## E.7 Alaoglu's Theorem

If $X$ is a normed space, then the closed unit ball in $X$ or $X^*$ is compact if and only if $X$ is finite-dimensional (Problem A.25). Even so, Alaoglu's Theorem states *that the closed unit ball in $X^*$ is compact in the weak* topology.* We will prove this theorem in this section.

### E.7.1 Product Topologies

For the case of two topological spaces $X$ and $Y$, the product topology on $X \times Y$ was defined in Section A.6. We review here some facts about the product topology on arbitrary products of topological spaces.

**Definition E.44 (Product Topology).** Let $J$ be a nonempty index set, and for each $j \in J$ let $X_j$ be a nonempty topological space. Let $X$ be the Cartesian product of the $X_j$:

$$X = \prod_{j \in J} X_j = \bigl\{\{x_j\}_{j\in J} : x_j \in X_J \text{ for } j \in J\bigr\},$$

where a sequence $F = \{x_j\}_{j\in J}$ denotes the mapping

$$\begin{aligned} F\colon J &\to \bigcup_{j\in J} X_j, \\ j &\mapsto x_j. \end{aligned}$$

The Axiom of Choice states that $X$ is nonempty. The *product topology* on $X$ is the topology generated by the collection

$$\mathcal{B} = \biggl\{\prod_{j\in J} U_j : U_j \text{ open in } X_j, \text{ and } U_j = X_j \text{ except for finitely many } j\biggr\}.$$

Since $\mathcal{B}$ is closed under finite intersections, it forms a base for the product topology. Thus, the open sets in $X$ are unions of elements of $\mathcal{B}$.

For each $j$, we define the *canonical projection* of $X$ onto $X_j$ to be the mapping $\pi_j\colon X \to X_j$ defined by

$$\pi_j\bigl(\{x_i\}_{i\in J}\bigr) = x_j.$$

If $U_j$ is an open subset of $X_j$ and we define $U_i = X_i$ for all $i \neq j$, then

$$\pi_j^{-1}(U_j) = \prod_{i\in J} U_i, \tag{E.8}$$

which is open in $X$. Hence each $\pi_j$ is a continuous map. Moreover, if $\mathcal{T}$ is any other topology on $X$ such that each canonical projection $\pi_j$ is continuous then $\mathcal{T}$ must contain all of the sets given by equation (E.8). Since finite intersections of those sets form the base $\mathcal{B}$, we conclude that $\mathcal{T}$ must contain the product topology on $X$. Thus, *the product topology on $X$ is the weakest topology such that each canonical projection $\pi_j$ is continuous.* This is another example of a general kind of weak topology determined by the requirement that a given class of mappings on $X$ be continuous.

Tychonoff's Theorem is a fundamental result on compact sets in the product topology. The proof uses the Axiom of Choice, see [Fol99]. In fact, Kelley proved in 1950 that Tychonoff's Theorem is *equivalent* to the Axiom of Choice [Kel50].

**Theorem E.45 (Tychonoff's Theorem).** *For each $j \in J$, let $X_j$ be a topological space. If each $X_j$ is compact, then $X = \prod_{j\in J} X_j$ is compact in the product topology.*

### E.7.2 Statement and Proof of Alaoglu's Theorem

Now we can prove Alaoglu's Theorem (which is also known as the *Banach–Alaoglu Theorem*).

**Theorem E.46 (Alaoglu's Theorem).** *Let $X$ be a normed linear space, and let*

$$B^* \;=\; \{\mu \in X^* : \|\mu\| \le 1\}$$

*be the closed unit ball in $X^*$. Then $B^*$ is compact in $X^*$ with respect to the weak* topology on $X^*$.*

*Proof.* For each $x \in X$, let

$$D_x \;=\; \{z \in \mathbb{C} : |z| \le \|x\|\}$$

be the closed unit ball of radius $\|x\|$ in the complex plane. Each $D_x$ is compact in $\mathbb{C}$, so Tychonoff's Theorem implies that $D = \prod_{x\in X} D_x$ is compact in the product topology.

The elements of $D$ are sequences $\mu = \{\mu_x\}_{x\in X}$ where $\mu_x \in D_x$ for each $x$. More precisely, $\mu$ is a mapping of $X$ into $\bigcup_{x\in X} D_x = \mathbb{C}$ that satisfies $|\mu_x| \le \|x\|$ for all $x \in X$. Thus $\mu$ is a functional on $X$, although it need not be linear. Since $\mu$ is a functional, we adopt our standard notation and write $\langle x, \mu\rangle = \mu_x$. Then we have

$$|\langle x, \mu\rangle| \;\le\; \|x\|, \qquad x \in X.$$

If $\mu$ is linear then we have $\|\mu\| \le 1$, so $\mu \in B^*$. In fact $B^*$ consists exactly of those elements of $D$ that are linear.

Our next goal is to show that $B^*$ is closed with respect to the product topology restricted to $D$. Suppose that $\{\mu_i\}_{i\in I}$ is a net in $B^*$ and $\mu_i \to \mu \in D$. Since the canonical projections are continuous in the product topology, we have

$$\langle x, \mu_i\rangle \;=\; \pi_x(\mu_i) \;\to\; \pi_x(\mu) \;=\; \langle x, \mu\rangle, \qquad x \in X. \tag{E.9}$$

In particular, given $x, y \in X$ and $a, b \in \mathbb{C}$, we have

$$\langle ax+by, \mu_i\rangle \;\to\; \langle ax+by, \mu\rangle.$$

However, each $\mu_i$ is linear since it belongs to $B^*$, so we also have

$$\langle ax+by, \mu_i\rangle \;=\; a\,\langle x, \mu_i\rangle + b\,\langle y, \mu_i\rangle \;\to\; a\,\langle x, \mu\rangle + b\,\langle y, \mu\rangle.$$

Therefore $\langle ax+by, \mu\rangle = a\,\langle x, \mu\rangle + b\,\langle y, \mu\rangle$, so $\mu$ is linear and therefore belongs to $B^*$. Hence $B^*$ is a closed subset of $D$. Since $D$ is compact in the product topology, we conclude that $B^*$ is also compact in the product topology (see Problem A.21).

Now we will show that the product topology on $D$ restricted to $B^*$ is the same as the weak* topology on $X$ restricted to $B^*$. To do this, let $\mathcal{T}$ denote the product topology on $D$ restricted to $B^*$, and let $\sigma$ denote the weak* topology on $X$ restricted to $B^*$. Suppose that $\{\mu_i\}_{i\in I}$ is a net in $B^*$ and $\mu_i \to \mu$ with respect to the product topology on $D$. We saw above that that this implies that $\mu \in B^*$ and $\mu_i \xrightarrow{\text{w}^*} \mu$, see equation (E.9). Hence every subset of $B^*$ that is closed with respect to $\sigma$ is closed with respect to $\mathcal{T}$, so we have $\sigma \subseteq \mathcal{T}$.

Conversely, fix any $x \in X$ and suppose that $\{\mu_i\}_{i\in I}$ is a net in $B^*$ such that $\mu_i \xrightarrow{\text{w}^*} \mu$. Then

$$\pi_x(\mu_i) \;=\; \langle x, \mu_i\rangle \;\to\; \langle x, \mu\rangle \;=\; \pi_x(\mu),$$

so the canonical projection $\pi_x$ is continuous with respect to the weak* topology restricted to $B^*$. However, $\mathcal{T}$ is the weakest topology with respect to which each canonical projection is continuous, so $\mathcal{T} \subseteq \sigma$.

Thus, $\mathcal{T} = \sigma$. Since we know that $B^*$ is compact with respect to $\mathcal{T}$, we conclude that it is also compact with respect to $\sigma$. That is, $B^*$ is compact with respect to the weak* topology on $X^*$ restricted to $B^*$, and this implies that it is compact with respect to the weak* topology on $X^*$. $\square$

As a consequence, if $X$ is reflexive then the closed unit ball in $X^*$ is weakly compact. In particular, the closed unit ball in a Hilbert space is weakly compact. On the other hand, the space $c_0$ is not reflexive, and its closed unit ball is not weakly compact (Problem E.9).

### E.7.3 Implications for Separable Spaces

Alaoglu's Theorem has some important consequences for separable spaces. Although we will restrict our attention here to normed spaces, many of these results hold more generally, see [Rud91].

We will need the following lemma.

**Lemma E.47.** *Let $\mathcal{T}_1$, $\mathcal{T}_2$ be topologies on a set $X$ such that:*

(a) *$X$ is Hausdorff with respect to $\mathcal{T}_1$,*

(b) *$X$ is compact with respect to $\mathcal{T}_2$, and*

(c) $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

*Then* $\mathcal{T}_1 = \mathcal{T}_2$.

*Proof.* Suppose that $F \subseteq X$ is $\mathcal{T}_2$-closed. Then $F$ is $\mathcal{T}_2$-compact since $X$ is $\mathcal{T}_2$-compact (see Problem A.21). Suppose that $\{U_\alpha\}_{\alpha\in J}$ is any cover of $F$ by sets that are $\mathcal{T}_1$-open. Then each of these sets is also $\mathcal{T}_2$-open, so there must exist a finite subcollection that covers $F$. Hence $F$ is $\mathcal{T}_1$-compact, and therefore is $\mathcal{T}_1$-closed since $\mathcal{T}_1$ is Hausdorff (again see Problem A.21). Consequently, $\mathcal{T}_2 \subseteq \mathcal{T}_1$. □

Now we can show that a weak*-compact subset of the dual space of a separable normed space is metrizable.

**Theorem E.48.** *Let $X$ be a separable normed space, and fix $K \subseteq X^*$. If $K$ is weak*-compact, then the weak* topology of $X^*$ restricted to $K$ is metrizable.*

*Proof.* Let $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $X$.

Each $x \in X$ determines a seminorm $\rho_x$ on $X^*$ given by

$$\rho_x(\mu) \;=\; |\langle x,\mu\rangle|, \qquad \mu \in X^*.$$

The family of seminorms $\{\rho_x\}_{x\in X}$ induces the weak* topology $\sigma(X^*,X)$ on $X^*$. The subfamily $\{\rho_{x_n}\}_{n\in\mathbb{N}}$ also induces a topology on $X^*$, which we will call $\mathcal{T}$. Since this is a smaller family of seminorms, we have $\mathcal{T} \subseteq \sigma(X^*,X)$.

Suppose that $\mu \in X^*$ and $\rho_{x_n}(\mu) = 0$ for every $n \in \mathbb{N}$. Then we have $\langle x_n,\mu\rangle = 0$ for every $n$. Since $\{x_n\}_{n\in\mathbb{N}}$ is dense in $X$ and $\mu$ is continuous, this implies that $\mu = 0$. Consequently, by Exercise E.17, the topology $\mathcal{T}$ is Hausdorff. Thus $\mathcal{T}$ is a Hausdorff topology induced from a countable family of seminorms, so Exercise E.24 tells us that this topology is metrizable. Specifically, $\mathcal{T}$ is induced from the metric

$$\mathrm{d}(\mu,\nu) \;=\; \sum_{n=1}^{\infty} 2^{-n}\,\frac{\rho_{x_n}(\mu-\nu)}{1+\rho_{x_n}(\mu-\nu)}, \qquad \mu,\nu \in X^*.$$

Let $\mathcal{T}|_K$ and $\sigma(X^*,X)|_K$ denote these two topologies restricted to the subset $K$. Then we have that $K$ is Hausdorff with respect to $\mathcal{T}|_K$, and is compact with respect to $\sigma(X^*,X)|_K$. Lemma E.47 therefore implies that $\mathcal{T}|_K = \sigma(X^*,X)|_K$. The topology $\mathcal{T}_K$ is metrizable, as it is formed by restricting the metric d to $K$. Hence $\sigma(X^*,X)|_K$ is metrizable as well. □

In the course of the proof of Theorem E.48 we constructed a metrizable topology $\mathcal{T}$ on $X^*$, and showed that the restrictions of $\mathcal{T}$ and $\sigma(X^*,X)$ to any weak*-compact set are equal. This does not show that $\mathcal{T}$ and $\sigma(X^*,X)$ are equal. In fact, if $X$ is infinite-dimensional, the weak* topology on $X$ is not metrizable, see [Rud91, p. 70].

Now we can prove a stronger form of Alaoglu's Theorem for separable normed spaces. Specifically, we show that if $X$ is normed and separable, then any bounded sequence in $X^*$ has a weak*-convergent subsequence.

**Theorem E.49.** *If $X$ is a separable normed space then the closed unit ball in $X^*$ is sequentially weak*-compact. That is, if $\{\mu_n\}_{n\in\mathbb{N}}$ is any sequence in $X^*$ with $\|\mu_n\| \le 1$ for $n \in \mathbb{N}$, then there exists a subsequence $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ and $\mu \in X^*$ such that $\mu_{n_k} \xrightarrow{\text{w}^*} \mu$.*

*Proof.* By Alaoglu's Theorem, the closed unit ball $B^*$ in $X^*$ is weak*-compact. Since $X$ is separable, Theorem E.48 implies that the weak* topology on $B^*$ is metrizable. Finally, since $B^*$ is a compact subset of a metric space, Theorem A.70 implies that it is sequentially compact. □

**Corollary E.50.** *If $X$ is a reflexive normed space then the closed unit ball in $X^*$ is sequentially weakly compact. In particular, the closed unit ball in a separable Hilbert space is sequentially weakly compact.*

~~In general, however, a weakly compact subset of $X$ need not be metrizable in the weak topology, even if $X$ is separable. For example, [Con90, Prop. 5.2] shows that the weak topology on closed unit ball in $\ell^1$ is not metrizable.~~

These statements are not accurate, please ignore. CH

There are several deep results on weak compactness that we will not elaborate upon. For example, the *Eberlein-Šmulian Theorem* states that a subset of a Banach space is weakly compact if and only if it is weakly sequentially compact, see [Con90, Thm. 13.1].

## Additional Problems

**E.7.** This problem will use Alaoglu's Theorem to construct an element of $(\ell^\infty)^*$ that does not belong to $\ell^1$.

(a) For each $n \in \mathbb{N}$, define $\mu_n \colon \ell^\infty \to \mathbb{C}$ by

$$\langle x, \mu_n\rangle \;=\; \frac{x_1 + \cdots + x_n}{n}, \qquad x = (x_1, x_2, \dots) \in \ell^\infty.$$

Show that $\mu_n \in (\ell^\infty)^*$ and $\|\mu_n\| \le 1$.

(b) Use Alaoglu's Theorem to show that there exists a $\mu \in (\ell^\infty)^*$ that is an accumulation point of $\{\mu_n\}_{n\in\mathbb{N}}$.

(c) Show that $\mu \ne \widehat{x}$ for any $x \in \ell^1$, where $\widehat{x}$ is the image of $x$ under the natural embeddings of $\ell^1$ into $(\ell^1)^{**} = (\ell^\infty)^*$.

**E.8.** Suppose that $S$ is a subspace of $C[0,1]$ and that $S$ is closed as a subspace of $L^2[0,1]$. Prove the following statements.

(a) $S$ is closed in $C[0,1]$, and the $L^\infty$ and $L^2$ norms are equivalent on $S$.

(b) For each $y \in [0,1]$, there exists $k_y \in L^2[0,1]$ such that $f(y) = \int f(x)\, k_y(x,y)\, dy$.

(c) Weak convergence in $S$ coincides with strong convergence in $S$.

(d) The closed unit ball in $S$ is strongly compact, and therefore $S$ is finite-dimensional.

**E.9.** Show that the closed unit ball $\{x \in c_0 : \|x\|_\infty \le 1\}$ in $c_0$ is not weakly compact.