A.7 Convergence and Continuity in Topological Spaces

In large part, the importance of topologies in this volume is that they provide notions of convergence and continuity. Indeed, a basic philosophy that we will expand upon in this section is that topologies and convergence criteria are equivalent. Thus, though many of the spaces that we will encounter are defined in terms of norms or families of seminorms (i.e., convergence criteria), this is equivalent to defining them in terms of a topology.

A.7.1 Convergence

In metric spaces, convergence is defined with respect to sequences indexed by the natural numbers (Definition A.2). In a general topological space, convergence must be formulated in terms of nets instead of countable sequences.

Definition A.43 (Directed Sets, Nets). A *directed set* is a set I together with a relation $\leq$ on I such that:

(a) $\leq$ is reflexive: $i \leq i$ for all $i \in I$,

(b) $\leq$ is transitive: $i \leq j$ and $j \leq k$ implies $i \leq k$, and

(c) for any $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

A *net* in a set X is a sequence $\{x_i\}_{i \in I}$ of elements of X indexed by a directed set $(I, \leq)$.

Remark A.44. By definition, a sequence $\{x_i\}_{i \in I}$ is shorthand for the function $x\colon I \to X$ defined by $x(i) = x_i$ for $i \in I$. In particular, unlike a set, a sequence allows repititions of the x_i. Technically, we should be careful to distinguish between a sequence $\{x_i\}_{i \in I}$ and a set $\{x_i : i \in I\}$, but it is usually clear from context whether a sequence or a set is meant.

The set of natural numbers $I = \mathbb{N}$ under the usual ordering is one example of a directed set, and hence every ordinary sequence indexed by the natural numbers is a net. Another typical example is $I = \mathcal{P}(X)$, the power set of X, ordered by *reverse inclusion*, i.e.,

$$U \leq V \iff V \subseteq U.$$

Definition A.45 (Convergence with respect to a Net). Let X be a topological space, let $\{x_i\}_{i \in I}$ be a net in X, and let $x \in X$ be given. Then we say that $\{x_i\}_{i \in I}$ *converges* to x, and write $x_i \to x$, if for any open neighborhood U of x there exists $i_0 \in I$ such that

$$i \geq i_0 \implies x_i \in U.$$

Next, we will define the notion of accumulation points of a subset of a generic topological space and see how this definition can be reformulated in terms of nets. We will also see that topologies induced from a metric have the advantage that we only need to use convergence of ordinary sequences indexed by $\mathbb{N}$ instead of general nets.

Definition A.46 (Accumulation Point). Let E be a subset of a topological space X. Then a point $x \in X$ is an *accumulation point* of E if every open neighborhood of x contains a point of E other than x itself, i.e.,

$$U \text{ open and } x \in U \implies E \cap (U \backslash \{x\}) \neq \emptyset.$$

Lemma A.47. *If E is a subset of a topological space X and $x \in X$, then the following statements are equivalent.*

(a) *x is an accumulation point of E.*

(b) *There exists a net $\{x_i\}_{i \in I}$ contained in $E \backslash \{x\}$ such that $x_i \to x$.*

If X is a metric space, then these statements are also equivalent to the following.

(c) *There exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ contained in $E \backslash \{x\}$ such that $x_n \to x$.*

Proof. (a) $\Rightarrow$ (b). Assume that x is an accumulation point of E. Define

$$I = \bigl\{U \subseteq X : U \text{ is open and } x \in U\bigr\}.$$

Exercise: Show that I is a directed set when ordered by reverse inclusion.

For each $U \in I$, by definition of accumulation point there exists a point $x_U \in E \cap (U \backslash \{x\})$. Then $\{x_U\}_{U \in I}$ is a net in $E \backslash \{x\}$, and we claim that $x_U \to x$. To see this, fix any open neighborhood V of x. Set $U_0 = V$, and suppose that $U \geq U_0$. Then, by definition, $U \in I$ and $U \subseteq U_0$. Hence $x_U \in U \subseteq U_0 = V$. Therefore $x_U \to x$.

(b) $\Rightarrow$ (a). Suppose that $\{x_i\}_{i \in I}$ is a net in $E \backslash \{x\}$ and $x_i \to x$. Let U be any open neighborhood of x. Then there exists an i_0 such that $x_i \in U$ for all $i \geq i_0$. Since $x_i \neq x$, this implies that $x_i \in E \cap (U \backslash \{x\})$ for all $i_0 \geq i$.

(a) $\Rightarrow$ (c), assuming X is metric. Suppose that x is an accumulation point of E. For each $n \in \mathbb{N}$, the open ball $B_{1/n}(x)$ is an open neighborhood of x, and hence there must exist some $x_n \in E \cap (B_{1/n}(x) \backslash \{x\})$. Therefore $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in $E \backslash \{x\}$, and since $\mathrm{d}(x, x_n) < 1/n$, we have $x_n \to x$.

(c) $\Rightarrow$ (b), assuming X is metric. This follows from the fact that every countable sequence $\{x_n\}_{n \in \mathbb{N}}$ is a net. ☐

We can now give an equivalent formulation of closed sets in terms of nets and accumulation points.

Exercise A.48. Given a subset E of a topological space, prove that the following statements are equivalent.

(a) E is closed, i.e., $X \backslash E$ is open.

(b) If x is an accumulation point of E, then $x \in E$.

(c) If $\{x_i\}_{i \in I}$ is a net in E and $x_i \to x \in X$, then $x \in E$.

If X is a metric space, show that these are also equivalent to the following statement.

(d) If $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in E and $x_n \to x \in X$, then $x \in E$.

Now we can quantify the philosophy that topologies and convergence criteria are equivalent. For arbitrary topologies, this requires that we use convergence with respect to nets, but for topologies induced from a metric we are able to use convergence of ordinary sequences indexed by the natural numbers.

Exercise A.49. Given two topologies $\mathcal{T}_1$, $\mathcal{T}_2$ on a set X, prove that the following statements are equivalent.

(a) $\mathcal{T}_1 = \mathcal{T}_2$, i.e.,

$$U \text{ is open with respect to } \mathcal{T}_1 \iff U \text{ is open with respect to } \mathcal{T}_2.$$

(b) If $\{x_i\}_{i \in I}$ is a net in E and $x \in X$, then

$$x_i \to x \text{ with respect to } \mathcal{T}_1 \iff x_i \to x \text{ with respect to } \mathcal{T}_2.$$

If $\mathcal{T}_1$ is induced from a metric d_1 on X, and $\mathcal{T}_2$ is induced from a metric d_2 on X, show that these statements are also equivalent to the following.

(c) If $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in E and $x \in X$, then

$$\lim_{n \to \infty} \mathrm{d}_1(x_n, x) = 0 \iff \lim_{n \to \infty} \mathrm{d}_2(x_n, x) = 0.$$

Example A.50. An example of a topological space where it is important to distinguish between convergence of ordinary sequences and convergence with respect to nets is the sequence space ℓ^1 under the weak topology. This topology will be defined precisely in Section E.6, but the important point for us at the moment is that it can be shown that if $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in ℓ^1 and $x_n \to 0$ with respect to the weak topology, then $x_n \to 0$ in norm, i.e., $\|x - x_n\|_1 \to 0$ (see [Con90, Prop. 5.2]). However, the weak topology on ℓ^1 is *not* the same as the topology induced by the norm $\|\cdot\|_1$. In particular, there exists a net $\{x_i\}_{i \in I}$ in ℓ^1 such that $x_i \to 0$ with respect to the weak topology, but $x_i \not\to 0$ with respect to the norm topology. The moral is that when discussing convergence in a topological space that is not a metric space, it is important to consider nets instead of ordinary sequences.

A.7.2 Continuity

Our next goal is to reformulate continuity of a function in terms of convergence of nets or sequences. Recall that if $f\colon X \to Y$ and $V \subseteq Y$, then the preimage of V is $f^{-1}(V) = \{x \in X : f(x) \in V\}$.

Definition A.51 (Continuity). Let X, Y be topological spaces. Then a function $f\colon X \to Y$ is *continuous* if

$$V \text{ is open in } Y \implies f^{-1}(V) \text{ is open in } X.$$

We say that f is a *topological isomorphism* or a *homeomorphism* if f is a bijection and both f and f^{-1} are continuous.

It will be convenient to restate continuity in terms of continuity at a point.

Definition A.52 (Continuity at a Point). Let X, Y be topological spaces and let $x \in X$ be given. Then a function $f\colon X \to Y$ is *continuous at* x if for each open neighborhood V of $f(x)$ in Y, there exists an open neighborhood U of x in X such that $U \subseteq f^{-1}(V)$.

Exercise A.53. Prove that f is continuous if and only if f is continuous at each $x \in X$.

Now we can formulate continuity in terms of preservation of convergence of nets. For the case of a metric space, this reduces to preservation of convergence of sequences.

Lemma A.54. *If X, Y be topological spaces, $f\colon X \to Y$, and $x \in X$ are given, then the following statements are equivalent.*

(a) *f is continuous at x.*

(b) *For any net $\{x_i\}_{i \in I}$ in X,*

$$x_i \to x \text{ in } X \implies f(x_i) \to f(x) \text{ in } Y.$$

If X, Y are metric spaces, then these are also equivalent to the following.

(c) *For any sequence $\{x_n\}_{n \in \mathbb{N}}$ in X,*

$$x_n \to x \text{ in } X \implies f(x_n) \to f(x) \text{ in } Y.$$

Proof. (a) $\Rightarrow$ (b). Assume that f is continuous at $x \in X$. Let $\{x_i\}_{i \in I}$ be any net in X such that $x_i \to x$. Let V be any open neighborhood of $f(x)$. Then by definition of continuity at a point, there exists an open neighborhood U of x that is contained in $f^{-1}(V)$. Hence, by definition of $x_i \to x$, there exists an $i_0 \in I$ such that $x_i \in U \subseteq f^{-1}(V)$ for all $i \geq i_0$. Hence $f(x_i) \in V$ for all $i \geq i_0$, which means that $f(x_i) \to f(x)$.

(b) $\Rightarrow$ (a). Suppose that statement (a) fails, i.e., f is not continuous at $x \in X$. Then by definition there exists an open neighborhood V of $f(x)$ such that $f^{-1}(V)$ is not open in X. Hence no open neighborhood U of x can be contained in $f^{-1}(V)$. Therefore each open neighborhood U of x must contain some point $x_U \in U \backslash f^{-1}(V)$. In particular, $f(x_U) \notin V$, and since $f(x) \in V$, we must have $x_U \neq x$. Now let

$$I = \{U : U \text{ is an open neighborhood of } x\}.$$

Then I is a directed set when ordered by reverse inclusion, so $\{x_U\}_{U \in I}$ is a net in X. We claim that $x_U \to x$. To see this, let W be any open set containing x, and set $U_0 = W \in I$. If $U \geq U_0$, then by definition U is an open neighborhood of x and $U \subseteq U_0$. Hence $x_U \in U \subseteq U_0 = W$. Therefore $x_U \in W$ for all $U \geq U_0$, and this means $x_U \to x$.

However, $f(x_U)$ does not converge to $f(x)$ because V is an open neighborhood of $f(x)$ but V contains no points $f(x_U)$. Hence statement (b) fails.

(c) $\Rightarrow$ (a), assuming X is metric. The proof of (b) $\Rightarrow$ (a) can essentially be repeated. Alternatively, given an open $V \subseteq Y$, we can show that $X \backslash f^{-1}(V)$ is closed by showing that it contains all of its accumulation points, as follows.

Suppose that x is an accumulation point of $X \backslash f^{-1}(V)$. Then by Exercise A.48, there exist $x_n \in X \backslash f^{-1}(V)$ such that $x_n \to x$. By statement (c), this implies $f(x_n) \to f(x)$. However, $f(x_n) \notin V$ and $Y \backslash V$ is a closed set. Therefore we must have $f(x) \in Y \backslash V$, and hence $x \in X \backslash f^{-1}(V)$. Thus $X \backslash f^{-1}(V)$ is closed, so $f^{-1}(V)$ is open. □

A.7.3 Equivalent Norms

Next we consider the equivalence of convergence criteria and topologies for the case of normed spaces.

Definition A.55. Suppose that X is a normed linear space with respect to a norm $\|\cdot\|_a$ and also with respect to another norm $\|\cdot\|_b$. Then we say that these norms are *equivalent* if there exist constants $C_1, C_2 > 0$ such that

$$\forall\, f \in X, \quad C_1\, \|f\|_a \;\leq\; \|f\|_b \;\leq\; C_2\, \|f\|_a. \tag{A.4}$$

We write $\|\cdot\|_a \asymp \|\cdot\|_b$ to denote that $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms.

Theorem A.56. *Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms on a vector space X. Then the following statements are equivalent.*

(a) *$\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms.*

(b) *$\|\cdot\|_a$ and $\|\cdot\|_b$ induce the same topologies on X.*

(c) *$\|\cdot\|_a$ and $\|\cdot\|_b$ define the same convergence criterion. That is, if $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in X and $x \in X$, then*

$$\lim_{n\to\infty} \|x - x_n\|_a = 0 \iff \lim_{n\to\infty} \|x - x_n\|_b = 0.$$

Proof. (a) $\Rightarrow$ (c). This is immediate from the definition of equivalent norms.

(c) $\Rightarrow$ (b). This follows from Exercise A.49.

(b) $\Rightarrow$ (a). Assume that statement (b) holds. Let $B_r^a(x)$ and $B_r^b(x)$ denote the open balls of radius r centered at $x \in X$ with respect to $\|\cdot\|_a$ and $\|\cdot\|_b$, respectively. Since $B_1^a(0)$ is open with respect to $\|\cdot\|_a$, statement (b) implies that $B_1^a(0)$ is open with respect to $\|\cdot\|_b$. Therefore, since $0 \in B_1^a(0)$, there must exist some $r > 0$ such that $B_r^b(0) \subseteq B_1^a(0)$.

Now choose any $x \in X$ and any $\varepsilon > 0$. Then

$$\frac{(r-\varepsilon)}{\|x\|_b}\, x \;\in\; B_r^b(0) \;\subseteq\; B_1^a(0),$$

so

$$\left\| \frac{(r-\varepsilon)\, x}{\|x\|_b} \right\|_a \;<\; 1.$$

Rearranging, this implies $(r-\varepsilon)\,\|x\|_a < \|x\|_b$. Since this is true for every ε, we conclude that $r\,\|x\|_a \le \|x\|_b$.

A symmetric argument, interchanging the roles of the two norms, shows that there exists an $s > 0$ such that $\|x\|_b \le s\,\|x\|_a$ for every $x \in X$. Hence the two norms are equivalent. □

Given any finite-dimensional vector space X, we can define many norms on X. In particular, the following norms are analogues of the ℓ^p norms defined in Section A.4.

Exercise A.57. Let $\mathcal{B} = \{x_1, \dots, x_d\}$ be any basis for a finite-dimensional vector space X, and let $x = \sum_{k=1}^d c_k(x)\, x_k$ denote the unique expansion of $x \in X$ with respect to this basis (the vector $[x]_{\mathcal{B}} = (c_1(x), \dots, c_d(x))$ is called the *coordinate vector* of x with respect to the basis $\mathcal{B}$). Show that

$$\|x\|_p \;=\; \begin{cases} \displaystyle\left(\sum_{k=1}^d |c_k(x)|^p\right)^{1/p}, & 1 \le p < \infty, \\ \displaystyle\max_k |c_k(x)|, & p = \infty, \end{cases}$$

are norms on X, and that X is complete with respect to each of these norms. Note that $\|x\|_p$ is simply the ℓ^p norm of the coordinate vector $[x]_{\mathcal{B}}$.

It is not difficult to see that all of the norms defined in Exercise A.57 are equivalent. Although we will not prove it, it is an important fact that *all* norms on a finite-dimensional space are equivalent.

Theorem A.58. *If X is a finite-dimensional vector space, then any two norms on X are equivalent. In particular, if $\|\cdot\|$ is any norm on X and $\|\cdot\|_p$ is any one of the norms constructed in Exercise A.57, then $\|\cdot\| \asymp \|\cdot\|_p$.*

Additional Problems

A.16. (a) Let X be a Hausdorff topological space. Show that if a net $\{x_i\}_{i\in I}$ converges in X, then the limit is unique.

(b) Show that if X is not Hausdorff, then there exists a net $\{x_i\}_{i\in I}$ in X that has two distinct limits.

A.17. Let $\{x_i\}_{i\in I}$ be a net in a Hausdorff topological space X. Show that if $x_i \to x$ in X, then either:

(a) there exists an open neighborhood U of x and some $i_0 \in I$ such that $x_i = x$ for all $i \geq i_0$, or

(b) every open neighborhood U of x contains infinitely many distinct x_i, i.e., the set $\{x_i : i \in I \text{ and } x_i \in U\}$ is infinite.

A.18. Show that if (X, d_1) and (Y, d_2) are metric spaces, then

$$\mathrm{d}\big((f_1, g_1),\, (f_2, g_2)\big) \;=\; \mathrm{d}_1(f_1, f_2) + \mathrm{d}_2(g_1, g_2)$$

defines a metric on $X \times Y$ that induces the product topology. Conclude that convergence in $X \times Y$ is componentwise convergence, i.e., $(f_n, g_n) \to (f, g)$ in $X \times Y$ if and only if $f_n \to f$ in X and $g_n \to g$ in Y.

A.19. Let X be a vector space with a metric d. We say that the metric is *translation-invariant* if $\mathrm{d}(f + h, g + h) = \mathrm{d}(f, g)$ for every $f,\ g,\ h \in X$. Show that vector addition is continuous in this case, i.e., $(f, g) \to f + g$ is a continuous mapping of $X \times X$ into X.

A.20. Let X, Y be topological spaces. Let $\{(f_i, g_i)\}_{i\in I}$ be any net in $X \times Y$, and suppose $(f, g) \in X \times Y$. Show that $(f_i, g_i) \to (f, g)$ with respect to the product topology on $X \times Y$ if and only if $f_i \to f$ in X and $g_i \to g$ in Y.