Open Mapping and Closed Graph Theorems

By definition, if $A : X \longrightarrow Y$ is continuous, then

$$U \text{ open in } Y \implies A^{-1}(U) \text{ open in } X.$$

However, it need <u>not</u> be true that a continuous ~~map~~ function will map an open set in X to an open set in Y.

Example

$f(x) = \sin x$ is a continuous map of $\mathbb{R}$ into $\mathbb{R}$.

However,

$$f\big((0, 2\pi)\big) = [-1, 1].$$

Definition

If X, Y are topological spaces, then $A : X \to Y$ is an <u>open mapping</u> if

$$U \text{ open in } X \implies A(U) \text{ open in } Y.$$

Remark

If A^{-1} exists, then

A is an open mapping $\Leftrightarrow$ A^{-1} is continuous.

Thus if A is a continuous bijection that is an open mapping, then it preserves topologies.

Such a map is called

- a topological isomorphism
- a homeomorphism
- a continuously invertible map

etc.

Open Mapping Theorem

If X, Y are Banach spaces and if $A: X \to Y$ is a continuous, linear, surjection, then A is an open mapping.

Before giving the proof, let us examine some of the consequences of the Open Mapping Theorem.

Corollary: Inverse Mapping Theorem

Let X, Y be Banach spaces. Then

$$A: X \to Y \text{ is a continuous bijection} \implies A^{-1}: Y \to X \text{ is a continuous bijection.}$$

In particular, in this case we have that A is a topological isomorphism, & furthermore,

$$\forall x \in X, \qquad \frac{\|x\|}{\|A^{-1}\|} \leq \|Ax\| \leq \|A\| \, \|x\|.$$

Proof:

Suppose U is an open subset of X. Then by the open mapping theorem, & the fact that A is a bijection,

$$(A^{-1})^{-1}(U) = A(U)$$

is an open subset of Y. Hence the preimage under A^{-1} of an open set is open, which implies that A^{-1} is continuous.

Finally, $\|Ax\| \leq \|A\| \, \|x\|$ follows by definition, & also $\|x\| = \|A^{-1}Ax\| \leq \|A^{-1}\| \, \|Ax\|$ for every $x \in X$. ∎

<u>Exercise</u>

Let X, Y be Banach spaces, and let $A \in \mathcal{B}(X, Y)$.
Show TFAE.

a. $\exists c > 0$ s.t. $\|Ax\| \geq c\|x\|$ $\forall x \in X$.

b. $\ker(A) = \{0\}$ and range(A) is closed in Y.

Thus, if A is a continuous bijection of a Banach space X onto a Banach space Y, then X & Y are topologically isomorphic. In particular,

$$\begin{aligned}
U \text{ open in } X &\iff A(U) \text{ open in } Y. \\
U \text{ open in } Y &\implies A^{-1}(U) \text{ open in } X \\
x_n \to x \text{ in } X &\iff Ax_n \to Ax \text{ in } Y \\
y_n \to y \text{ in } Y &\implies A^{-1}y_n \to A^{-1}y \text{ in } X.
\end{aligned}$$

Corollary

Suppose that X is a normed vector space that is complete (a Banach space) with respect to two norms $\|\cdot\|$ & $|||\cdot|||$. If $\exists C>0$ st.

$$\|x\| \le C\, |||x||| \quad \forall x \in X,$$

then these norms are equivalent, i.e., $\exists c>0$ st.

$$c\, |||x||| \le \|x\| \quad \forall x \in X.$$

Proof

Consider the identity map

$$\begin{aligned}
I : (X, |||\cdot|||) &\longrightarrow (X, \|\cdot\|) \\
x &\longmapsto x.
\end{aligned}$$

By hypothesis,

$$\|I(x)\| \leq C\,|||x|||$$

so I is bounded. But I is a bijection,

so the inverse mapping theorem implies that

$$I^{-1} : (X, \|\cdot\|) \longrightarrow (X, |||\cdot|||)$$

is bounded. Hence

$$|||x||| = |||I^{-1}(x)||| \leq \underbrace{|||I^{-1}|||}_{\text{operator norm of } I^{-1}} \|x\|$$

so we have $c\,|||x||| \leq \|x\| \;\; \forall x \in X$ with $c = \frac{1}{\|I^{-1}\|}$. $\blacksquare$

Note that

$$\|I^{-1}\| = \sup_{\|x\|=1} |||I^{-1}x||| = \sup_{\|x\|=1} |||x|||.$$

Example

Consider $I : (\ell^1, \|\cdot\|_1) \longrightarrow (\ell^1, \|\cdot\|_\infty)$
$$x \longmapsto x$$

We have

$$\|Ix\|_\infty = \|x\|_\infty \leq \|x\|_1 ,$$

so I is bounded & $\|I\| = 1$.

Now consider

$$I^{-1} : (\ell^1, \|\cdot\|_\infty) \longrightarrow (\ell^1, \|\cdot\|_1)$$
$$x \longmapsto x$$

We have for $x_n = (1, \dots, 1, 0, 0, \dots)$ that

$$\|I^{-1}x\|_1 = \|x\|_1 = n \quad \text{but} \quad \|x\|_\infty = 1$$

Hence I^{-1} is unbounded.

Thus, the identity map need not be bounded if we use different norms for the domain & codomain.

Note: $(\ell^1, \|\cdot\|_1)$ is complete
$(\ell^1, \|\cdot\|_\infty)$ is incomplete.

Exercise

Show that $(C[0,1], \|\cdot\|_p)$ is incomplete for $1 \le p < \infty$.

Hint: Show $\|\cdot\|_p$ & $\|\cdot\|_\infty$ are not equivalent norms on $C[0,1]$, & apply a preceding corollary.

Exercises

Let X, Y be normed spaces.

a. If $E \subseteq X$ & $c > 0$, then $\overline{cE} = c\overline{E}$, where

$$cE = \{cx : x \in E\}.$$

b. If $E_1, E_2 \subseteq X$, then $\overline{E_1 + E_2} \subseteq \overline{E_1} + \overline{E_2}$ where

$$E_1 + E_2 = \{x_1 + x_2 : x_1 \in E_1, x_2 \in E_2\}.$$

c. If $A : X \to Y$ & $E_n \subseteq X$, then

$$A(\bigcup E_n) = \bigcup A(E_n)$$

d. If $A : X \to Y$ is linear & $c > 0$, then

$$A(cE) = c\,A(E)$$

The following lemma forms the main part of the proof of the Open Mapping Theorem.

<u>Lemma</u>

Let X, Y be Banach spaces, & let $A: X \to Y$ be a continuous surjection. Then $\exists r > 0$ s.t.

$$B_r^Y(0) \subseteq A(B_1^X(0))$$

<u>Proof:</u>

First we will show that $\overline{A(B_1^X(0))}$ contains an open ball in Y centered at 0. That is, we will show $\exists r > 0$ s.t.

$$B_r^Y(0) \subseteq \overline{A(B_1^X(0))}.$$

To see this, note first that

$$X = \bigcup_{k=1}^{\infty} B_k^X(0).$$

Since A is surjective, this implies that

$$\begin{aligned} Y = A(X) &= A\left(\bigcup_{k=1}^{\infty} B_k^X(0)\right) \\ &= \bigcup_{k=1}^{\infty} A(B_k^X(0)) \\ &\subseteq \bigcup_{k=1}^{\infty} \overline{A(B_k^X(0))} \\ &\subseteq Y. \end{aligned}$$

But Y is <u>complete</u>, so by the Baire Category Theorem, at least one set $\overline{A(B_k^X(0))}$ must contain an open ball. That is, $\exists\, k \in \mathbb{N}$, $\exists\, s > 0$, $\exists\, y \in Y$ s.t.

$$B_s^Y(y) \subseteq \overline{A(B_k^X(0))}.$$

By rescaling & translating, it follows that

$$B^Y_{\frac{s}{2k}}(0) \subseteq \overline{A(B_1^X(0))}.$$

To see this directly, suppose that $z \in B^Y_{\frac{s}{2k}}(0)$, i.e., $\|z\| < \frac{s}{2k}$. Then

$$2kz + y \in B_s^Y(y) \subseteq \overline{A(B_k^X(0))}.$$

Hence $\exists\, x_n \in X$ with $\|x_n\| < k$ such that

$$Ax_n \longrightarrow 2kz + y.$$

Since A is surjective, we know that $y = Ax$ for some $x \in X$. Further, $y \in B_s^Y(y) \subseteq \overline{A(B_k^X(0))}$ so we can assume that $\|x\| \leq k$ (exercise: fill in details).

Hence

$$A\left(\frac{x_n - x}{2k}\right) = \frac{1}{2k} A x_n - \frac{1}{2k} y \longrightarrow z.$$

Since

$$\left\| \frac{x_n - x}{2k} \right\| \leq \frac{\|x_n\| + \|x\|}{2k} < \frac{k + k}{2k} = 1,$$

we conclude that

$$z \in \overline{A(B_1^X(0))}.$$

Thus we have shown that

$$B_r^Y(0) \subseteq \overline{A(B_1^X(0))} \qquad (*)$$

where $r = \delta/2k$.

By rescaling, it follows from $(*)$ that

$$\begin{aligned} B^Y_{r/2^n}(0) &= \frac{1}{2^n} B_r^Y(0) \\ &\subseteq \frac{1}{2^n} \overline{A(B_1^X(0))} \\ &= \overline{A\left(\frac{1}{2^n} B_1^X(0)\right)} \\ &= \overline{A(B^X_{1/2^n}(0))}. \end{aligned}$$

We claim next that

$$B^Y_{r/2}(0) \subseteq A(B^X_1(0)).$$

To see this, suppose $y \in B^Y_{r/2}(0)$, i.e., $\|y\| < \frac{r}{2}$.

Since

$$y \in B^Y_{r/2}(0) \subseteq \overline{A(B^X_{1/2}(0))},$$

$\exists\ x_1 \in B^X_{\frac{1}{2}}(0)$ s.t.

$$\|y - Ax_1\| < \frac{r}{4}.$$

Then

$$y - Ax_1 \in B^Y_{\frac{r}{4}}(0) \subseteq \overline{A(B^X_{\frac{1}{4}}(0))}$$

So $\exists\ x_2 \in B^X_{\frac{1}{4}}(0)$ s.t.

$$\|(y - Ax_1) - Ax_2\| < \frac{r}{8}.$$

Repeating, we obtain $x_n \in B^X_{\frac{1}{2^n}}(0)$ s.t.

$$\left\| y - \sum_{k=1}^{n} Ax_k \right\| < \frac{r}{2^{n+1}}. \qquad (**)$$

Let $z_n = \sum_{k=1}^{n} x_k$. Then by (**),

$$\| y - Az_n \| < \frac{r}{2^{n+1}},$$

so $Az_n \longrightarrow y$. Further,

$$\| z_{n+1} - z_n \| = \| x_{n+1} \| < \frac{1}{2^{n+1}}$$

so by an old exercise, $\{z_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X. But X is complete, so $\exists\, z \in X$ such that $z_n \longrightarrow z$. Further,

$$\| z \| = \lim_{n \to \infty} \| z_n \| \leq \lim_{n \to \infty} \sum_{k=1}^{n} \| x_k \|$$

$$= \| x_1 \| + \lim_{n \to \infty} \sum_{k=2}^{n} \| x_k \|$$

$$< \frac{1}{2} + \sum_{k=2}^{\infty} \frac{1}{2^k}$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1.$$

Thus $z \in B_1^X(0)$, so $Az \in A(B_1^X(0))$.

But we have both $Ax_n \to y$ & $Ax_n \to Az$

(since $x_n \to x$ & A is continuous), so

$y = Az \in A(B_1^X(0))$. Thus, we've shown that

$$B_{r/2}^Y(0) \subseteq A(B_1^X(0)). \quad \blacksquare$$

<u>Remark</u>

By rescaling, it follows under the same hypotheses that

$$\forall r>0, \ \exists s>0 \text{ s.t. } \quad B_s^Y(0) \subseteq A(B_r^X(0)).$$

Open Mapping Theorem

Let X, Y be Banach spaces, &
$A: X \to Y$ a continuous linear surjection.
Then A is an open mapping.

Proof:
Suppose that $U \subseteq X$ is open & choose any point $y \in A(U)$. Then $y = Ax$ for some $x \in X$, so $\exists\, r > 0$ s.t. $B_r^X(x) \subseteq U$.

By the Lemma, $\exists\, s > 0$ s.t. $B_s^Y(0) \subseteq A(B_r^X(0))$.

Hence

$$
\begin{aligned}
B_s^Y(y) = B_s^Y(Ax) &= B_s^Y(0) + Ax \\
&\subseteq A(B_r^X(0)) + Ax \\
&= A(B_r^X(0) + x) \qquad A \text{ linear} \\
&= A(B_r^X(x)) \\
&\subseteq A(U).
\end{aligned}
$$

Hence $A(U)$ is open. ∎

Remark

Recall that we proved that if X, Y are Banach spaces, then

$$X \times Y = \{(x,y) : x \in X, y \in Y\}$$

is a Banach space, under any of the equivalent norms

$$\left(\|x\|_X^p + \|y\|_Y^p \right)^{1/p}, \qquad 1 \leq p < \infty$$

$$\max\{\|x\|_X, \|y\|_Y\}, \qquad p = \infty.$$

Consequently,

$$(x_n, y_n) \longrightarrow (x,y) \text{ in } X \times Y \iff \begin{array}{l} x_n \to x \text{ in } X \\ \& \ y_n \to y \text{ in } Y \end{array}$$

Closed Graph Theorem

Let X, Y be Banach spaces, & assume $A: X \to Y$ is linear.
Define
$$\text{graph}(A) = \{(x, Ax) \in X \times Y : x \in X\}.$$
Then:
$$A \text{ is continuous} \iff \text{graph}(A) \text{ is closed in } X \times Y.$$

Remark
graph(A) is closed if & only if it contains all its limit points. Thus

$$\text{graph}(A) \text{ is closed} \iff \left(\begin{array}{l} x_n \to x \in X \\ \& \ Ax_n \to y \in Y \end{array} \implies y = Ax \right).$$

Proof:

$\Rightarrow$ Suppose that A is continuous. Suppose that $x_n \to x \in X$ & $Ax_n \to y \in Y$. Since A is continuous, $x_n \to x$ implies $Ax_n \to Ax$. By uniqueness of limits, $y = Ax$.

$\Leftarrow$ Suppose that graph(A) is closed in $X \times Y$. Define

$$|||x||| = \|x\|_X + \|Ax\|_Y = \|(x, Ax)\|_{X \times Y}.$$

We claim that $(X, |||\cdot|||)$ is a Banach space.

Exercise: $|||\cdot|||$ is a norm on X.

So, we just have to show that X is complete with respect to $|||\cdot|||$. Suppose that $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy w.r.t. $|||\cdot|||$. Since

$$\|x_m - x_n\|_X \leq |||x_m - x_n|||,$$

we have that $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy w.r.t. $\|\cdot\|_X$. But $(X, \|\cdot\|_X)$ is a Banach space, $\exists\, x \in X$ s.t. $x_n \to x$ w.r.t. $\|\cdot\|_X$, i.e., $\|x - x_n\|_X \to 0$.

But we also have

$$\|Ax_m - Ax_n\|_Y \leq |||x_m - x_n|||,$$

so $\{Ax_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y.
Hence $\exists y \in Y$ s.t. $Ax_n \to y$ w.r.t. $\|\cdot\|_Y$ i.e.,
$\|y - Ax_n\|_Y \to 0$.

By hypothesis, graph(A) is closed, so we have $y = Ax$. Therefore

$$|||x - x_n||| = \|x - x_n\|_X + \|Ax - Ax_n\|_Y$$

$$\longrightarrow 0 \quad \text{as } n \to \infty.$$

Thus $x_n \to x$ w.r.t. $|||\cdot|||$, so $(X, |||\cdot|||)$ is complete.

Thus X is complete w.r.t. both $\|\cdot\|_X$ & $|||\cdot|||$, & also

$$\|x\|_X \leq |||x||| \quad \forall x \in X.$$

A corollary to the Inverse Mapping Theorem therefore implies that $\exists c > 0$ s.t. $c|||x||| \leq \|x\|_X \;\; \forall x \in X$. Hence

$$\|Ax\|_Y \leq |||x||| \leq \frac{1}{c}\|x\|_X,$$

so A is bounded. ∎

This example shows that the assumption in the Closed Graph Theorem that X is a Banach space is necessary.

Example

Consider $\quad D : \left(C^1[0,1], \|\cdot\|_\infty\right) \longrightarrow \left(C[0,1], \|\cdot\|_\infty\right)$

$$f \longmapsto f'$$

Note that $C^1[0,1] = \{ f : [0,1] \to \mathbb{F} : f \text{ is differentiable } \& \ f, f' \in C[0,1] \}$

is not complete under the norm $\|\cdot\|_\infty$. On the other hand, $C[0,1]$ is complete w.r.t. $\|\cdot\|_\infty$. Further we know by a previous exercise that D is <u>unbounded</u>. Yet we claim that D has a closed graph.

To see this, suppose that $f_n \in C'[0,1]$ are such that $f_n \to f$ in L^∞-norm & $Df_n = f_n' \longrightarrow g$ in L^∞-norm. Then since $[0,x]$ is a finite interval, we have by the Uniform Convergence Theorem that

$$\int_0^x f_n'(t)\,dt \longrightarrow \int_0^x g(t)\,dt.$$

But on the other hand, by the Fundamental Theorem of Calculus,

$$\int_0^x f_n'(t)\,dt = f_n(x) - f(0) \longrightarrow f(x) - f(0).$$

Hence

$$f(x) = \int_0^x g(t)\,dt + f(0),$$

so $f \in C^1[0,1]$ and $Df = f' = g$. Therefore D has a closed graph. ∎