3.7. The Fourier Transform of Tempered Distributions.

One of the main uses of the space of tempered distributions is that we can define the F.T. of any tempered distribution. As usual, we simply extend a function definition to functionals. In particular, if $f, g \in L^2(\mathbb{R})$ then the F.T. is defined by the Parseval formula

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle.$$

<u>Definition</u>

Let $\mu \in \mathcal{S}'(\mathbb{R})$ be given. Then $\hat{\mu} : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ is defined by

$$\forall f \in \mathcal{S}(\mathbb{R}), \quad \langle \hat{f}, \hat{\mu} \rangle = \langle f, \mu \rangle.$$

Note that since the F.T. maps $\mathcal{S}(\mathbb{R})$ onto itself, this implicitly defines the action of $\hat{\mu}$ on all of $\mathcal{S}(\mathbb{R})$.

Alternatively, we could define $\hat{\mu}$ by

$$\forall f \in \mathcal{S}(\mathbb{R}), \quad \langle f, \hat{\mu} \rangle = \langle \check{f}, \mu \rangle.$$

Example

Consider the δ distribution. If $f \in \mathcal{S}(\mathbb{R})$, then

$$\begin{aligned}
\langle f, \hat{\delta} \rangle &= \langle \check{f}, \delta \rangle \\
&= \check{f}(0) \\
&= \int f(x)\, dx \\
&= \langle f, \underline{1} \rangle.
\end{aligned}$$

Thus, distributionally, $\hat{\delta}$ is the constant 1.

(Note the interchange of "decay" with "smoothness" in this identification!).

Of course, we have not yet shown that $\mu \in \mathcal{S}'(\mathbb{R})$ implies $\hat{\mu} \in \mathcal{S}'(\mathbb{R})$. This is our next task.

Exercise

For $\xi \in \mathbb{R}$, define $e_\xi(x) = e^{2\pi i \xi x}$.

Then $e_\xi \in \mathcal{S}'(\mathbb{R})$ since it is bounded.

Compute $\hat{e}_\xi$.

Exercise

Given $\xi \in \mathbb{R}$, compute $(\sin 2\pi \xi x)^\wedge$

We know that the F.T. maps $\mathcal{S}(\mathbb{R})$ onto itself. The following exercise shows that the F.T. is a <u>continuous</u> map of $\mathcal{S}(\mathbb{R})$ onto itself, w.r.t. the topology of $\mathcal{S}(\mathbb{R})$.

<u>Exercise</u>
Suppose that $f_k, f \in \mathcal{S}(\mathbb{R})$ and $f_k \to f$ in $\mathcal{S}(\mathbb{R})$. Prove that $\hat{f}_k \to \hat{f}$ in $\mathcal{S}(\mathbb{R})$.

"By duality" we obtain the following corresponding result for tempered distributions.

Theorem

If $\mu \in \mathcal{S}'(\mathbb{R})$ then $\hat{\mu} \in \mathcal{S}'(\mathbb{R})$.

Proof:

Suppose that $\mu \in \mathcal{S}'(\mathbb{R})$, and that $f_k \to f$ in $\mathcal{S}(\mathbb{R})$.

Then by the preceding exercise, $\check{f}_k \to \check{f}$ in $\mathcal{S}(\mathbb{R})$. Hence,

$$\langle f_k, \hat{\mu} \rangle = \langle \check{f}_k, \mu \rangle \longrightarrow \langle \check{f}, \mu \rangle = \langle f, \hat{\mu} \rangle.$$

Therefore $\hat{\mu} \in \mathcal{S}'(\mathbb{R})$. ∎

Exercise

Show that $\mu \longmapsto \hat{\mu}$ is a continuous bijection of $\mathcal{S}'(\mathbb{R})$ onto itself.

Further, show that if we define $\check{\mu} \in \mathcal{S}'(\mathbb{R})$ by $\langle \check{f}, \check{\mu} \rangle = \langle f, \mu \rangle$ for $f \in \mathcal{S}(\mathbb{R})$, then $\check{\hat{\mu}} = \hat{\check{\mu}} = \mu$ for $\mu \in \mathcal{S}'(\mathbb{R})$.

Exercise

Show that if $n \geq 0$ then

$$\widehat{\delta^{(n)}} = (2\pi i x)^n,$$

i.e., $\widehat{\delta^{(n)}}$ corresponds to the distribution given by the function $(2\pi i x)^n$.

We define the translation, modulation & dilation of tempered distributions as usual, by duality:

$$\langle f, T_a\mu \rangle = \langle T_{-a}f, \mu \rangle$$

$$\langle f, M_b\mu \rangle = \langle M_{-b}f, \mu \rangle$$

$$\langle f, D_\lambda\mu \rangle = \langle \lambda D_{1/\lambda}f, \mu \rangle$$

<u>Exercise</u>

Prove that if $\mu \in \mathcal{S}'(\mathbb{R})$, then for $a, b \in \mathbb{R}$, $\lambda > 0$,

$$(T_a\mu)^\wedge = M_{-a}\hat{\mu}$$

$$(M_b\mu)^\wedge = T_b\hat{\mu}$$

$$(D_\lambda\mu)^\wedge = \lambda D_{1/\lambda}\hat{\mu}$$

Exercise

Given $\theta \in \mathcal{S}(\mathbb{R})$ & $\mu \in \mathcal{S}'(\mathbb{R})$. Prove that

a. $(\theta\mu)^\wedge = \hat{\theta} * \hat{\mu}$

b. $(\theta * \mu)^\wedge = \hat{\theta}\,\hat{\mu}$

The basic properties of the F.T. extend from functions to distributions.

Exercise

assume polynomial growth of derivatives

a. Prove that if $\theta \in C^{\infty}(\mathbb{R})$ & $\mu \in \mathcal{S}'(\mathbb{R})$, then $\theta\mu \in \mathcal{S}'(\mathbb{R})$, where

$$\langle f, \theta\mu \rangle = \langle f\bar{\theta}, \mu \rangle, \qquad f \in \mathcal{S}(\mathbb{R}).$$

b. Prove that if $\mu \in \mathcal{S}'(\mathbb{R})$, then

$$(D\mu)^{\wedge} = 2\pi i \xi \, \hat{\mu}$$

The notation $2\pi i \xi \hat{\mu}$ means $\theta\hat{\mu}$ where $\theta(\xi) = 2\pi i \xi$. Often we abuse notation & write $2\pi i \xi \, \hat{\mu}(\xi)$ for $\theta\hat{\mu}$.

c. Prove that if $\mu \in \mathcal{S}'(\mathbb{R})$, then

$$(-2\pi i x \mu)^{\wedge} = D\hat{\mu}.$$

Remark

The Paley-Wiener Theorem extends from compactly supported functions in $L^2(\mathbb{R})$ to compactly supported distributions $\mu \in \mathcal{E}'(\mathbb{R})$. We will prove here only a portion of the Paley-Wiener Theorem. (see Rudin, Functional Analysis, 2nd ed., p. 199 for the full Theorem).

Theorem

If $\mu \in \mathcal{E}'(\mathbb{R})$, then $\hat{\mu} \in C^\infty(\mathbb{R})$, & if $e_\xi(x) = e^{2\pi i \xi x}$ then $\hat{\mu}(\xi) = \overline{\langle e_\xi, \mu \rangle}$, $\xi \in \mathbb{R}$.

Proof:

Suppose that $\mu \in \mathcal{E}'(\mathbb{R})$, i.e., μ is a compactly supported distribution. By an earlier theorem, $\mathcal{E}'(\mathbb{R}) = C^\infty(\mathbb{R})^*$, so we can define

$$F(\xi) = \overline{\langle e_\xi, \mu \rangle}, \qquad \xi \in \mathbb{R}$$

where $e_\xi(x) = e^{2\pi i \xi x} \in C^\infty(\mathbb{R})$.

Let $\theta \in C_c^\infty(\mathbb{R})$ be s.t. $\theta = 1$ on $\mathrm{supp}(\mu)$.

Then $\theta\mu = \mu$ by an earlier exercise.

Also, since $\hat{\theta} \in \mathcal{S}(\mathbb{R})$ & $\hat{\mu} \in \mathcal{S}'(\mathbb{R})$, we have by earlier exercises that $\hat{\mu} = (\theta\mu)^\wedge = \hat{\theta} * \hat{\mu} \in C^\infty(\mathbb{R})$.

Further,

$$
\begin{aligned}
\hat{\mu}(\xi) = (\hat{\theta} * \hat{\mu})(\xi) &= \int \hat{\theta}(\xi - \eta)\, \hat{\mu}(\eta)\, d\eta \\
&= \int \overline{\tilde{\hat{\theta}}(\eta - \xi)}\, \hat{\mu}(\eta)\, d\eta \\
&= \overline{\langle T_\xi \tilde{\hat{\theta}}, \hat{\mu} \rangle} \\
&= \overline{\langle T_\xi \hat{\bar{\theta}}, \hat{\mu} \rangle} \\
&= \overline{\langle (T_\xi \hat{\bar{\theta}})^\vee, \mu \rangle} \\
&= \overline{\langle M_\xi \bar{\theta}, \mu \rangle} \\
&= \overline{\langle e_\xi \bar{\theta}, \mu \rangle} \\
&= \overline{\langle e_\xi, \theta\mu \rangle} \\
&= \overline{\langle e_\xi, \mu \rangle} \\
&= F(\xi). \quad \blacksquare
\end{aligned}
$$

Exercise

a. Let $\mu \in \mathcal{S}'(\mathbb{R})$ be given. Recall that $\bar{\mu}$ & $\tilde{\mu}$ are defined by $\langle f, \bar{\mu} \rangle = \overline{\langle \bar{f}, \mu \rangle}$ & $\langle f, \tilde{\mu} \rangle = \overline{\langle \tilde{f}, \mu \rangle}$.

Prove that $\hat{\tilde{\mu}} = \bar{\hat{\mu}}$.

b. Suppose $\mu \in \mathcal{E}'(\mathbb{R})$ & $\nu \in \mathcal{S}'(\mathbb{R})$. Prove that $\mu * \nu \in \mathcal{S}'(\mathbb{R})$, and that

$$(\mu * \nu)^\wedge = \hat{\mu} \hat{\nu}$$

(note that $\hat{\mu} \in C^\infty(\mathbb{R})$ since μ has compact support).

Exercise

Prove that if $\mu \in \mathcal{E}'(\mathbb{R})$, then

$$\operatorname{supp}(\mu) = \{0\} \iff \hat{\mu} \text{ is a nonzero polynomial.}$$

What if $\operatorname{supp}(\mu) = \{a\}$?

Recall that $L^p(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$ for all $1 \le p \le \infty$.

<u>Definition</u>

For $1 \le p \le \infty$, we set

$$\mathcal{F}L^p(\mathbb{R}) = \{\hat{g} : g \in L^p(\mathbb{R})\},$$

with norm

$$\|\hat{g}\|_{\mathcal{F}L^p} = \|g\|_p, \quad g \in L^p(\mathbb{R}).$$

Note that $\mathcal{F}L^1(\mathbb{R}) = A(\mathbb{R})$, & $\mathcal{F}L^p(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$ $\forall p$.

The space $\mathcal{F}L^\infty(\mathbb{R})$ is called the space of <u>pseudomeasures</u> on $\mathbb{R}$.

<u>Exercise</u>

a. Show that $\mathcal{F}L^p(\mathbb{R})$ is a Banach space for each $1 \le p \le \infty$.

b. Show $\mathcal{F}L^p(\mathbb{R})^* = \mathcal{F}L^{p'}(\mathbb{R})$, $1 \le p < \infty$.

c. Suppose $g \in L^\infty(\mathbb{R})$, so $\hat{g} \in \mathcal{F}L^\infty(\mathbb{R})$. Show that $M_b g \xrightarrow{w^*} 0$ as $b \to \infty$ in $\mathcal{F}L^\infty(\mathbb{R})$.

The following exercise will compute the F.T. of the Heaviside function $H = \chi_{[0,\infty)}$. An earlier exercise showed that $H' = \delta$, so we might hope that, at least formally,

$$1 = \hat{\delta} = \widehat{H'} = 2\pi i \xi \hat{H}(\xi),$$

or $$\hat{H}(\xi) = \frac{1}{2\pi i \xi}.$$

Unfortunately, we know that the function $1/\xi$ does not define a distribution. But the principal value $\mathrm{pv}(1/\xi)$ is a distribution, & perhaps it is a suitable replacement for $1/\xi$.

Exercise

a. Prove that

$$\langle f, \mathrm{pv}(\tfrac{1}{x})\rangle = \lim_{T\to\infty} \int_{\frac{1}{T}<|x|<T} \frac{f(x)}{x}\,dx$$

defines a tempered distribution (an earlier exercise showed that $\mathrm{pv}(\frac{1}{x}) \in \mathcal{D}'(\mathbb{R})$; this proof is similar).

b. Given $f \in \mathcal{S}(\mathbb{R})$, write $f = \check{f}^{\wedge}$, so

$$\langle f, \mathrm{pv}(\tfrac{1}{x})\rangle = \lim_{T\to\infty} \int_{\frac{1}{T}<|x|<T} \frac{\check{f}^{\wedge}(x)}{x}\,dx$$

$$= \lim_{T\to\infty} \int_{\frac{1}{T}<|x|<T} \int \frac{\check{f}(\xi)\, e^{-2\pi i \xi x}}{x}\,d\xi\,dx.$$

We will exchange the limit & the integral in a specific order. First write

$$E_T(\xi) = \int_{\frac{1}{T}<|x|<T} \frac{e^{-2\pi i \xi x}}{x}\,dx,$$

and show that

$$E_T(\xi) = -i\int_{-T}^{T} \frac{\sin 2\pi \xi x}{x}\,dx + i\int_{-1/T}^{1/T} \frac{\sin 2\pi \xi x}{x}\,dx.$$

c. Show that

$$\lim_{T\to\infty} E_T(\xi) = -2\pi i H(\xi) + \pi i = \begin{cases} \pi i, & \xi < 0 \\ -\pi i, & \xi \geq 0 \end{cases}$$

Hint: We know the value of the following improper Riemann integral:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \lim_{T\to\infty} \int_{-T}^{T} \frac{\sin x}{x}\, dx = \pi$$

d. Show that for each fixed $T > 0$,

$$\int_{\frac{1}{T}<|x|<T} \int \frac{\check{f}(\xi)\, e^{-2\pi i \xi x}}{x}\, d\xi\, dx = \int \check{f}(\xi)\, E_T(\xi)\, d\xi$$

e. Show that

$$\sup_{\substack{T>0 \\ \xi\in\mathbb{R}}} |E_T(\xi)| < \infty.$$

Hint: $E_T(\xi)$ converges as $T\to\infty$, so for each ξ $\exists\, C_\xi$ s.t. $|E_T(\xi)| \leq C_\xi$ $\forall T$. Show that all the C_ξ are equal. by making a change of variables.

f. Prove that

$$\langle f, pv(\tfrac{1}{x})\rangle = \lim_{T\to\infty} \int \check{f}(\xi) E_T(\xi)\, d\xi$$

$$= -2\pi i \langle \check{f}, H\rangle + \pi i \langle \check{f}, 1\rangle$$

g. Prove that

$$\hat{H} = \frac{1}{2\pi i} pv(\tfrac{1}{x}) + \frac{1}{2}\delta$$

Exercise (See Kammler, p. 467)

Since $g(x) = |x|^{-1/2} \in L^1_{loc}(\mathbb{R})$ and has polynomial growth at ∞, it is a tempered distribution. This exercise will show that

$$\hat{g}(\xi) = |\xi|^{-1/2} = g(\xi).$$

a. Show that $$\int_0^\infty x^{-1/2} \cos 2\pi|\xi|x \, dx$$

$$= \left(\frac{2}{\pi|\xi|}\right)^{1/2} \int_0^\infty \cos u^2 \, du.$$

Hint: Change of variables $u^2 = 2\pi|\xi|x$.

Remark: $\int_0^\infty \cos u^2 \, du$ is a Fresnel integral.

It can be shown that

$$\int_0^\infty \cos u^2 \, du = \int_0^\infty \sin u^2 \, du = \frac{1}{2}\sqrt{\frac{\pi}{2}},$$

as a consequence of the fact that

$$\int_{-\infty}^\infty e^{it^2} dt = (\pi i)^{1/2} = \pi^{1/2} \frac{1+i}{\sqrt{2}}.$$

For a proof using contour integration, see Benedetto, Harmonic Analysis, p. 135.

b. Prove that if $f \in \mathcal{S}(\mathbb{R})$, then

$$\langle \hat{f}, \hat{g} \rangle = 2 \int \hat{f}(\xi) \int_0^\infty x^{-1/2} \cos 2\pi |\xi| x \, dx \, d\xi,$$

and use this to show that $\hat{g}(\xi) = |\xi|^{-1/2}$.

~~Exam~~ Exercise (See Kammler, p.420; Benedetto p.137)
In this exercise we will compute the distributional F.T. of the "chirp"

$$g(x) = e^{\pi i x^2} \in C_b^\infty(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R}).$$

a. Show that $\widehat{g'} = -D\hat{g}$.

Hint: An earlier exercise showed how to compute the distributional F.T. $(2\pi i x\mu)^\wedge$.

b. Show that $D(e^{\pi i \xi^2}\hat{g}) = 0$. Apply an earlier exercise to conclude that $\hat{g}(\xi) = c\,e^{-\pi i \xi^2}$ for some scalar $c \in \mathbb{C}$.

c. Now we compute c. Let $\phi(x) = e^{-\pi x^2}$. Show that

$$\langle \phi, g\rangle = \langle \hat{\phi}, \hat{g}\rangle = \bar{c}\,\overline{\langle \phi, g\rangle}.$$

d. Let $z = \langle \phi, g\rangle$. Show that $z^2 = \frac{1-i}{2}$.

e. Show $z^2 = \bar{c}\,|z|^2$ & conclude that $c = \frac{1-i}{\sqrt{2}}$.
Thus

$$\hat{g}(\xi) = \frac{1-i}{\sqrt{2}}\,e^{-\pi i \xi^2} = \overline{\frac{1+i}{\sqrt{2}}\,g(\xi)}$$

Exercise (See Benedetto, p. 136).

Let $k(x) = i^{-1/2} e^{\pi i x^2}$. The preceding exercise computed the distributional F.T. of k. k is called a "chirp" because its "frequency" increases with time x (see plot). Note that $k \notin L^1(\mathbb{R}), L^2(\mathbb{R})$, but as an improper Riemann integral we have $\int_{-\infty}^{\infty} k(x)\, dx = 1$. Because of its lack of decay & rapid oscillations, it seems pointless to try to use k to form an approximate identity. Even so, define $k_\lambda(x) = \lambda k(\lambda x)$, and prove that $k_\lambda \to \delta$ in $\mathcal{S}'(\mathbb{R})$.

Exercise

Prove that
$$\sup_{a<b} \left| \int_a^b e^{-ix^2}\, dx \right| < \infty.$$

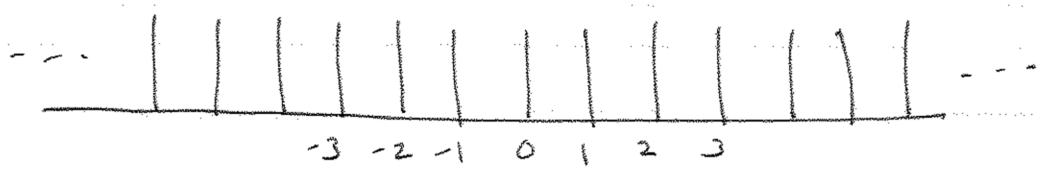
Hint: Consider $1 \leq a < b < \infty$. Multiply & divide by $2x$.

Exercise

Let $\mu = \sum_{n\in\mathbb{Z}} \delta_n$, i.e.,

$$\forall f \in \mathcal{S}(\mathbb{R}), \qquad \langle f, \mu \rangle = \sum_{n\in\mathbb{Z}} f(n).$$

Pictorially, we may (with some poetic license), imagine that μ "looks" like this:



This picture inspires many names for μ, including:

- the delta train or train of deltas (or Diracs)
- the Dirac comb
- the Shah distribution, because the picture is suggestive of the Cyrillic letter "sha": Ш. For this reason, μ is often denoted by the symbol Ш.

a. Prove that $\mu \in \mathcal{S}'(\mathbb{R})$.

b. Prove that

$$\left(\sum_{n\in\mathbb{Z}} \delta_n\right)^\wedge = \sum_{n\in\mathbb{Z}} \delta_n .$$

Explain why this is the Poisson Summation Formula for μ.

Exercise

a. In an earlier exercise, you showed that the δ distribution satisfies a one-term refinement equation of the form

$$\delta(x) = 2\delta(2x)$$

(interpreted distributionally). Suppose that $\mu \in \mathcal{S}'(\mathbb{R})$. Show that

$$\mu(x) = 2\mu(2x) \iff \hat{\mu}(\xi) = \hat{\mu}(\xi/2)$$

(again interpreting these distributionally).

b. Find distributional solutions to $\mu(x) = 2\mu(2x)$ other than the δ distribution. *Hint:* What if $\hat{\mu}$ is a <u>function</u>?

c. Prove that if $\mu \in \mathcal{S}'(\mathbb{R})$ satisfies $\mu(x) = 2\mu(2x)$ and $\hat{\mu}$ is a function that is continuous at $\xi = 0$, then $\mu = c\delta$ where c is a scalar.

Exercise

In this exercise we will study the <u>refinement equation</u>

$$\varphi(x) = \sum_{k=0}^{N} c_k \varphi(2x-k). \qquad (*)$$

Solutions to such equations play important roles in wavelet theory & in subdivision schemes.

a. Suppose that $\varphi \in \mathcal{S}'(\mathbb{R})$. Show that φ satisfies $(*)$ (in the sense of distributions) if & only if

$$\hat{\varphi}(\xi) = m_0(\xi/2)\, \hat{\varphi}(\xi/2) \qquad (**)$$

(again in the sense of distributions), where

$$m_0(\xi) = \frac{1}{2} \sum_{k=0}^{N} c_k\, e^{-2\pi i k \xi}.$$

b. Iterating equation $(**)$, we have for any N that

$$\hat{\varphi}(\xi) = m_0(\xi/2)\, m_0(\xi/4) \cdots m_0(\xi/2^N)\, \hat{\varphi}(\xi/2^N).$$

Let $P_N(\xi) = \prod_{j=1}^{N} m_0(\xi/2^j)$. Prove that

$$P(\xi) = \prod_{j=1}^{\infty} m_0(\xi/2^j) = \lim_{N\to\infty} P_N(\xi)$$

converges uniformly on each compact set in $\mathbb{R}$, as long as $m_0(0) = \frac{1}{2} \sum_{k=0}^{N} c_k = \underline{1}$.

Hint: Show that $\exists C > 0$ s.t.

$$|m_0(\xi) - 1| \leq C|\xi|,$$

and use the fact that $1 + |\xi| \leq e^{|\xi|}$.

c. Prove that P is a continuous function with at most polynomial growth at ∞. (Assume $\sum_{k=0}^{N}{}' c_k = 2$ from now on)

Hint: How does $\| P \cdot \chi_{[-2,2]} \|_\infty$ compare to $\| P \cdot \chi_{[-1,1]} \|_\infty$?

d. Prove that $\varphi = \check{P}$ is a solution to $(*)$ in the sense of distributions.

e. Show that $\check{P}_N$ is supported within $[0,$ [redacted] $N]$.

f. Show that $\check{P}_N \longrightarrow \check{P}$ in $\mathcal{S}'(\mathbb{R})$.

g. Show that $\varphi = \check{P}$ is a compactly supported distribution, with $\mathrm{supp}(\varphi) \subseteq$ [redacted] $[0, N]$.

[This approach to the support of φ is due to Deslaurier & Dubuc; there are other approaches.]

h. Thus, as long as $\sum_{k=0}^{N} c_k = 2$, the refinement equation has a compactly supported distributional solution. Suppose that $\psi \in \mathcal{S}'(\mathbb{R})$ was also a distributional solution. Prove that if $\hat{\varphi}$ is a continuous function, then ψ is a scalar multiple of φ. Thus, up to scalar multiples, there is a unique compactly supported solution to the refinement equation. [Due to Daubechies/Lagarias]

i. Prove that $D\varphi$ satisfies the R.E.

$$D\varphi(x) = 2 \sum_{k=0}^{N} c_k \, D\varphi(2x-k).$$

Exercise [Daubechies]

In this exercise we will derive some sufficient conditions for the existence of solutions to the refinement equation (*). We impose the requirements

$$\sum c_k = 2, \qquad \sum{}' c_{2k} = \sum{}' c_{2k+1} = 1.$$

a. Show $m_0(0) = 1$ & $m_0(\tfrac{1}{2}) = 0$.

b. Show $m_0(\xi) = \dfrac{1+e^{2\pi i \xi}}{2} Q(\xi)$ where Q is a trigonometric polynomial (finite linear combination of $e^{2\pi i n \xi}$, $n \in \mathbb{Z}$).

Hint: Write m_0 as a polynomial in the variable $z = e^{-2\pi i \xi}$.

c. Let φ be the compactly supported distributional solution constructed in the preceding exercise. Show

$$\hat{\varphi}(\xi) = \frac{\sin \pi \xi}{\pi \xi} \prod_{j=1}^{\infty} Q(\xi/2^j).$$

Hint: Prove Viète's formula:

$$\prod_{j=1}^{\infty} \cos(\xi/2^j) = \frac{\sin \xi}{\xi}.$$

d. Prove $\exists R > 0$ s.t.

$$\| P \cdot \chi_{[-2^n, 2^n]} \|_\infty \leq R \|Q\|_\infty^n$$

and show this implies $\exists C > 0$ s.t.

$$|P(\xi)| \leq C |\xi|^{\log_2 \|Q\|_\infty}$$

e. Show $\exists C'$ s.t.

$$|P(\xi)|^2 \leq C' \left| \frac{\sin \pi\xi}{\pi\xi} \right|^{1+p}$$

where $p = -2 \log_2 \|Q\|_\infty$.

f. Show that if $\|Q\|_\infty < 2^{1/2}$ then $P \in L^2(\mathbb{R})$

and therefore $\varphi = \check{P} \in L^2(\mathbb{R})$.

g. Impose the additional requirement that

$m_0'(\frac{1}{2}) = \cdots = m_0^{(L)}(\frac{1}{2})$. Show the results of this

exercise generalize, e.g.,

$$\hat{\varphi}(\xi) = \frac{\sin \pi\xi}{\pi\xi} \prod_{j=1}^{\infty} Q_L(\xi/2^j)$$

& if $\|Q_L\|_\infty < 2^{L-1/2}$ then $\varphi \in L^2(\mathbb{R})$.

1h. Investigate some special cases, e.g., $N=3$.

See Wim Swelden's <u>wavelet applet</u>.

<u>Exercise</u>

This exercise gives another approach to proving the existence of solutions to the Refinement equation. This result is due to Mallat, with the particular technique used here due to Lawton.

Throughout, assume

$$\sum' c_k = 2 \quad \& \quad \sum' c_{2k} = \sum' c_{2k+1} = 1.$$

a. Suppose that $\varphi \in L^2(\mathbb{R})$ is a solution to the R.E., & $\{T_n\varphi\}_{n\in\mathbb{Z}}$ is an ON system.

Show

$$\sum_k' c_k\, c_{k+2l} = 2\,\delta_{l,0} \quad \forall l \in \mathbb{Z} \qquad \text{(A)}$$

($c_k = 0$ for $k \neq 0, \dots, N$).

b. Show (A) is equivalent to

$$|m_0(\xi)|^2 + |m_0(\xi+\tfrac{1}{2})|^2 = 1, \quad \xi \in \mathbb{R}.$$

c. Suppose that

$$|m_0(\xi)|^2 + |m_0(\xi + \tfrac{1}{2})|^2 \leq 1 \quad \forall \xi \in \mathbb{R}. \quad (\mathcal{B})$$

Set

$$p_N(\xi) = P_N(\xi) \cdot \chi_{[-2^{N-1}, 2^{N-1}]}(\xi)$$

Note that $p_N \to P$ uniformly on each compact set. Prove that

$$\|p_N\|_2 \leq \|p_{N-1}\|_2 .$$

Hint: Write

$$\|p_N\|_2^2 = \int_0^{2^{N-1}} \left| \prod_{j=1}^{N} m_0(\xi/2^j) \right|^2 d\xi$$

$$+ \int_0^{2^{N-1}} \left| \prod_{j=1}^{N} m_0\left(\frac{\xi + 2^{N-1}}{2^j}\right) \right|^2 d\xi,$$

use the fact that m_0 is 1-periodic, & use $(\mathcal{B})$.

d. Thus $\{p_N\}_{N \in \mathbb{N}}$ is a bounded sequence in $L^2(\mathbb{R})$. Therefore, by Alaoglu's Theorem (see the Appendix), it has a weakly convergent subsequence, i.e., $p_N \xrightarrow{\omega} \psi$ for some $\psi \in L^2(\mathbb{R})$. Prove that

$\Psi = P$, & hence $\varphi = \check{P} \in L^2(\mathbb{R})$. is a compactly support function that satisfies the R.E.

<u>Remark</u>

The implication

$$\{T_n\varphi\}_{n\in\mathbb{Z}} \text{ ON} \implies |m_0(\xi)|^2 + |m_0(\xi+\tfrac{1}{2})|^2 = 1$$

does not reverse, e.g., consider $\varphi = \chi_{[0,3]}$, which satisfies

$$\varphi(x) = \varphi(2x) + \varphi(2x-3).$$

However, "almost all" choices of c_k satisfy the converse implication. Further, Lawton proved that

$$|m_0(\xi)|^2 + |m_0(\xi+\tfrac{1}{2})|^2 = 1 \implies \forall f \in \overline{\operatorname{span}}\{T_n\varphi\}_{n\in\mathbb{Z}},$$
$$f = \sum_{n\in\mathbb{Z}}{}' \langle f, T_n\varphi\rangle T_n\varphi$$

That is, $\{T_n\varphi\}_{n\in\mathbb{Z}}$ is a <u>Parseval frame</u> — but this does not imply $\{T_n\varphi\}_{n\in\mathbb{Z}}$ is ON.

[References/Examples].