mathbf{N}$ be an ultrafilter. Then the following statements hold.*

(a) *Every bounded sequence of complex scalars $\{c_k\}_{k\in\mathbf{N}}$ converges with respect to p to some $c \in \mathbf{C}$.*

(b) *p-limits are unique.*

(c) *If $p = e_n$ is a principal ultrafilter, then $p\text{-}\lim c_k = c_n$.*

(d) *If $\{c_k\}_{k\in\mathbf{N}}$ is a convergent sequence in the usual sense, p is a free ultrafilter, and $\lim_{k\to\infty} c_k = c$, then $p\text{-}\lim c_k = c$.*

(e) *If $\{c_k\}_{k\in\mathbf{N}}$ is a bounded sequence and p is a free ultrafilter, then $p\text{-}\lim_{k\in\mathbf{N}} c_k$ is an accumulation point of $\{c_k\}_{k\in\mathbf{N}}$.*

(f) *If c is an accumulation point of a bounded sequence $\{c_k\}_{k\in\mathbf{N}}$, then there exists a free ultrafilter p such that $p\text{-}\lim c_k = c$. In particular, there exists an ultrafilter p such that $p\text{-}\lim c_k = \limsup c_k$, and there exists an ultrafilter q such that $q\text{-}\lim c_k = \liminf c_k$.*

(g) *p-limits are linear, i.e., $p\text{-}\lim(ac_k + bd_k) = a\, p\text{-}\lim c_k + b\, p\text{-}\lim d_k$.*

(h) *p-limits respect products, i.e., $p\text{-}\lim(c_k d_k) = \big(p\text{-}\lim c_k\big)\big(p\text{-}\lim d_k\big)$.*

Acknowledgments

We gratefully acknowledge conversations with Karlheinz Gröchenig and Massimo Fornasier on localization of frames, and thank them for providing us with preprints of their articles. We thank Hans Feichtinger, Norbert Kaiblinger, Gitta Kutyniok, and Henry Landau for conversations regarding the details of our arguments, and also acknowledge helpful conversations with Akram Aldroubi, Carlos Cabrelli, Mark Lammers, Ursula Molter, and Kasso Okoudjou.

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