3.5 "Generalized Function" properties: Differentiation

Many properties valid for functions in $C_c^\infty(\mathbb{R})$ can be extended to its dual space $\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$. For example, we can define the derivative of any distribution.

<u>Motivation</u>

Suppose that $g \in C^1(\mathbb{R}) \subseteq L^1_{loc}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$. Then g determines a distribution T_g. But furthermore, g' is a continuous function, so $g' \in C(\mathbb{R}) \subseteq L^1_{loc}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$. Thus g' determines a distribution $T_{g'}$, and by integration by parts we have for $f \in C_c^\infty(\mathbb{R})$ that

$$\begin{aligned}
\langle f, T_{g'} \rangle &= \int_{-\infty}^{\infty} f(x)\,\overline{g'(x)}\,dx \\
&= f(x)\overline{g(x)}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\overline{g(x)}\,dx \\
&= 0 - \langle f', T_g \rangle.
\end{aligned}$$

We <u>define</u> the distribution $(T_g)'$ to be given by

$$\langle f, T_g' \rangle \overset{def}{=} -\langle f', T_g \rangle = -\langle f, T_{g'} \rangle.$$

That is, we <u>define</u> $T_g' = -T_{g'}$.

Definition

The <u>derivative</u> of $T \in \mathcal{D}'(\mathbb{R})$ is $T' : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ given by

$$\langle f, T' \rangle = - \langle f', T \rangle, \quad f \in C_c^\infty(\mathbb{R}).$$

In essence, we <u>declare</u> that integration by parts is valid for distributions.

<u>Exercise</u>

a. Show that if $T \in \mathcal{D}'(\mathbb{R})$ then $T' \in \mathcal{D}'(\mathbb{R})$.

b. Show that if $g \in C^1(\mathbb{R})$ then ordinary & distributional differentiation are equal, ie,

$$(T_g)' = T_{g'}.$$

c. Show that $\langle f, \delta' \rangle = -f'(0)$, & $\operatorname{supp}(\delta') = \{0\}$.

d. Show that δ is infinitely differentiable in the sense of distributions, & find a formula for $\delta^{(n)}$.

e. Show that every $T \in \mathcal{D}'(\mathbb{R})$ is infinitely differentiable in the sense of distributions.

f. Let H be the Heaviside function: $H = \chi_{[0,\infty)} \in L^1_{loc}(\mathbb{R})$.

Show that the <u>distributional</u> derivative of H is

$H' = \delta$, but the <u>pointwise a.e.</u> derivative is $H' = 0$ a.e.

Notation

To distinguish between the ordinary & distributional derivatives of functions, we will write

$$Dg = \text{distributional derivative of } g \in L^1_{loc}(\mathbb{R})$$

$$g' = \text{pointwise a.e. derivative of a function that is differentiable a.e.}$$

Theorem (See Benedetto, p. 81)

Assume g is continuously differentiable on $\mathbb{R} \setminus \{0\}$, & has a jump discontinuity at 0 of height $\sigma = g(0+) - g(0-)$. Then, as distributions,

$$Dg = g' + \sigma \delta$$

That is,

$$\forall f \in C, \quad \langle f, Dg \rangle = \int f(x) \overline{g'(x)}\, dx + \bar{\sigma} f(0).$$

Proof: Exercise.

Hints: Write

$$\langle f, Dg \rangle = -\langle f', g \rangle = -\int_{-\infty}^{0} f'(x) \overline{g(x)}\, dx - \int_{0}^{\infty} f'(x) \overline{g(x)}\, dx$$

(More precisely, these are improper Riemann integrals.)

Since f' & g are continuously differentiable on $\mathbb{R} \setminus \{0\}$, you can apply integration by parts. Recall that f is compactly supported, hence fg is as well.

Exercise (see Benedetto, p. 82)

Define

$$g(x) = \begin{cases} x\, e^{ie^{-x^2}}, & x > 0 \\ 0, & x \le 0. \end{cases}$$

Show that g is continuous on $\mathbb{R}$, g is continuously differentiable on $\mathbb{R} \setminus \{0\}$, but the pointwise a.e. derivative g' does not belong to $L^1_{loc}(\mathbb{R})$. How does this relate to the preceding exercise?

See Benedetto p. 85
Folland p. 290

Exercise

Let $g(x) = \ln|x|$ (defined everywhere except $x=0$).

Show that $\ln|x| \in L^1_{loc}(\mathbb{R})$.

Hints: Given $0 < a < 1 < b < \infty$, write

$$\int_a^b |\ln|x||\, dx = -\int_a^1 \ln x\, dx + \int_1^b \ln x\, dx.$$

Integrate using the fact that $\frac{d}{dx}(x\ln x - x) = \ln x$.

Taking the limit as $a \to 0$, conclude that

$$\int_0^b |\ln|x||\, dx < \infty \quad \forall b > 1.$$

Example

Since $\ln|x| \in L^1_{loc}(\mathbb{R})$, it has a distributional derivative. We will show that

$$D(\ln|x|) = pv\left(\frac{1}{x}\right).$$

By definition, if $f \in C_c^\infty(\mathbb{R})$, then

$$\langle f, D(\ln|x|)\rangle = -\langle f', \ln|x|\rangle = -\int f'(x) \ln|x|\, dx.$$

Let R be s.t. $supp(f) \subseteq [-R, R]$.

If $0 < a < 1 < R < b$, then

$$\int_a^b f'(x) \ln x \, dx = f(b) \ln b - f(a) \ln x - \int_a^b \frac{f(x)}{x} dx$$

$$= 0 - a \ln a \, \frac{f(a) - f(0)}{a - 0} - f(0) \ln a - \int_a^b \frac{f(x)}{x} dx$$

and similarly

$$\int_{-b}^{-a} f'(x) \ln(-x) \, dx = f(-a) \ln a - f(-b) \ln b - \int_{-b}^{-a} \frac{f(x)}{x} dx$$

$$= -a \ln a \, \frac{f(-a) - f(0)}{-a - 0} + f(0) \ln a - 0 - \int_{-b}^{-a} \frac{f(x)}{x} dx$$

Hence

$$\int_{a<|x|<b} f'(x) \ln|x| \, dx = -a \ln a \left[\frac{f(a) - f(0)}{a - 0} + \frac{f(-a) - f(0)}{-a - 0} \right] - \int_{a<|x|<b} \frac{f(x)}{x} dx$$

Thus

$$-\lim_{\substack{a \to 0^+ \\ b \to \infty}} \int_{a<|x|<b} f'(x) \ln|x| = \left(\lim_{a \to 0^+} a \ln a \right) \left[\lim_{a \to 0^+} \frac{f(a) - f(0)}{a - 0} + \lim_{a \to 0^+} \frac{f(-a) - f(0)}{-a - 0} \right] + \lim_{\substack{a \to 0^+ \\ b \to \infty}} \int_{a<|x|<b} \frac{f(x)}{x} dx$$

$$= 0 \cdot \left(f'(0) + f'(0)\right) + \left\langle f, \, pv\left(\tfrac{1}{x}\right)\right\rangle$$

Thus

$$\left\langle f, \, D(\ln|x|)\right\rangle = \left\langle f, \, pv\left(\tfrac{1}{x}\right)\right\rangle \quad \forall f \in C_c^\infty(\mathbb{R}). \quad \blacksquare$$

Exercises

Compute the distributional derivatives of the following functions.

a. $\chi_{[-T,T]}$.

b. $H(x)\cos x$, where $H = \chi_{[0,\infty)}$ is the Heaviside function.

c. $x H(x)$.

d. $|x|$

e. $|\cos x|$

Exercise

Define $x^n \delta^{(n)}(x)$ in the usual way, i.e.,

$$\langle f, x^n \delta^{(n)}(x) \rangle = \langle x^n f(x), \delta^{(n)} \rangle, \quad f \in C_c^\infty(\mathbb{R}).$$

Prove that $x^n \delta^{(n)}(x) = (-1)^n n! \, \delta$.

Exercise

Show $\exists g \in C^\infty(\mathbb{R})$ s.t. $D^2 g = \delta$.

<u>Exercise</u> (See Rudin, F.A., 2^{nd} ed, p. 178)

a. Let $\mu \in \mathcal{D}'(\mathbb{R})$ & $\theta \in C_c^\infty(\mathbb{R})$ be given. Show

$$\theta = 0 \text{ on } \operatorname{supp}(\mu) \not\Rightarrow \theta\mu = 0.$$

Hint: Consider $\theta\delta'$.

b. Let $\mu \in \mathcal{D}'(\mathbb{R})$ be s.t. $\operatorname{supp}(\mu) \subseteq [-1,1]$. Show

$$\theta = 0 \text{ on } [-1,1] \Rightarrow \theta\mu = 0.$$

Explain the difference from part a.

Exercise

Let $\mu \in \mathcal{D}'(\mathbb{R})$ & $f \in C_c^\infty(\mathbb{R})$ be given. Recall that

$$(f * \mu)(x) = \overline{\langle T_x \tilde{f}, \mu \rangle}, \qquad \tilde{f}(x) = \overline{f(-x)}.$$

~~[illegible]~~ An earlier exercise shows

$$f * \mu \in C^\infty(\mathbb{R}) \quad \& \quad (f * \mu)' = f' * \mu.$$

Prove that

$$f * D\mu = f' * \mu = (f * \mu)' = D(f * \mu)$$

Exercise: Product Rule

Let $\mu \in \mathcal{D}'(\mathbb{R})$ & $\theta \in C^\infty(\mathbb{R})$. Recall that $\theta\mu$ is a distribution defined by $\langle f, \theta\mu \rangle = \langle f\bar{\theta}, \mu \rangle$ for $f \in C_c^\infty(\mathbb{R})$. Prove that

$$D(\theta\mu) = \theta'\mu + \theta D\mu.$$

Exercise (See Rudin, FA, 2nd ed., p. 180)

Given $\mu \in \mathcal{D}'(\mathbb{R})$, prove that

$$\frac{\mu - T_a\mu}{a} \longrightarrow D\mu \quad \text{in } \mathcal{D}'(\mathbb{R}) \text{ as } a \to 0.$$

Lemma

Suppose that $g \in C(\mathbb{R})$ is s.t. $Dg \in C(\mathbb{R}) \in C(\mathbb{R})$.

Then $g \in C^1(\mathbb{R})$ & $g' = Dg$.

Remark

Since $C(\mathbb{R}) \subseteq L^1_{loc}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$, we know that g has a distributional derivative. By saying that $Dg \in C(\mathbb{R})$, we are saying that $\exists$ a continuous function G that acts as a distribution just as Dg does, i.e.,

$$\forall f \in C_c^\infty(\mathbb{R}), \quad \langle f, Dg \rangle \overset{\text{def}}{=} -\underbrace{\langle f', g \rangle}_{\text{integral}} = \underbrace{\langle f, G \rangle}_{\text{integral}}$$

Proof:

Note that if g did belong to $C^1(\mathbb{R})$, then by the Fundamental Theorem of Calculus we would have

$$g(x+h) - g(x) = \int_x^{x+h} g'(t)\, dt$$

$$= \int_0^h g'(x+t)\, dt$$

$$= h \int_0^1 g'(x+ht)\, dt.$$

We do not yet know that g' exists, but we claim that

$$(*) \quad g(x+h) - g(x) = \int_x^{x+h} Dg(t)\,dt = h\int_0^1 Dg(x+ht)\,dt.$$

Before proving the claim, observe that if $f \in C_c^\infty(\mathbb{R})$,

then $\tilde{f} * g \in C^\infty(\mathbb{R})$, & by the FTC we have

$$\begin{aligned} h\int_0^1 (\tilde{f}*g)'(ht)\,dt &= \int_0^h (\tilde{f}*g)'(t)\,dt \\ &= (\tilde{f}*g)(h) - (\tilde{f}*g)(0) \\ &= (\tilde{f}*g)(h) - \overline{\langle f,g\rangle}. \end{aligned}$$

~~Hence~~ By an earlier exercise, $(\tilde{f}*g)' = \tilde{f} * Dg$, so we have

$$\begin{aligned} \langle f,g\rangle &= \overline{(\tilde{f}*g)(h)} - h\int_0^1 \overline{(\tilde{f}*Dg)(ht)}\,dt \\ &= \int f(-x)\,\overline{g(h-x)}\,dx - h\int_0^1\int f(-x)\,\overline{Dg(ht-x)}\,dx\,dt \\ &= \int f(x)\,\overline{g(h+x)}\,dx - h\int\int_0^1 f(x)\,\overline{Dg(ht+x)}\,dt\,dx \end{aligned}$$

Exercise: Apply Fubini's Theorem to justify the ~~[illegible]~~ interchange in the order of integration.

Since $\langle f,g\rangle = \int f(x)\,\overline{g(x)}\,dx$, rearranging the above calculation gives

$$(**) \quad \int f(x) \left[\overline{g(x+h) - g(x) - h\int_0^1 Dg(x+ht)\, dt} \right] dx = 0.$$

Since the term in brackets is a continuous function of x, & since $(**)$ holds for every $f \in C_c^\infty(\mathbb{R})$, the term in brackets must be identically zero. This proves that the claimed equation $(*)$ holds, & hence

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} Dg(t)\, dt. \qquad (***)$$

Since Dg is continuous, it follows from the Fundamental Theorem of Calculus that the RHS of $(***)$ approaches $Dg(x)$ as $h \to 0$. Thus

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{1}{h}\int_x^{x+h} Dg(t)\, dt = Dg(x).$$

Exercise

In the course of the preceding proof, we showed that if $g \in C(\mathbb{R})$ & $f \in C_c^\infty(\mathbb{R})$, then

$$\langle f, g\rangle = \overline{(\tilde{f} * g)(h)} - h\int_0^1 \overline{(\tilde{f} * Dg)(ht)}\, dt.$$

Prove that this extends to arbitrary distributions, i.e., show that if $\mu \in \mathcal{D}'(\mathbb{R})$ & $h \in \mathbb{R}$ then

$$\langle f, \mu\rangle = \overline{(\tilde{f} * \mu)(h)} - h\int_0^1 (\tilde{f} * D\mu)(ht)\, dt.$$

Hint: By an earlier exercise, $\tilde{f} * D\mu = (\tilde{f} * \mu)'$, & $\tilde{f} * \mu \in C^\infty(\mathbb{R})$ so the FTC is valid.

See Leib & Loss
Analysis 2nd ed.
p. 143 - 146

<u>Theorem</u>
Let $\mu \in \mathcal{D}'(\mathbb{R})$ be given. Then

$$\mu \in C^1(\mathbb{R}) \iff D\mu \in C(\mathbb{R}).$$

Moreover, in this case we have $D\mu = \mu'$.

<u>Proof:</u>
$\Rightarrow$ Suppose μ is continuously differentiable on $\mathbb{R}$, and fix any $f \in C_c^\infty(\mathbb{R})$. Then integration by parts is valid & gives (say $\operatorname{supp}(f) \subseteq [-T,T]$)

$$\begin{aligned}
\langle f, D\mu \rangle &= -\langle f', \mu \rangle \\
&= -\int_{-T}^{T} f'(x)\,\overline{\mu(x)}\,dx \\
&= -f(x)\mu(x)\Big|_{-T}^{T} + \int_{-T}^{T} f(x)\,\overline{\mu'(x)}\,dx \\
&= 0 + \langle f, \mu' \rangle.
\end{aligned}$$

Hence $D\mu = \mu' \in C(\mathbb{R})$ (equality of distributions).

$\Leftarrow$. Suppose that $\mu \in \mathcal{D}'(\mathbb{R})$ & $D\mu \in C(\mathbb{R})$.

Our first goal is to show that $\mu \in C(\mathbb{R})$.

Once this is done, an earlier lemma implies that

$\mu \in C^1(\mathbb{R})$ & $D\mu = \mu'$, so the proof is complete.

From an earlier exercise, we have for any

$f \in C_c^\infty(\mathbb{R})$ that $\forall x \in \mathbb{R}$,

$$\langle f, \mu \rangle = \overline{(\tilde{f} * \mu)(x)} - \int_0^1 x \overline{(\tilde{f} * \mu)'(xy)}\, dy,$$

and also $(\tilde{f} * \mu)' = \tilde{f} * D\mu$ by an earlier exercise.

Choose any function $k \in C_c^\infty(\mathbb{R})$ with $\int k = 1$.

Then

$$\begin{aligned}
\langle f, \mu \rangle &= \int \langle f, \mu \rangle k(x)\, dx \\
&= \int k(x) \overline{(\tilde{f} * \mu)(x)}\, dx - \int k(x) \int_0^1 x \overline{(\tilde{f} * D\mu)(xy)}\, dy\, dx \\
&= \langle k, \tilde{f} * \mu \rangle - \int \int_0^1 \int k(x)\, x\, \overline{\tilde{f}(t)\, D\mu(xy - t)}\, dt\, dy\, dx
\end{aligned}$$

$$= \langle k * f, \mu \rangle - \int f(-t) \int_0^1 \int x k(x) \overline{D\mu(xy-t)}\, dx\, dy\, dt$$

$$= \langle f, \tilde{k} * \mu \rangle - \int f(t) \int_0^1 \int x k(x) \overline{D\mu(xy+t)}\, dx\, dy\, dt$$

$$= \int f(t) \overline{\left[\tilde{k} * \mu(t) - \int_0^1 x \overline{k(x)} D\mu(xy+t)\, dx\, dy \right]}\, dt$$

Hence, as distributions, μ equals the function

$$g(t) = (\tilde{k} * \mu)(t) - \int_0^1 x \overline{k(x)} D\mu(xy+t)\, dx\, dy.$$

The first term on the right belongs to $C^\infty(\mathbb{R})$, & the second is a continuous function of t, so $\mu = g \in C(\mathbb{R})$ & the proof is complete. ■

Exercise: Use Fubini's Theorem to justify the interchange of integrals in the proof.

<u>Exercise</u>: Show that if $\mu \in \mathcal{D}'(\mathbb{R})$ & $D\mu = 0$, then μ equals a constant function.

Motivation

If $g \in C(\mathbb{R})$ then $g \in L^1_{loc}(\mathbb{R})$, hence determines a distribution, & has distributional derivatives $D^k g$ of every order $k \geq 0$, even if g is not differentiable.

The next result is a partial converse to this statement. Namely, every distribution is "locally" a derivative of some continuous function.

In order to prove this, we recall a result proved in the appendix giving an analogue of "continuity = boundedness" for distributionals. Specifically, it is shown in the Appendix that a linear map

$\mu: C_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}$ is continuous if & only if

$$\forall \text{compact } K \subseteq \mathbb{R},\ \exists N_K > 0,\ \exists C_K > 0 \text{ s.t.}$$

$$f \in C_c^\infty(\mathbb{R}),\ supp(f) \subseteq K \implies |\langle f, \mu \rangle| \leq C_K \|f\|_{N_K},$$

Where

$$\|f\|_N = \max\{\|f\|_\infty, \|f'\|_\infty, \ldots, \|f^{(N)}\|_\infty\}.$$

If a <u>single</u> N will work for every compact set K (with different C_K) then the smallest such N is called the <u>order</u> of μ.

Theorem

Let $T \in \mathcal{D}'(\mathbb{R})$ & a compact $K \subseteq \mathbb{R}$ be given.

Then $\exists\, g \in C(\mathbb{R})$, $\exists\, k \geq 0$ s.t.

$$\forall f \in C_c^\infty(\mathbb{R}) \text{ with } \operatorname{supp}(f) \subseteq K, \quad \langle f, T \rangle = \langle f, D^k g \rangle.$$

Remark: $\langle f, D^k g \rangle = (-1)^k \langle f^{(k)}, g \rangle = (-1)^k \int f^{(k)}(x) \overline{g(x)}\, dx.$

Proof:

By dilating T we may assume $K \subseteq [0,1]$ (exercise).

Then $\exists\, N \geq 0$, $\exists\, C > 0$ s.t. if $f \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}(f) \subseteq K$,

$$|T(f)| \leq C \|f\|_N.$$

For such an f, we have by the Mean Value Theorem that if

$0 \leq x \leq 1$ then $\exists\, c \in (0, x)$ s.t.

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} \qquad (\text{since } \operatorname{supp}(f) \subseteq [0,1])$$

Hence $\quad |f(x)| = |x|\,|f'(c)| \leq |f'(c)| \leq \|f'\|_\infty.$

This is also valid for $x \notin [0,1]$ since $f(x) = 0$ in that case.

Thus

$$\|f\|_\infty \leq \|f'\|_\infty.$$

We can repeat the argument to obtain

$$\|f\|_\infty \leq \|f'\|_\infty \leq \cdots \leq \|f^{(N)}\|_\infty,$$

and hence $\|f\|_N \overset{\text{def}}{=} \max\{\|f\|_\infty, \ldots, \|f^{(N)}\|_\infty\} = \|f^{(N)}\|_\infty$.

Further, by the Fundamental Theorem of Calculus,

$$f^{(N)}(x) = \int_0^x f^{(N+1)}(t)\,dt,$$

so

$$\|f^{(N)}\|_\infty \leq \sup_{0 \leq x \leq 1} \int_0^x |f^{(N+1)}(t)|\,dt = \|f^{(N+1)}\|_1.$$

Define

$$C_c^\infty(K) = \{f \in C_c^\infty(\mathbb{R}) : \operatorname{supp}(f) \subseteq K\}$$

$$L^1_K(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : \operatorname{supp}(f) \subseteq K\}.$$

Then $D^{N+1} : C_c^\infty(K) \longrightarrow C_c^\infty(K)$ is linear & injective (exercise). Let

$$Y = \operatorname{range}(D^{N+1}) \subseteq C_c^\infty(K).$$

Define

$$\begin{aligned} \Lambda : Y &\longrightarrow \mathbb{C} \\ D^{N+1}f &\longmapsto -\langle f, T\rangle. \end{aligned}$$

Then $\quad |\langle D^{N+1} f, \Lambda \rangle|$

$$= |\langle f, T \rangle|$$

$$\leq C \, \|f\|_N$$

$$\leq C \, \|f^{(N+1)}\|_1$$

$$= C \, \|D^{N+1} f\|_1 .$$

Thus, considering Y as a subspace of $L^1_K(\mathbb{R})$ under the L^1-norm, $\Lambda : Y \to \mathbb{C}$ is continuous. The Hahn-Banach Theorem therefore implies that Λ extends to a continuous mapping on all of $L^1_K(\mathbb{R})$. Thus $\Lambda \in L^1_K(\mathbb{R})^* = L^\infty_K(\mathbb{R})$, i.e., $\exists\, \psi \in L^\infty(\mathbb{R})$ with $\operatorname{supp}(\psi) \subseteq K$ s.t.

$$\langle f, \Lambda \rangle = \int f(x)\, \overline{\psi(x)}\, dx, \qquad f \in L^1_K(\mathbb{R}).$$

Define

$$g(x) = \int_0^x \psi(t)\, dt.$$

Then $g \in C(\mathbb{R})$, and for $f \in C_c^\infty(K)$ we have

$$\int D^{N+2} f(x)\, \overline{g(x)}\, dx = - \int D^{N+1} f(x)\, \overline{\Psi(x)}\, dx$$

$$= - \langle D^{N+1} f, \Lambda \rangle$$

$$= \langle f, T \rangle. \blacksquare$$

Thus

$$\langle f, T \rangle = \langle D^{N+2} f, g \rangle = \langle f, (-1)^{N+2} D^{N+2} g \rangle$$

<u>Exercise</u>

Show that if $T \in \mathcal{D}'(\mathbb{R})$ has order N, then

$\exists\, g \in C(\mathbb{R})$ s.t. $T = D^{N+2} g$.

Hint: For each compact K, let g_K be the function constructed in the preceding theorem. Show that $g_K = g_{K'}$ on $K \cap K'$.

<u>Remark</u>: It is shown in ~~the Appendix~~ a previous section that all compactly supported distributions have finite order.

<u>Exercise</u> (See Rudin, F.A., 2nd ed, p.178)

Suppose $f_k \in L^1_{loc}(\mathbb{R})$, & $\forall$ compact $K \subseteq \mathbb{R}$,

$$\|f_k \cdot \chi_K\|_1 \longrightarrow 0 \quad \text{as } k \to \infty.$$

Prove that $D^N f_k \longrightarrow 0$ in $\mathcal{D}'(\mathbb{R})$ as $k \to \infty$

for each $N \geq 0$.

Recall that $AC_{loc}(\mathbb{R})$ was defined in Chapter 1.

Definition
Let I be an interval in $\mathbb{R}$ (bounded or unbounded, including the possibility that $I = \mathbb{R}$). Then f has <u>bounded variation on I</u> if $\exists\, M>0$ s.t.

$\forall\, x_0 < x_1 < \cdots < x_N$ with $x_0, \ldots, x_N \in I$,

$$\sum_{j=1}^{N} |f(x_j) - f(x_{j-1})| \le M.$$

We let $BV(I)$ be the collection of all such f.

We say that $f \in BV_{loc}(\mathbb{R})$ if $f \in BV[a,b]$ for all $a<b$.

Exercise
a. Prove that if $g \in BV(\mathbb{R}) \cap AC_{loc}(\mathbb{R})$, then $Dg = g'$ in the sense of distributions.

b. Prove that

$$L^1(\mathbb{R}) = \{Dg : g \in BV(\mathbb{R}) \cap AC_{loc}(\mathbb{R})\}.$$

Our next result will illustrate the role of the order of a distribution. First we need the following lemma, e.g., see Rudin, Functional Analysis, 2nd ed., Lemma 3.9. (also Lieb & Loss)

<u>Lemma</u>

Let X be a vector space & suppose $\mu_1, \dots, \mu_N : X \to \mathbb{C}$ are linear. Set

$$K = \bigcap_{k=1}^{N} \ker(\mu_k).$$

Then TFAE.

a. $\mu = \sum_{k=1}^{N} c_k \mu_k$ for some $c_k \in \mathbb{C}$

b. $\exists C > 0$ s.t. $\forall x \in X$,

$$|\langle x, \mu \rangle| \leq C \max_{k=1,\dots,N} |\langle x, \mu_k \rangle|$$

c. $x \in K \implies \langle x, \mu \rangle = 0.$

<u>Proof:</u>

Exercises: $a \Rightarrow b$ & $b \Rightarrow c$.

$c \Rightarrow a$. Assume c holds, & define

$$\pi : X \longrightarrow \mathbb{C}^N$$
$$x \longmapsto (\langle x, \mu_1 \rangle, \ldots, \langle x, \mu_N \rangle)^T.$$

Then define

$$T : \pi(X) \longrightarrow \mathbb{C}$$
$$\pi(x) \longmapsto \langle x, \mu \rangle$$

Exercise: Show that the hypothesis of statement c implies that T is well-defined, i.e.,

$$\pi(x) = \pi(x') \implies \langle x, \mu \rangle = \langle x', \mu \rangle.$$

Exercise: Show T is linear.

Exercise: Since $\pi(X)$ is a finite-dimensional subspace of $\mathbb{C}^N$, show $\exists$ linear $\tilde{T} : \mathbb{C}^N \longrightarrow \mathbb{C}$ s.t. $\tilde{T}|_{\pi(X)} = T$.

Now, since $\tilde{T}$ is linear, it is given by a $1 \times N$ matrix, i.e., $\exists\, c_1, \ldots, c_N \in \mathbb{C}$ s.t.

$$\langle u, \tilde{T} \rangle = [c_1 \cdots c_N] \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} = \sum_{k=1}^{N} c_k u_k$$

for $u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \in \mathbb{C}^N$.

Hence for $x \in X$ we have

$$
\begin{aligned}
\langle x, \mu \rangle &= \langle \pi(x), T \rangle \\
&= \langle \pi(x), \tilde{T} \rangle \\
&= \begin{bmatrix} c_1 & \cdots & c_N \end{bmatrix} \begin{bmatrix} \langle x, \mu_1 \rangle \\ \vdots \\ \langle x, \mu_N \rangle \end{bmatrix} \\
&= \sum_{k=1}^{N} c_k \langle x, \mu_k \rangle \\
&= \left\langle x, \sum_{k=1}^{N} \overline{c_k} \mu_k \right\rangle . \qquad \blacksquare
\end{aligned}
$$

We will show that if a distribution is supported on a single point, then it must be a finite linear combination of derivatives of δ's at that point. To prove this we first need the following lemma.

<u>Lemma</u>

If $f \in C_c^\infty(\mathbb{R})$ satisfies $f(0) = f'(0) = \cdots = f^{(N)}(0) = 0$, then $\forall \eta > 0 \ \exists \delta > 0$ s.t.

$$(*) \quad |x| < \delta, \ n = 0, \ldots, N \implies |f^{(n)}(x)| \leq \eta |x|^{N-n}.$$

<u>Proof:</u>

Since $f^{(n)}$ is continuous & vanishes at the origin, $\exists \delta > 0$ s.t.

$$(**) \quad |x| < \delta, \ n = 0, \ldots, N \implies |f^{(n)}(x)| \leq \eta.$$

We now proceed to prove $(*)$ by induction. If $n = N$, then $(*)$ is just $(**)$. So, suppose that $(*)$ holds for some $0 < n \leq N$. Then given $|x| < \delta$, we have by the Mean Value Theorem that $\exists$ c between 0 & x s.t.

$$\left| \frac{f^{(n-1)}(x)}{x} \right| = \left| \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x - 0} \right|$$

$$= |f^{(n)}(c)|$$

$$\leq \eta \, |c|^{N-n}$$

$$\leq \eta \, |x|^{N-n}.$$

Hence

$$|f^{(n-1)}(x)| \leq \eta \, |x|^{N-n+1} = \eta \, |x|^{N-(n-1)},$$

which completes the induction. ∎

Now we can prove that a distribution supported on a set $\{a\}$ is a linear combination of derivatives of a δ supported at a. By translating, it suffices to consider $a = 0$.

Theorem

Given $\mu \in \mathcal{D}'(\mathbb{R})$ TFAE:

a. $\operatorname{supp}(\mu) = \{0\}$.

b. $\exists N \geq 0$, $\exists c_0, \dots, c_N \in \mathbb{C}$ not all zero s.t.

$$\mu = \sum_{n=0}^{N} c_n D^n \delta.$$

Proof:

$b \Rightarrow a$. Exercise.

$a \Rightarrow b$. Suppose $\operatorname{supp}(\mu) = \{0\}$. By a previous theorem, since μ has compact support, it has finite order, i.e., $\exists N \geq 0$, $\exists C > 0$ s.t.

$$\forall f \in C_c^\infty(\mathbb{R}), \quad |\langle f, \mu \rangle| \leq C \|f\|_N = C \max_{n=0,\dots,N} \|f^{(n)}\|_\infty.$$

Let
$$K = \bigcap_{n=0}^{N} \ker(D^n \delta).$$

We will show that

$$f \in K \implies \langle f, \mu \rangle = 0. \quad (*)$$

It then follows from a preceding lemma that

$\mu = \sum_{k=0}^{N} c_n D^n \delta$ for some scalars c_n.

These scalars cannot all be zero, for if they were then we would have $\mu = 0$ and $\operatorname{supp}(\mu) = \emptyset$. Hence the proof is complete once we show that $(*)$ holds.

So, suppose $f \in K$, i.e., $f^{(n)}(0) = 0$ for $n = 0, \ldots, N$. Choose any $\eta > 0$. By the preceding lemma, $\exists\, r > 0$ s.t.

$$|x| < r, \quad n = 0, \ldots, N \implies |f^{(n)}(x)| \le \eta\, |x|^{N-n}.$$

WLOG, assume $r < 1$.

Let $\theta \in C_c^\infty(\mathbb{R})$ be such that $\operatorname{supp}(\theta) \subseteq (-1,1)$ and $\theta = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$.

Set $\theta_r(x) = \theta(x/r)$, and note that

$$\|\theta_r^{(j)}\|_\infty = \frac{1}{r^j}\, \|\theta^{(j)}\|_\infty, \qquad j \ge 0.$$

Then for $|x| < r$ and $n = 0, \ldots, N$ we have $r^{N-n} \le 1$, so

$$\left| (f\bar{\theta}_r)^{(n)}(x) \right| \le \sum_{j=0}^{n} \binom{n}{j} |f^{(j)}(x)| \, |\theta_r^{(n-j)}(x)|$$

$$\le \sum_{j=0}^{n} \binom{n}{j} \eta |x|^{N-j} \frac{1}{r^{n-j}} \|\theta^{(n-j)}\|_\infty$$

$$\le \sum_{j=0}^{n} \binom{n}{j} \eta \, r^{N-j} \frac{1}{r^{n-j}} \|\theta^{(n-j)}\|_\infty$$

$$\le \eta \sum_{j=0}^{n} \binom{n}{j} \|\theta^{(n-j)}\|_\infty$$

$$= C_n' \eta,$$

where $C_n' = \sum_{j=0}^{n} \binom{n}{j} \|\theta^{(n-j)}\|_\infty$ is a constant independent of η. Since $\operatorname{supp}(f\theta_r) \subseteq (-r, r)$, this implies that

$$\|(f\bar{\theta}_r)^{(n)}\|_\infty \le C_n' \eta, \qquad n = 0, \dots, N.$$

Now, since $\theta_r = 1$ on a neighborhood of $\operatorname{supp}(\mu)$, we have $\theta_r \mu = \mu$ by an earlier exercise. Consequently,

$$\begin{aligned} |\langle f, \mu\rangle| &= |\langle f, \theta_r \mu\rangle| \\ &= |\langle f\bar{\theta}_r, \mu\rangle| \\ &\leq C \, \| f\bar{\theta}_r \|_N \\ &\leq C \sum_{n=0}^{N} \| (f\bar{\theta}_r)^{(n)} \|_\infty \\ &\leq \eta \, C \sum_{n=0}^{N} C_n' . \end{aligned}$$

Since η is arbitrary, this implies $\langle f, \mu\rangle = 0$, so (*) holds and the proof is complete. ∎

Exercise

Suppose that $\mu \in \mathcal{D}'(\mathbb{R})$. Show that

$$f = 0 \text{ on } \operatorname{supp}(\mu) \not\Rightarrow \langle f, \mu\rangle = 0.$$

The following exercises will provide conditions on f that do imply that $\langle f, \mu\rangle = 0$.

Also, by an earlier theorem, $\exists\, C>0$ s.t. $\forall \varphi \in C_c^\infty(\mathbb{R})$, $|\langle \varphi, \mu\rangle| \le C \sum_{n=0}^{N} \|\varphi^{(n)}\|_\infty$

<u>Exercise</u> (See Rudin, F.A., 2nd ed., p.179).

Let $\mu \in \mathcal{E}'(\mathbb{R}) = C^\infty(\mathbb{R})^*$ and $f \in C^\infty(\mathbb{R})$ be given. Let N be the order of μ. Show that if $f^{(n)} = 0$ on $\mathrm{supp}(\mu)$ for $n = 0, \dots, N$, then $\langle f, \mu \rangle = 0$.

Hint: Follow the same techniques used in the preceding theorem & lemma. Also, $\mathrm{supp}(\mu)$ is compact – every open cover has a finite subcover.

<u>Exercise</u> (See Rudin, p.179)

Let $\mu \in \mathcal{D}'(\mathbb{R})$ & $f \in C_c^\infty(\mathbb{R})$ be given. Show that if $f^{(n)} = 0$ on $\mathrm{supp}(\mu)$ $\forall n \ge 0$, then $\langle f, \mu \rangle = 0$.

Hint: Choose a smooth cutoff function θ.

We can use the preceding exercises to give another proof that a distribution supported on a single point is a finite linear combination of δ distributions at that point.

<u>Theorem</u>

Given $\mu \in \mathcal{D}'(\mathbb{R})$, TFAE:

a. $\operatorname{supp}(\mu) = \{0\}$,

b. $\exists N \geq 0$, $\exists c_0, \ldots, c_N \in \mathbb{C}$ not all zero s.t.

$$\mu = \sum_{n=0}^{N} c_n D^n \delta.$$

<u>Proof.</u>

$b \Rightarrow a$. Exercise.

$a \Rightarrow b$. Suppose $\operatorname{supp}(\mu) = \{0\}$.

By a previous theorem, since μ has compact support it has finite order, say order N. Since $\mu \in \mathcal{E}'(\mathbb{R}) = C^\infty(\mathbb{R})^*$, $\langle X^n, \mu \rangle$ exists for all $n \geq 0$. Fix any $f \in C_c^\infty(\mathbb{R})$. Since f is C^∞, we can write it in a Taylor expansion as

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n + r(x)$$

where $r \in C^\infty(\mathbb{R})$. Further, for $n = 0, \dots, N$ we have

$$r^{(n)}(0) = f^{(n)}(0) - \frac{d^n}{dx^n}\left(\sum_{k=0}^{N} \frac{f^{(k)}(0)}{n!} x^k\right)(0)$$

$$= f^{(n)}(0) - f^{(n)}(0)$$

$$= 0.$$

Hence by the preceding exercise, ~~[illegible]~~

$\langle r, \mu \rangle = 0$, so

$$\langle f, \mu \rangle = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} \langle x^n, \mu \rangle$$

$$= \sum_{n=0}^{N} \frac{(-1)^n \langle x^n, \mu \rangle}{n!} \langle f, D^n \delta \rangle$$

$$= \left\langle f, \sum_{n=0}^{N} \frac{(-1)^n \overline{\langle x^n, \mu \rangle}}{n!} D^n \delta \right\rangle.$$

Thus

$$\mu = \sum_{n=0}^{N} \frac{(-1)^n \overline{\langle x^n, \mu \rangle}}{n!} D^n \delta. \blacksquare$$