Example: The Trigonometric System

Theorem

Let $\mathcal{E} = \{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$. Easy fact: this is an ON system in $L^2[0,1]$.

a. $\mathcal{E}$ is an ONB for $L^2[0,1]$

b. $\mathcal{E}$ is a Schauder basis for $L^p[0,1]$ if $1 < p < \infty$.

However, for $p \neq 2$ it is only a conditional basis: the partial sums converge if we index $\mathbb{Z}$ as $\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \dots\}$, but do not converge for every ordering of $\mathbb{Z}$.

c. $\mathcal{E}$ is complete & minimal in $L^1[0,1]$ and in $C[0,1]$, but is not a Schauder basis for either space.

Remark: Here $C[0,1]$ requires continuity in the torus sense of $[0,1]$, i.e., we must have $f(0) = f(1)$.

Remark: The interval $[0,1]$ can be replaced by any interval of length 1.

Remark: We can extend to $L^2(\mathbb{R})$:

$$\left\{ e^{2\pi i n x} \chi_{[k,k+1]}(x) \right\}_{n,k \in \mathbb{Z}}$$

forms an ONB for $L^2(\mathbb{R})$.

Unfortunately, it's not a very good one because of the discontinuities.

Example: The Haar System

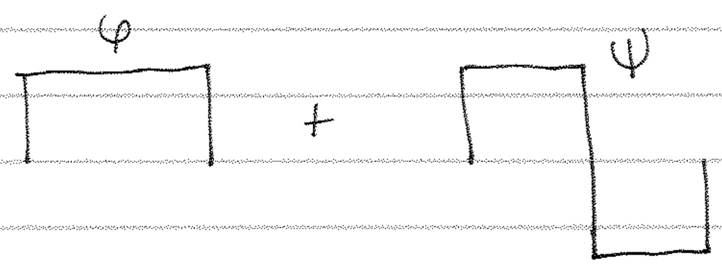
Set

$$\varphi = \chi_{[0,1]} \quad \& \quad \psi = \chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]}.$$

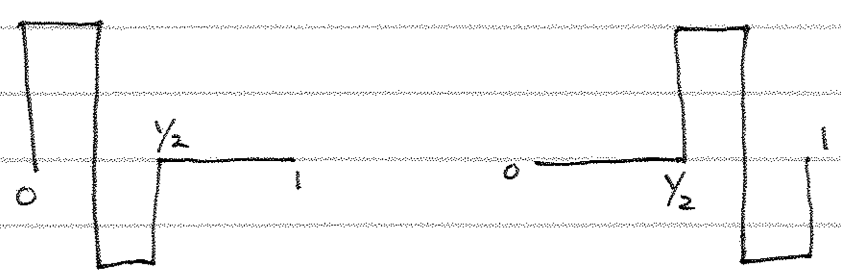
The Haar system is

$$\{\varphi\} \cup \left\{ 2^{n/2}\, \psi(2^n x - k) \right\}_{n \geq 0,\ k = 0, \ldots, 2^n - 1}.$$

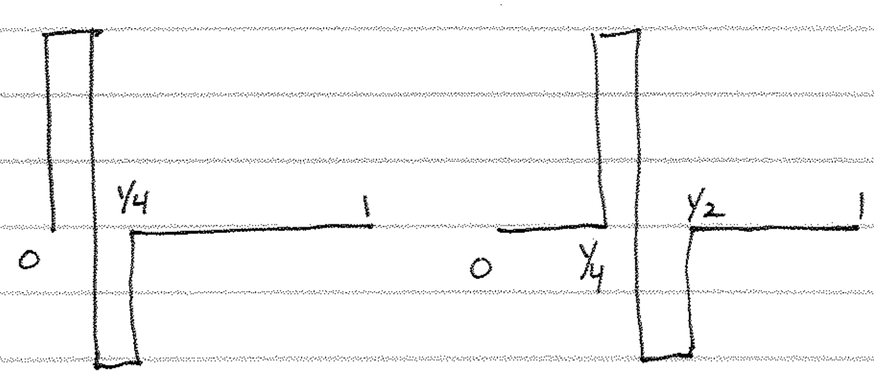
The elements of the Haar system are:



+ two half-size ψ's



+ 4 quarter-size ψ's



etc

Easy: The Haar system is ON in $L^2[0,1]$.

Not too hard: The Haar system is complete in $L^2[0,1]$ and therefore is an ONB for $L^2[0,1]$.

<u>Theorem</u>

The Haar system is a bounded unconditional basis for $L^p[0,1]$ for each $1 < p < \infty$.

<u>Remark:</u>

Can extend to $L^2(\mathbb{R})$ in two ways:

$$\{\varphi(x-k)\}_{k\in\mathbb{Z}} \cup \{2^{n/2}\psi(2^n x-k)\}_{n\geq 0,\, k\in\mathbb{Z}}$$

is an ONB for $L^2(\mathbb{R})$, and so is

$$\{2^{n/2}\psi(2^n x-k)\}_{n,k\in\mathbb{Z}}$$

Still, the elements of this basis are discontinuous functions. <u>Wavelet bases</u> (discovered ≈ 1986) are ONB for $L^2(\mathbb{R})$ with the same translation/dilation structure but generated by smoother functions.

Discussion of a wavelet basis

Another example is the *Daubechies D_4 function.* This is the compactly supported function that satisfies the four-term *refinement equation*

$$\begin{aligned} D_4(x) \;&=\; \frac{1+\sqrt{3}}{4}D_4(2x) + \frac{3+\sqrt{3}}{4}D_4(2x-1) \\ &\quad + \frac{3-\sqrt{3}}{4}D_4(2x-2) + \frac{1-\sqrt{3}}{4}D_4(2x-3). \end{aligned} \tag{B.15}$$

Thus, D_4 exhibits a kind of self-similarity, as it equals a linear combination of four smaller, shifted copies of itself. It can be shown that there exists a unique (up to scale) compactly supported function that satisfies this refinement equation, and furthermore this solution is Hölder continuous precisely for those exponents α that lie in the range

$$0 \;<\; \alpha \;\le\; -\log_2\Big(\frac{1+\sqrt{3}}{4}\Big) \;\approx\; 0.5500\ldots,$$

see [Dau92]. The Cantor–Lebesgue function also satisfies a refinement equation, based on dilation by 3 rather than 2, see equation (B.11).

Exercise B.102. For this exercise, assume that equation (B.15) has a solution that is continuous and compactly supported.

(a) Show that $\operatorname{supp}(D_4) \subseteq [0,3]$.

(b) Combine part (a) with the refinement equation to find the values of $D_4(k)$ for all integer k.

(c) Now compute the values $D_4(k/2)$ for $k \in \mathbb{Z}$ by considering $x = k/2$ in equation (B.15). Iterating this, we can obtain the values $D_4(k/2^j)$ for any $k \in \mathbb{Z}$, $j \in \mathbb{N}$. Plot the Daubechies D_4 function.

When suitably normalized, D_4 has the interesting property that its integer translates are orthonormal, i.e., $\{D_4(x-k)\}_{k\in\mathbb{Z}}$ forms an orthonormal system in $L^2(\mathbb{R})$. The Daubechies D_4 function is but one refinable function that has orthonormal integer translates. Each such *orthonormal scaling function* leads to a second function, called the *wavelet*, which generates an orthonormal basis for *all* of $L^2(\mathbb{R})$ via the operations of translation and dilation. For D_4, the corresponding wavelet is

$$\begin{aligned} W_4(x) \;&=\; \frac{1-\sqrt{3}}{4}D_4(2x) - \frac{3-\sqrt{3}}{4}D_4(2x-1) \\ &\quad + \frac{3+\sqrt{3}}{4}D_4(2x-2) - \frac{1+\sqrt{3}}{4}D_4(2x-3). \end{aligned}$$

This function has the property that

$$\big\{2^{n/2}W_4(2^n x - k)\big\}_{n,k\in\mathbb{Z}}$$

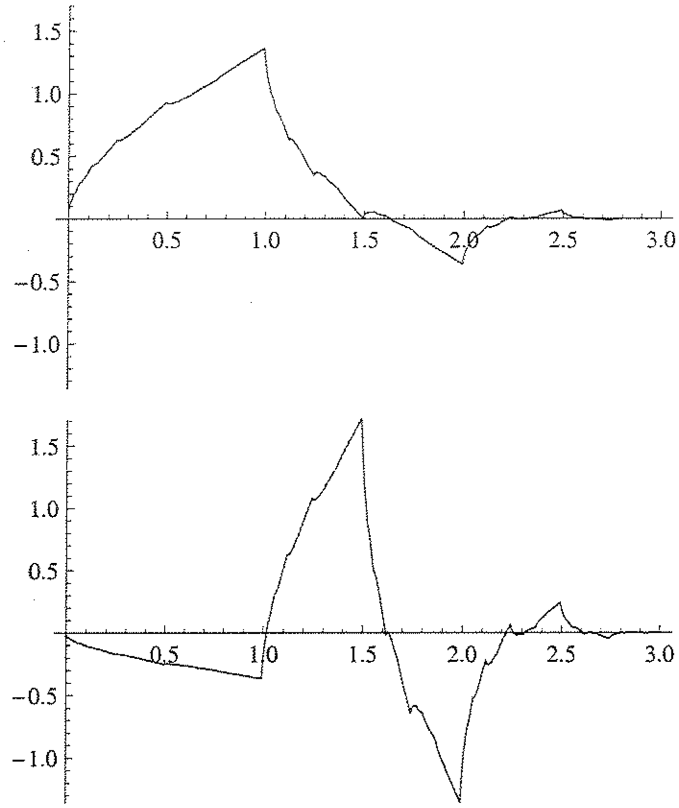


Fig. B.3. Top: The Daubechies D_4 scaling function. Bottom: The corresponding wavelet W_4.

forms an orthonormal basis for $L^2(\mathbb{R})$. Equivalently,

$$\{D_2(x-k)\}_{k\in\mathbb{Z}} \cup \{2^{n/2}W_4(2^n x - k)\}_{n\geq 0,\, k\in\mathbb{Z}}$$

forms an orthonormal basis for $L^2(\mathbb{R})$, compare Problem B.29.

Why the subscript 4? The Daubechies scaling function D_4 is the second of an infinite family of functions $\{D_{2N}\}_{N\in\mathbb{N}}$, each of which satisfies a $2N$-term refinement equation, is supported in $[0, 2N-1]$, and has orthonormal integer translates. Moreover, the smoothness of D_{2N} increases with N. In particular, $D_2 = \chi_{[0,1]}$ is discontinuous, while D_4 is continuous. Each of these scaling functions has an associated wavelet W_{2N} whose integer translates and dyadic dilations form an orthonormal basis for $L^2(\mathbb{R})$.

The first wavelet, the function $W_2 = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, was introduced by Haar in his 1910 Ph.D. thesis [Haa10], and is called the *Haar wavelet*, see Problem B.29. It was not until much later that other wavelets, including the Daubechies scaling functions and wavelets in particular, were discovered. We will not be able to do justice to wavelet theory here, but only mention that the main papers in the development of wavelet theory, including a translation of Haar's original paper into English, appear in the reprint volume [HW06]. More fundamentally, we refer to Daubechies' classic text [Dau92] for complete details on scaling functions and wavelets, and to the texts by Mallat [Mal98] and Strang and Nguyen [SN96] for their relation to signal processing. The text by Walnut [Wal02] is an accessible introduction to wavelet theory and its applications.

Additional Problems

B.29. Let $\chi = \chi_{[0,1)}$ be the box function, and let $\psi = W_2 = \chi_{[0,1/2)} - \chi_{[1/2,1)}$ be the *Haar wavelet.* Prove that

$$\big\{\chi(x-k)\big\}_{k\in\mathbb{Z}} \cup \big\{2^{n/2}\psi(2^n x - k)\big\}_{n\geq 0,\, k\in\mathbb{Z}}$$

forms an orthonormal basis for $L^2(\mathbb{R})$. Observe that χ satisfies the refinement equation

$$\chi(x) \;=\; \chi(2x) + \chi(2x-1),$$

while ψ is determined from χ by the equation

$$\psi(x) \;=\; \chi(2x) - \chi(2x-1).$$