3.2 Distributions

Distributions, or generalized functions, are continuous linear functionals on the following spaces:

$$C_c^\infty(\mathbb{R}) = \left\{ f:\mathbb{R}\to\mathbb{C} \ : \ f \text{ is infinitely differentiable \& compactly supported} \right\}$$

$$\mathcal{S}(\mathbb{R}) = \left\{ f:\mathbb{R}\to\mathbb{C} \ : \ f \text{ is infinitely differentiable \&} \ x^m f^{(n)}(x) \in L^\infty(\mathbb{R}) \ \forall m,n \geq 0 \right\}$$

$$C^\infty(\mathbb{R}) = \left\{ f:\mathbb{R}\to\mathbb{C} \ : \ f \text{ is infinitely differentiable} \right\}.$$

<u>Note:</u> We are not requiring functions in $C^\infty(\mathbb{R})$ to be bounded (or their derivatives), unlike the space $C_b^\infty(\mathbb{R})$.

These spaces are topological vector spaces, whose topologies are determined by a family of seminorms.* Their dual spaces are spaces of distributions.

* See the Appendix for more details on topological vector spaces.

Since

$$\begin{array}{ll} C_c(\mathbb{R}) & \text{smallest domain - imposes least restriction} \\ & \quad \text{on continuity of a linear functional} \\ \cap & \\ \mathcal{S}(\mathbb{R}) & \\ \cap & \\ C^\infty(\mathbb{R}) & \text{largest domain — imposes greatest restrictions} \\ & \quad \text{on continuity} \end{array}$$

if there is some connection between the topologies on these spaces then we can hope that

$$\begin{array}{ll} C_c(\mathbb{R})^* \overset{\text{def}}{=} & \text{space of distributions } \mathcal{D}'(\mathbb{R}) \\ \cup & \\ \mathcal{S}(\mathbb{R})^* \overset{\text{def}}{=} & \text{space of tempered distributions } \mathcal{S}'(\mathbb{R}) \\ \cup & \\ C^\infty(\mathbb{R})^* \overset{\text{def}}{=} & \text{space of compactly supported} \\ & \quad \text{distributions } \mathcal{E}'(\mathbb{R}) \end{array}$$

We will see that these inclusions are correct.

Unfortunately, these spaces are not Banach spaces, so we must first review the definition of their topologies, or, equivalently, the definition of convergence in these spaces. (We did this for $\mathcal{S}(\mathbb{R})$ in an earlier section).

Convergence in $C_c^\infty(\mathbb{R})$ is defined as follows.

<u>Definition</u>
Given $f_k, g \in C_c^\infty(\mathbb{R})$, we say that $f_k \to g$ in $C_c^\infty(\mathbb{R})$ if

a. $\exists$ compact $K \subseteq \mathbb{R}$ s.t. $\operatorname{supp}(f_n) \subseteq K$, $\forall n$, &

b. $\forall n \geq 0$, $\lim\limits_{k\to\infty} \| g^{(n)} - f_k^{(n)} \|_\infty = 0$.

<u>Remark</u>
A basic philosophy is that "topology is equivalent to a convergence criterion". Above we give a convergence criterion, & there is an equivalent formulation in terms of a topology. Unfortunately, this equivalence is more complicated for $C_c^\infty(\mathbb{R})$ than it is for $\mathcal{S}(\mathbb{R})$ or $C^\infty(\mathbb{R})$. The topology on $\mathcal{S}(\mathbb{R})$ is induced by the countable family of seminorms

$$\rho_{mn}(f) = \| x^m f^{(n)}(x) \|_\infty, \quad m, n \geq 0.$$

The corresponding convergence criterion is

$$f_k \to g \text{ in } \mathcal{S}(\mathbb{R}) \text{ if } \forall m,n \geq 0, \ \rho_{mn}(f_k - g) \to 0.$$

Because there are countably many seminorms for $\mathcal{S}(\mathbb{R})$, & because the topology is Hausdorff, ie.,

$$\rho_{mn}(f) = 0 \ \forall m,n \geq 0 \implies f = 0,$$

the Schwartz space $\mathcal{S}(\mathbb{R})$ is metrizable, leading to an easy connection between topology & convergence.

Likewise $C^\infty(\mathbb{R})$ is determined by a family of seminorms, namely

$$\rho_{K,n}(f) = \| f^{(n)} \cdot \chi_K \|_\infty, \quad n \geq 0, \ K \text{ compact}.$$

WLOG we can reduce this to the countable family

$$\rho_{mn}(f) = \| f^{(n)} \cdot \chi_{[-m,m]} \|_\infty, \quad m,n \geq 0,$$

and $C^\infty(\mathbb{R})$ is metrizable as well.

The situation for $C_c^\infty(\mathbb{R})$ is more complicated. In technical terms, $C_c^\infty(\mathbb{R})$ is the <u>inductive limit</u> of the spaces

$$C_c^\infty([-m,m]) = \{ f \in C_c^\infty(\mathbb{R}) : \operatorname{supp}(f) \subseteq [-m,m] \}.$$

Each individual space $C_c^\infty([-m,m])$ is determined by the countable family of seminorms

$$\rho_{mn}(f) = \| f^{(n)} \cdot \chi_{[-m,m]} \|_\infty, \qquad n \geq 0,$$

and is metrizable. The inductive limit of these topologies is roughly determined by the requirement that a function on $C_c^\infty(\mathbb{R})$ be continuous if & only if each restriction to $C_c^\infty([-m,m])$ be continuous. This leads to the definition of convergence given above.

These issues, & in particular the equivalence between topology & convergence, is presented in detail in the Appendix. Here we will simply take the convergence criterion given above as a definition.

The remainder of this section will focus on $C_c^\infty(\mathbb{R})$ & its dual space $\mathcal{D}'(\mathbb{R})$, & later sections will consider $\mathcal{S}(\mathbb{R})$, $C^\infty(\mathbb{R})$, and their duals.

~~We will concentrate in this section on the space of distributions $\mathcal{D}'(\mathbb{R})$. See the appendix for expanded discussion of these definitions.~~

~~Definition~~

~~Given $f_n, g \in C_c^\infty(\mathbb{R})$, we say that $f_n \to g$ in $C_c^\infty(\mathbb{R})$ if~~

~~a. $\exists$ compact $K \subseteq \mathbb{R}$ s.t. $\mathrm{supp}(f_n) \subseteq K$ $\forall n$,~~

~~b. $\forall k \geq 0$, $\lim\limits_{n\to\infty} \| g^{(k)} - f_n^{(k)} \|_\infty = 0$.~~

Definition

a. A linear functional $T: C_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}$ is <u>continuous</u> if

$$f_n \longrightarrow 0 \text{ in } C_c^\infty(\mathbb{R}) \implies \langle f_n, T\rangle \longrightarrow 0$$

b. The space of distributions on $\mathbb{R}$ is

$$\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^* = \left\{ T: C_c^\infty(\mathbb{R}) \to \mathbb{C} \; : \; T \text{ is a continuous linear functional} \right\}$$

c. A distribution $T \in \mathcal{D}'(\mathbb{R})$ is <u>positive</u>, denoted $T \geq 0$, if

$$\forall f \in C_c^\infty(\mathbb{R}), \quad f \geq 0 \implies \langle f, T\rangle \geq 0$$

Exercise:

$\mathcal{D}'(\mathbb{R})$ is a vector space.

Exercise

Let $k \in C_c^\infty(\mathbb{R})$ be s.t. $\int k = 1$, and set $k_\lambda(x) = \lambda k(\lambda x)$.

Prove that

$$\forall f \in C_c^\infty(\mathbb{R}), \quad f * k_\lambda \longrightarrow f \text{ in } C_c^\infty(\mathbb{R}), \text{ as } \lambda \to \infty.$$

Conclude that if $\mu \in \mathcal{D}'(\mathbb{R})$, then

$$\langle f * k_\lambda, \mu \rangle \longrightarrow \langle f, \mu \rangle \quad \text{as } \lambda \to \infty$$

Exercise

Define $\delta : C_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}$ by

$$\langle f, \delta \rangle = f(0), \quad f \in C_c^\infty(\mathbb{R}).$$

Prove that $\delta \in \mathcal{D}'(\mathbb{R})$, & δ is a positive distribution ($\delta \geq 0$)

Exercise

Prove that differentiation is a continuous operation on $C_c^\infty(\mathbb{R})$, i.e.,

$$f_k \longrightarrow f \text{ in } C_c^\infty(\mathbb{R}) \implies f_k' \longrightarrow f' \text{ in } C_c^\infty(\mathbb{R}).$$

Notation: We let $\psi T = T\psi$

<u>Exercise</u>: Products of distributions & smooth functions.

Given $T \in \mathcal{D}'(\mathbb{R})$ & $\psi \in C^\infty(\mathbb{R})$, define

$T\psi : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ by

$$\langle f, T\psi \rangle = \langle f\overline{\psi}, T \rangle, \quad f \in C_c^\infty(\mathbb{R}).$$

Show that $T\psi \in \mathcal{D}'(\mathbb{R})$.

<u>Remark</u>: Unfortunately, we cannot define the product of two arbitrary distributions - some restrictions are required.

However, this process of "imitating" functional definitions for distributions is often valid & leads to extensions of ideas to the setting of distributions.

<u>Exercise</u>

Define the translation $T_a\mu$, modution $M_b\mu$, and dilation $D_r\mu$ of a distribution μ, & prove they are distributions.

Explain why $\langle f, \overline{\mu} \rangle = \overline{\langle \overline{f}, \mu \rangle}$ is an appropriate definition of the complex conjugate of a distribution, & prove $\overline{\mu} \in \mathcal{D}'(\mathbb{R})$.

Motivation

If $g \in L^1_{loc}(\mathbb{R})$ & $f \in C_c^\infty(\mathbb{R})$, then $\left(\tilde{f}(x) = \overline{f(-x)} \right)$

$$(f*g)(x) = \int f(x-y)\, g(y)\, dy$$

$$= \overline{\int \tilde{f}(y-x)\, \overline{g(y)}\, dy}$$

$$= \overline{\langle T_x \tilde{f}, g \rangle}.$$

We use this formula to extend to convolutions with distributions.

Definition: Convolution

Given $\mu \in \mathcal{D}'(\mathbb{R})$ & $f \in C_c^\infty(\mathbb{R})$, we define

$f*\mu$ to be the function given by

$$(f*\mu)(x) = \overline{\langle T_x \tilde{f}, \mu \rangle}$$

Example

$$(f*\delta)(x) = \overline{\langle T_x \tilde{f}, \delta \rangle}$$

$$= \overline{T_x \tilde{f}(0)}$$

$$= f(x).$$

Thus $f*\delta = f$, i.e., δ is an identity for convolution. (at least on $C_c^\infty(\mathbb{R})$).

*Exercise**

Show that $(f * \delta')(x) = -f'(x)$.

Exercise

Show that convolution commutes with translation, i.e., if $f \in C_c^\infty(\mathbb{R})$, $\mu \in \mathcal{D}'(\mathbb{R})$, & $a \in \mathbb{R}$, then

$$T_a(f*\mu) = T_a f * \mu = f * T_a \mu.$$

* Note: δ' is the distribution whose rule is

$$\langle f, \delta' \rangle = -f'(0), \qquad f \in C_c^\infty(\mathbb{R}).$$

Exercise

Let $\mu \in \mathcal{D}'(\mathbb{R})$ & $f \in C_c^\infty(\mathbb{R})$ be given.

a. Show $f * \mu \in C(\mathbb{R})$.

Hint: Show $T_a f \to f$ in $C_c^\infty(\mathbb{R})$ as $a \to 0$.

b. Show $f * \mu \in C^1(\mathbb{R})$, & $(f*\mu)' = f' * \mu$.

Hint: Show $\frac{T_a f - f}{a} \to f'$ in $C_c^\infty(\mathbb{R})$ as $a \to 0$.

c. Show $f * \mu \in C^\infty(\mathbb{R})$.

d. Prove that $\forall x \in \mathbb{R}$,

$$\langle f, \mu \rangle = \overline{(\tilde{f} * \mu)(x)} - \int_0^1 x \, \overline{(\tilde{f} * \mu)'(xy)} \, dy$$

Hint: Simply apply the Fundamental Theorem of Calculus.

Exercise

Let $\delta_t = T_t \delta$, so $\langle f, \delta_t \rangle = f(t)$ for $f \in C_c^\infty(\mathbb{R})$.

Find a formula for $f * \delta_t$.

The following result is the analogue of "continuity = boundedness" for linear functions on a normed space. An expanded version of this result is given in the Appendix. We use the notation

$$\|f\|_N = \max_{n=0,\dots,N} \|f^{(n)}\|_\infty .$$

<u>Theorem</u>

If $\mu : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ is a linear functional, the TFAE.

a. μ is continuous, i.e., $\mu \in \mathcal{D}'(\mathbb{R})$.

b. $\forall$ compact $K \subseteq \mathbb{R}$, $\exists\, N_K \geq 0$, $\exists\, C_K > 0$ s.t.

$$f \in C^\infty(K) \implies |\langle f, \mu\rangle| \leq C_K \|f\|_{N_K}.$$

<u>Proof:</u>

$a \Rightarrow b$. Suppose $\mu \in \mathcal{D}'(\mathbb{R})$, & fix any compact $K \subseteq \mathbb{R}$. Suppose that $\nexists$ any C, N s.t.

$$f \in C^\infty(K) \implies |\langle f, \mu\rangle| \leq C \|f\|_N .$$

Then for each $C = N = k \in \mathbb{N}$ we can find $f_k \in C^\infty(K)$ s.t.

$$|\langle f_k, \mu\rangle| > k \|f_k\|_k .$$

In particular $|\langle f_k, \mu\rangle| > 0$, so we can define

$$\varphi_k = \frac{f_k}{\langle f_k, \mu \rangle}.$$

Then we have

$$1 = \langle \varphi_k, \mu \rangle > k \, \|\varphi_k\|_k ,$$

so $\|\varphi_k\|_k < \frac{1}{k}$. *Hence* $\varphi_k \to 0$ in $C_c^\infty(\mathbb{R})$, but this contradicts the fact that μ is continuous, which implies $\langle \varphi_k, \mu \rangle \longrightarrow 0$. $\blacksquare$

<u>Definition</u>

If a <u>single</u> N will work for all compact K (with possibly different C_K), then we say that μ has <u>finite order</u>, & the smallest such N is called the <u>order</u> of T.

<u>Exercise</u>

Show that δ has finite order.

<u>Exercise</u>

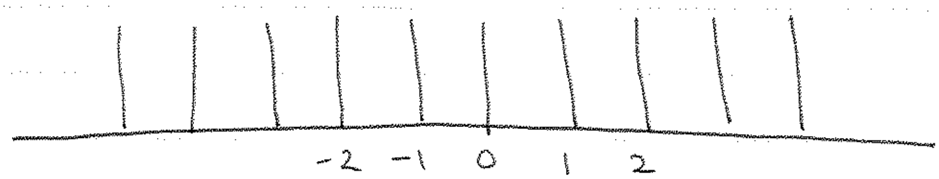
Show that $\langle f, \mu \rangle = \sum_{n=0}^{\infty} f^{(n)}(n)$ defines a distribution, & that μ has infinite order.

Exercise

Let $\mu = \sum_{n \in \mathbb{Z}}' \delta_n$, i.e.,

$$\forall f \in C_c^\infty(\mathbb{R}), \qquad \langle f, \mu \rangle = \sum_{n \in \mathbb{Z}}' f(n).$$

Pictorially, we may (with some poetic license) imagine that μ "looks" like this:



This picture inspires many names for μ, including:

- the delta train or train of deltas (or Diracs)
- the Dirac comb
- the Shah distribution, because the picture is suggestive of the Cyrillic letter "sha", which is written Ш. For this reason, μ is sometimes denoted by the symbol Ш.

a. Prove that $\mu \in \mathcal{D}'(\mathbb{R})$.

b. Prove that μ has order 0, but the constants C_K cannot be chosen independent of the compact set K.

Definition: Convergence of Distributions

If $\mu_n, \mu \in \mathcal{D}'(\mathbb{R})$, then we say that $\mu_n \to \mu$ in $\mathcal{D}'(\mathbb{R})$ if

$$\forall f \in C_c^\infty(\mathbb{R}), \quad \langle f, \mu_n \rangle \longrightarrow \langle f, \mu \rangle .$$

That is, convergence of distributions is convergence in the weak* topology.

Remark: This is the "natural" to put on a dual space in many ~~space~~ respects – the elements of the dual space are functionals on $C_c^\infty(\mathbb{R})$, & weak* convergence is simply "pointwise convergence" of these functions (with the "points" being the elements of $C_c^\infty(\mathbb{R})$).

Exercise

Let $\theta \in C_c^\infty(\mathbb{R})$ be s.t. $\theta = 1$ on a neighborhood of 0. Define $\theta_\lambda(x) = \theta(x/\lambda)$. Prove that if $\mu \in \mathcal{D}'(\mathbb{R})$ then $\theta_\lambda \mu \longrightarrow \mu$ in $\mathcal{D}'(\mathbb{R})$.

Example

A function $f: \mathbb{R} \to \mathbb{C}$ is refinable if it satisfies a refinement equation of the form

$$f(x) = \sum_{k \in \mathbb{Z}}' c_k f(2x-k)$$

for some scalars c_k (the dilation factor 2 is chosen for convenience only, other dilations may be used). See the figures for some examples of refinable functions.

A refinable function is a fixed point of the operator

$$Sf(x) = \sum_{k \in \mathbb{Z}}' c_k f(2x-k).$$

Often, the iteration $f_{n+1}(x) = S f_n(x)$ will converge to the solution f (if one exists). This exercise illustrates where things go wrong - this Cascade Algorithm does not converge in a functional sense. On the other hand, it does converge distributionally.

a. Show that $f = \chi_{[0,3]}$ is a solution to the refinement equation

$$f(x) = f(2x) + f(2x-3).$$

b. Let $f_0 = \chi_{[0,1]}$. Compute f_1, f_2, f_3 using the Cascade algorithm.

c. Show that

$$f_n = \sum_{k=0}^{2^n-1} \chi_{\left[\frac{3k}{2^n}, \frac{3k+1}{2^n}\right]}$$

d. Prove that $f_n \longrightarrow \frac{1}{3}\chi_{[0,3]}$ in $\mathcal{D}'(\mathbb{R})$, i.e.,

$$\forall h \in C_c^\infty(\mathbb{R}), \quad \langle h, f_n \rangle \longrightarrow \frac{1}{3}\int_0^3 h(x)\,dx.$$

Motivation

To motivate our next result, suppose that $f, g \in C_c^\infty(\mathbb{R})$, and that μ is a function, say $\mu \in L^1(\mathbb{R})$. Then

$$\begin{aligned}
\langle f, g*\mu \rangle &= \int f(x)\, \overline{(g*\mu)(x)}\, dx \\
&= \int f(x) \int \overline{g(x-y)}\, \overline{\mu(y)}\, dy\, dx \\
&= \int \left(\int f(x)\, \tilde{g}(y-x)\, dx \right) \overline{\mu(y)}\, dy \\
&= \int (f*\tilde{g})(y)\, \overline{\mu(y)}\, dy \\
&= \langle f*\tilde{g}, \mu \rangle,
\end{aligned}$$

where $\tilde{g}(x) = \overline{g(-x)}$. The interchange of integrals is justified by Fubini's Theorem.

Our next result shows that this equality extends to arbitrary distributions $\mu \in \mathcal{D}'(\mathbb{R})$.

Thus, this can be viewed as an equivalent definition of the convolution of $g \in C_c^\infty(\mathbb{R})$ with $\mu \in \mathcal{D}'(\mathbb{R})$

Theorem

If $\mu \in \mathcal{D}'(\mathbb{R})$, then

$$\forall f, g \in C_c^\infty(\mathbb{R}), \quad \langle f, g * \mu \rangle = \langle f * \tilde{g}, \mu \rangle.$$

Remark: By an earlier exercise, $g * \mu \in C^\infty(\mathbb{R})$, so the inner product $\langle f, g*\mu\rangle$ is well-defined since $f \in C_c^\infty(\mathbb{R})$. Likewise, $f * \tilde{g} \in C_c^\infty(\mathbb{R})$, so the evaluation $\langle f*\tilde{g}, \mu\rangle$ of μ at $f*\tilde{g}$ is defined.

Proof:

Let T be s.t. $\operatorname{supp}(f) \subseteq [-T, T]$. Let

$x_0 = -T < x_1 < \cdots < x_N = T$ be a regular partition

of $[-T,T]$, & set $\Delta x = \frac{2T}{N}$. Then, since $f \in C_c^\infty(\mathbb{R})$

& $g * \mu \in C^\infty(\mathbb{R})$, we can write $\langle f, g*\mu\rangle$ as a limit

of Riemann sums:

$$\begin{aligned}
\langle f, g*\mu\rangle &= \int_{-T}^{T} f(x) \langle T_x \tilde{g}, \mu\rangle \, dx \\
&= \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k) \langle T_{x_k}\tilde{g}, \mu\rangle \, \Delta x \\
&= \lim_{N\to\infty} \left\langle \sum_{k=1}^{N} f(x_k) T_{x_k}\tilde{g} \, \Delta x, \mu \right\rangle \\
&= \lim_{N\to\infty} \langle h_N, \mu\rangle
\end{aligned}$$

We claim that $h_N = \sum_{k=1}^{N} f(x_k) T_{x_k} \tilde{g} \, \Delta x \longrightarrow f * \tilde{g}$ in $C_c^\infty(\mathbb{R})$ as $N \to \infty$. First, since $x_k \in [-T, T]$, we have

$$\mathrm{supp}(h_N) \subseteq \mathrm{supp}(\tilde{g}) + [-T, T] = K,$$

which is a compact set independent of N.

Second, we can also write $f * \tilde{g}$ in terms of Riemann sums:

$$\begin{aligned} f * \tilde{g}(t) &= \int_{-T}^{T} f(x) \, \tilde{g}(t-x) \, dt \\ &= \int_{-T}^{T} f(x) \, T_x \tilde{g}(t) \, dt \\ &= \lim_{N \to \infty} \sum_{k=1}^{N} f(x_k) \, T_{x_k} \tilde{g}(t) \, \Delta x. \end{aligned}$$

In fact (exercise), because $f, g \in C_c^\infty(\mathbb{R})$ the convergence is uniform, i.e.,

$$\| f*\tilde{g} - h_N \|_\infty = \| f * \tilde{g} - \sum_{k=1}^{N} f(x_k) T_{x_k} \tilde{g} \cdot \Delta x \|_\infty \longrightarrow 0 \text{ as } N \to \infty.$$

And since differentiation commutes with convolution, by the same argument we have for any $m \geq 0$ that

$$\|(f * \tilde{g})^{(m)} - h_N^{(m)}\|_\infty$$

$$= \| f * \tilde{g}^{(m)} - \sum_{k=1}^{N} f(x_k)\, T_{x_k}\tilde{g}^{(m)} \cdot \Delta x \|_\infty$$

$$\longrightarrow 0 \quad \text{as } N \to \infty$$

Thus $h_N \longrightarrow f * \tilde{g}$ in $C_c^\infty(\mathbb{R})$. But μ is continuous, so this implies that

$$\langle f, g * \mu \rangle = \lim_{N\to\infty} \langle h_N, \mu \rangle = \langle f * \tilde{g}, \mu \rangle. \quad \blacksquare$$

We can use this theorem to obtain some interesting results. First, despite the fact that $C_c^\infty(\mathbb{R})$ appears to be a very small part of $\mathcal{D}'(\mathbb{R})$, we show that it is in fact dense in $\mathcal{D}'(\mathbb{R})$ in the topology of $\mathcal{D}'(\mathbb{R})$. That is, every distribution is a weak* limit of functions in $C_c^\infty(\mathbb{R})$.

Theorem

$C_c^\infty(\mathbb{R})$ is dense in $\mathcal{D}'(\mathbb{R})$ (in the $\mathcal{D}'(\mathbb{R})$ topology).

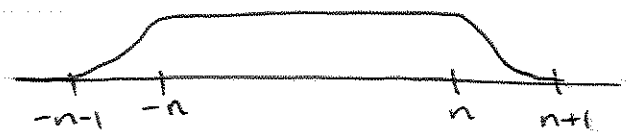
That is,

$$\forall \mu \in \mathcal{D}'(\mathbb{R}), \quad \exists \text{ net } \{f_i\}_{i\in I} \subseteq C_c^\infty(\mathbb{R}) \text{ s.t.}$$

$$\forall \varphi \in C_c^\infty(\mathbb{R}), \quad \langle \varphi, f_i \rangle \longrightarrow \langle \varphi, \mu \rangle.$$

Proof:

Let $\theta_n \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function:



i.e., $\operatorname{supp}(\theta_n) \subseteq [-n-1, n+1]$, $\quad 0 \le \theta_n \le 1$, &

$\theta_n = 1$ on $[-n,n]$. Then given $\mu \in \mathcal{D}'(\mathbb{R})$,

we have for any $\varphi \in C_c^\infty(\mathbb{R})$ that $\forall$ large enough n,

$$\langle \varphi, \mu\theta_n \rangle = \langle \varphi\bar{\theta}_n, \mu \rangle = \langle \varphi, \mu \rangle.$$

Specifically this holds $\forall n$ with $\operatorname{supp}(\varphi) \subseteq [-n,n]$.

In other words, $\mu\theta_n \xrightarrow{w^*} \mu$.

Now choose $k \in C_c^\infty(\mathbb{R})$ with $\int k = 1$, & set $k_\lambda(x) = \lambda k(\lambda x)$.

Let $I = \mathbb{N} \times (0,\infty)$, under the ordering

$$(n_1, \lambda_1) \leq (n_2, \lambda_2) \iff n_1 \leq n_2 \ \& \ \lambda_1 \leq \lambda_2.$$

Then $\{k_\lambda * \mu\theta_n\}_{(n,\lambda)\in I}$ is a net in $C_c^\infty(\mathbb{R})$.

Choose any $\varphi \in C_c^\infty(\mathbb{R})$.

Let n_0 be s.t. $\mathrm{supp}(\varphi) \subseteq [-n_0, n_0]$. Since $\{\tilde{k}_\lambda\}_{\lambda>0}$ is an approximate identity (as is $\{k_\lambda\}_{\lambda>0}$), $\varphi * \tilde{k}_\lambda \longrightarrow \varphi$ in $C_c^\infty(\mathbb{R})$. Choose $\varepsilon > 0$. As μ is continuous, this implies $\langle \varphi * \tilde{k}_\lambda, \mu\rangle \longrightarrow \langle \varphi, \mu\rangle$, so $\exists\ \lambda_0 > 1$ s.t.

$$\lambda > \lambda_0 \implies |\langle \varphi, \mu\rangle - \langle \varphi * \tilde{k}_\lambda, \mu\rangle| < \varepsilon.$$

Note that if $\lambda > \lambda_0 > 1$, then $\mathrm{supp}(\varphi * \tilde{k}_\lambda) \subseteq [-n_0 - 1, n_0 + 1]$.

Hence if $(n, \lambda) > (n_0 + 1, \lambda_0)$, then

$$|\langle \varphi, \mu\rangle - \langle \varphi, k_\lambda * \mu\theta_n\rangle|$$

$$= |\langle \varphi, \mu\rangle - \langle \varphi * \tilde{k}_\lambda, \mu\theta_n\rangle| \quad \text{(preceding theorem)}$$

$$= |\langle \varphi, \mu \rangle - \langle (\varphi * \tilde{k}_\lambda)\bar{\theta}_n, \mu \rangle|$$

$$= |\langle \varphi, \mu \rangle - \langle \varphi * \tilde{k}_\lambda, \mu \rangle|$$

$$< \varepsilon.$$

Thus $k_\lambda * \mu\theta_n \xrightarrow{\omega^*} \mu$. ∎

<u>Remark</u>

Weak* convergence is a "very weak" type of convergence, & thus the density of $C_c^\infty(\mathbb{R})$ in $\mathcal{D}'(\mathbb{R})$ is only a very weak statement.

However, our next result is very practical: it shows that classical & distributional differentiation of functions coincide for all continuously differentiable functions.

An earlier exercise shows that convolution commutes with translations. We now expand on that result.

<u>Theorem</u> (See Rudin, F.A., 2nd ed., p.173)

a. If $\mu \in \mathcal{D}'(\mathbb{R})$, then

$$L : C_c^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$$
$$f \longmapsto f * \mu$$

is a continuous linear map of $C_c^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R})$ that satisfies $LT_a = T_a L$ for all $a \in \mathbb{R}$.

b. If $L : C_c^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ is a continuous linear map s.t. $LT_a = T_a L$ $\forall a \in \mathbb{R}$, then $\exists$ unique $\mu \in \mathcal{D}'(\mathbb{R})$ s.t. $Lf = f * \mu$.

<u>Proof:</u>
a. We have seen before that L is well-defined and linear & we have

$$T_a(Lf) = T_a(f * \mu) = T_a f * \mu = L(T_a f).$$

So, it remains only to show that L is continuous.

Suppose that $f_k \longrightarrow 0$ in $C_c^\infty(\mathbb{R})$, i.e., $\exists$ compact $K \subseteq \mathbb{R}$ s.t. $\operatorname{supp}(f_k) \subseteq K$ $\forall k$, and

$\forall n \geq 0, \quad \| f_k^{(n)} \cdot \quad \|_\infty \longrightarrow 0.$

We must show that $f_k * \mu \longrightarrow 0$ in $C^\infty(\mathbb{R})$, i.e.,

for any compact $Q \subseteq \mathbb{R}$ we must show that

$$\forall n \geq 0, \quad \| (f_k * \mu)^{(n)} \chi_Q \|_\infty \longrightarrow 0.$$

Now, $(f_k * \mu)(x) = \overline{\langle T_x \tilde{f}_k, \mu \rangle}$, and

$\mathrm{supp}(T_x \tilde{f}) \subseteq x - K$. The set

$$Q - K = \{ x - y : x \in Q, y \in K \}$$

is compact, so since μ is continuous, by an earlier theorem $\exists N \geq 0$, $\exists C > 0$ s.t.

$$f \in C^\infty(Q-K) \Rightarrow |\langle f, \mu \rangle| \leq C \sum_{n=0}^{N} \| f^{(n)} \quad \|_\infty.$$

Hence if $x \in Q$ then since $\mathrm{supp}(T_x \tilde{f}) \subseteq x - K \subseteq Q - K$,

$$\begin{aligned} |(f_k * \mu)^{(m)}(x)| &= |(f_k^{(m)} * \mu)(x)| \\ &= | \langle T_x \widetilde{f_k^{(m)}}, \mu \rangle | \end{aligned}$$

$$\leq C \sum_{n=0}^{N} \| (T_x \tilde{f}_k^{(m)})^{(n)} \|_\infty$$

$$= C \sum_{n=0}^{N} \| T_x \tilde{f}_k^{(m+n)} \|_\infty$$

$$= C \sum_{n=0}^{N} \| f_k^{(m+n)} \|_\infty$$

Therefore

$$\sup_{x \in Q} |(f_k * \mu)^{(m)}(x)| \leq C \sum_{n=0}^{N} \| f_k^{(m+n)} \|_\infty$$

$$\longrightarrow 0 \quad \text{as } k \to \infty.$$

Thus L is continuous.

b. Suppose $L : C_c^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is linear & continuous & commutes with translations. Define $\mu : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ by

$$\langle f, \mu \rangle = \overline{L\tilde{f}(0)}, \qquad f \in C_c^\infty(\mathbb{R}).$$

Then μ is a linear functional on $C_c^\infty(\mathbb{R})$. If $f_k \to 0$ in $C_c^\infty(\mathbb{R})$ then $Lf_k \to 0$ in $C^\infty(\mathbb{R})$ since L is continuous. In particular, the convergence is uniform so

$$\langle f_k, \mu \rangle = L\tilde{f}_k(0) \longrightarrow 0.$$

Thus $\mu \in \mathcal{D}'(\mathbb{R})$. Further,

$$
\begin{aligned}
(f * \mu)(x) &= \overline{\langle T_x \tilde{f}, \mu \rangle} \\
&= L\left((T_x \tilde{f})^{\sim} \right)(0) \\
&= L(T_{-x} f)(0) \\
&= T_{-x}(Lf)(0) \\
&= Lf(x).
\end{aligned}
$$

Exercise: Show μ is unique. $\blacksquare$

<u>Corollary</u> (See Rudin, F.A., 2nd ed, p. 180)

Suppose $L: C_c^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ is a continuous linear map that commutes with differentiation, i.e., $LD = DL$. Then $\exists \mu \in \mathcal{D}'(\mathbb{R})$ s.t. $Lf = f * \mu$ for all $f \in C_c^\infty(\mathbb{R})$.

<u>Proof:</u>

Choose any $f \in C_c^\infty(\mathbb{R})$. Define

$$g(x) = T_{-x} L T_x f(0) = L T_x f(x)$$

By a previous exercise, we know that $\frac{T_{+h} f - f}{-h} \longrightarrow f'$ in $C_c^\infty(\mathbb{R})$ as $h \to 0$. Hence $\frac{T_{x+h} f - T_x f}{-h} \longrightarrow T_x f'$ in $C_c^\infty(\mathbb{R})$, so, since L is linear & continuous,

$$\frac{L T_{x+h} f - L T_x f}{-h} \longrightarrow L T_x f' \text{ in } C^\infty(\mathbb{R}).$$

In particular, the LHS is converging uniformly to the RHS, which implies

$$\frac{L T_{x+h} f(x+h) - L T_x f(x+h)}{-h} \longrightarrow L T_x f'(x)$$

Therefore, letting $h \to 0$,

$$\frac{g(x+h) - g(x)}{h} = \frac{L T_{x+h} f(x+h) - L T_x f(x)}{h}$$

$$= \frac{L T_{x+h} f(x+h) - L T_x f(x+h)}{h} + \frac{L T_x f(x+h) - L T_x f(x)}{h}$$

$$\longrightarrow - L T_x f'(x) + D L T_x f(x)$$

$$= - L D T_x f(x) + D L T_x f(x)$$

$$= 0$$

Hence $g'(x) = 0 \ \forall x$, so g is constant. Therefore

$$T_{-x} L T_x f(0) = g(x) = g(0) = L f(0).$$

Since this is true for every $f \in C_c^\infty(\mathbb{R})$, we can replace f by $T_y f$ to get $T_{-x} L T_x f(y) = L f(y) \ \forall y$.

Hence $T_{-x} L T_x = L$, & therefore the result follows from the preceding theorem. $\square$