# The Hahn-Banach Theorem & Its Consequences

Definition

Let $X$ be a vector space. Then a function $q : X \longrightarrow \mathbb{R}$ is sublinear if

a. $q(x+y) \leq q(x) + q(y) \quad \forall x, y \in X$

b. $q(cx) = c\,q(x) \quad \forall x \in X \;\; \forall c \geq 0.$

Example: Any norm or seminorm on $X$ is sublinear.

The Hahn-Banach Theorems are abstract theorems about sublinear functions (when $\mathbb{F} = \mathbb{R}$) or seminorms on $X$ (when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$).

We will state the general form of the H-B Theorems next. However, it is the corollaries of the H-B Theorems that are useful in practice, so we will explore these corollaries before giving the proof of H-B.

# Hahn-Banach Theorem for $\mathbb{F} = \mathbb{R}$

Let $X$ be a real vector space,
$q : X \to \mathbb{R}$ sublinear,
$M$ a subspace of $X$ (not necessarily closed)

Suppose $\lambda : M \to \mathbb{R}$ is a linear functional, &

$$\langle x, \lambda \rangle \le q(x) \quad \forall x \in M,$$

then $\exists$ linear functional $\Lambda : X \to \mathbb{R}$ s.t.

$$\Lambda|_M = \lambda \quad \& \quad \langle x, \Lambda \rangle \le q(x) \quad \forall x \in X.$$

For the case of complex vector spaces, we will need to impose a stronger condition, namely that $|\langle x, \lambda \rangle|$ is dominated by a seminorm, rather than just a sublinear function.

## Hahn-Banach Theorem for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Let $X$ be a vector space over $\mathbb{F}$
$p: X \to [0,\infty)$ a seminorm on $X$
$M$ a subspace of $X$.

Suppose $\lambda: M \to \mathbb{F}$ is a linear functional, &

$$|\langle x, \lambda \rangle| \leq p(x) \quad \forall x \in M.$$

Then $\exists$ linear functional $\Lambda: X \to \mathbb{F}$ s.t.

$$\Lambda\big|_M = \lambda \quad \& \quad |\langle x, \Lambda \rangle| \leq p(x) \ \forall x \in X.$$

We will give the proofs of the Hahn-Banach Theorems later, but first we examine some of its important implications. In practice, it is these corollaries which are applied, rather than the abstract form of the H-B. Theorem given above. Hence when invoking these corollaries it is customary to write "by the H.-B. Theorem" rather than "by a corollary to the H.-B. Theorem."

For most of these corollaries, the dominating sublinear function is actually a norm.

The first corollary is that any bounded linear functional on a subspace of a normed space has an extension to the full space. This is easy when the space is a Hilbert space (see the following remark), but far from obvious that such an extension should be possible in non-Hilbert spaces.

<u>Corollary</u> (Hahn-Banach)

Let $X$ be a normed linear space,
$M$ a subspace of $X$,
$\lambda \in M^*$ ($\lambda : M \to \mathbb{F}$ is a bounded linear functional).

Then $\exists \Lambda \in X^*$ s.t.

$$\Lambda|_M = \lambda \quad \& \quad \underbrace{\|\Lambda\|}_{\text{operator norm on } X} = \underbrace{\|\lambda\|}_{\text{operator norm on } M.}$$

<u>Proof:</u>

Set $p(x) = \|\lambda\| \, \|x\|$ for $x \in X$.

This is a seminorm (in fact, it is a norm if $\lambda \neq 0$).

Further

$\forall x \in M, \quad |\langle x, \lambda \rangle| \leq \|x\| \|\lambda\| = p(x).$

Hence by Hahn-Banach, $\exists$ linear functional $\Lambda : X \to \mathbb{F}$ s.t.

$$\underbrace{\Lambda\big|_M = \lambda}_{\text{implies } \|\Lambda\| \geq \|\lambda\|} \quad \& \quad \underbrace{|\langle x, \Lambda \rangle| \leq p(x) = \|\lambda\| \|x\| \ \forall x \in X.}_{\text{implies } \|\Lambda\| \leq \|\lambda\|}$$

Hence $\|\Lambda\| = \|\lambda\|$. ▨

Remark

For the Hilbert space case we can give a concrete construction of the functional $\Lambda$.

We are given a linear & continuous functional $\lambda : M \to \mathbb{F}$.

Exercise: Since $M$ is dense on $\overline{M}$, $\exists$ unique extension of $\lambda$ to a continuous linear functional on $\overline{M}$, and this extension has the same operator norm as $\lambda$. We therefore call this extension $\lambda$.

Now we extend to all of $H$ by making use of the fact that $H = \overline{M} \oplus M^\perp$. We declare $\lambda$ to be the zero function on $M^\perp$, i.e., given $z \in H$ we write $z$ uniquely as $z = x + y$ with $x \in \overline{M}$, $y \in M^\perp$, & we define $\Lambda : H \to \mathbb{F}$ by

$$\langle z, \Lambda \rangle = \langle x, \lambda \rangle.$$

In other words,

$$\Lambda = \lambda \circ P_{\bar{M}}$$

where $P_{\bar{M}}$ is the orthogonal projection of $H$ onto $\bar{M}$.

Exercise: Show that $\Lambda|_M = \lambda$ & $\|\Lambda\| = \|\lambda\|$.

It is remarkable that without the existence of any analogue of the orthogonal complement, a similar extension of bounded linear functions from a subspace to the full space is possible in any normed linear space.

Easy to show that the supremum is achieved, take $y = \frac{x}{\|x\|}$.

To motivate the next H.-B. corollary, consider the following easy Hilbert space exercise.

<u>Exercise</u>

If $H$ is a Hilbert space, then

$$\|x\| = \sup_{\|y\|=1} |\langle x,y\rangle| \quad \& \quad \|y\| = \sup_{\|x\|=1} |\langle x,y\rangle|.$$

Recall now that by the Riesz Representation Theorem for Hilbert Spaces, every bounded linear functional on $H$ is given by inner products with some element of $H$, & conversely. That is, $\mu \in H^*$ if & only if $\exists\, y \in H$ s.t. $\langle x,\mu\rangle = \langle x,y\rangle$ $\forall x \in H$. Therefore we can rewrite the conclusions of the preceding exercise as follows.

<u>Lemma</u>

Let $H$ be a Hilbert space.

a. If $x \in H$ then $\|x\| = \sup_{\mu \in H^*, \|\mu\|=1} |\langle x,\mu\rangle|$.

b. If $\mu \in H^*$ then $\|\mu\| = \sup_{x \in H, \|x\|=1} |\langle x,\mu\rangle|$.

Statement b is nothing more than the definition of the operator norms we compute the operator norm by "looking back" at the action of $\mu \in H^*$ on vectors $x \in H$.

Statement a is perhaps more surprising: we can compute the norm of $x \in H$ by "looking forward" at all the possible actions of $\mu \in H^*$ on $x$.

Again the real surprise is that there is an analogue of this for general normed spaces. The analogue of statement b is simply the definition of the operator norm: If $X$ is a normed space & $\mu \in X^*$, then

$$\|\mu\| = \sup_{x \in X,\ \|x\|=1} |\langle x, \mu \rangle|.$$

It is the analogue of statement a that is more interesting & surprisingly useful.

Corollary (Hahn-Banach)
If $X$ is a normed linear space & $x \in X$, then

$$\|x\| = \sup_{\mu \in X^*,\ \|\mu\|=1} |\langle x, \mu \rangle|.$$

Further, the supremum is achieved.

Proof:

Fix $x \in X$, and set $\alpha = \sup_{\mu \in X^*, \|\mu\|=1} |\langle x, \mu \rangle|$.

Then we certainly have

$$\alpha \leq \sup_{\mu \in X^*, \|\mu\|=1} \|x\| \, \|\mu\| = \|\mu\|.$$

Let $M = \text{span}\{x\}$. Define

$$\lambda : M \longrightarrow \mathbb{F}$$
$$cx \longmapsto c\|x\|.$$

Then $\lambda \in M^*$, with

$$\|\lambda\| = \sup_{m \in M, \|m\|=1} |\langle m, \lambda \rangle|$$

$$= \left| \left\langle \frac{x}{\|x\|}, \lambda \right\rangle \right|$$

$$= \frac{1}{\|x\|} \|x\|$$

$$= \underline{1}$$

Therefore, by a previous H.-B. Corollary,

$\exists \Lambda \in X^*$ with $\Lambda|_M = \lambda$ and $\|\Lambda\| = \|\lambda\| = 1$.

In particular, since $x \in M$ we have

$$\langle x, \Lambda \rangle = \langle x, \lambda \rangle = 1\,\|x\| = \|x\|,$$

so

$$\alpha = \sup_{\mu \in X^*, \|\mu\|=1} |\langle x, \mu \rangle|$$

$$\geq |\langle x, \Lambda \rangle|$$

$$= \|x\|.$$

Thus $\alpha = \|x\|$. ∎

<u>Remark / Exercise</u>

Review the notes on reflexive spaces. The above Corollary completes the proof that there is a natural embedding of $X$ into $X^{**}$. If this map is surjective, then we say $X$ is reflexive (and abuse notation by writing $X = X^{**}$).

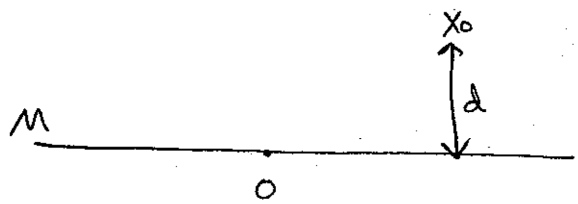
The next result is one of the most powerful & useful corollaries of the Hahn-Banach Theorem.

Corollary (Hahn-Banach).

Suppose $X$ is a normed space,
$M$ is a closed subspace of $X$,
$x_0 \in X \setminus M$
$d = \text{dist}(x_0, M)$ ($>0$ since $M$ is closed).

Then $\exists \mu \in X^*$ s.t.

$$\langle x_0, \mu \rangle = 1$$




$$\mu|_M = 0$$

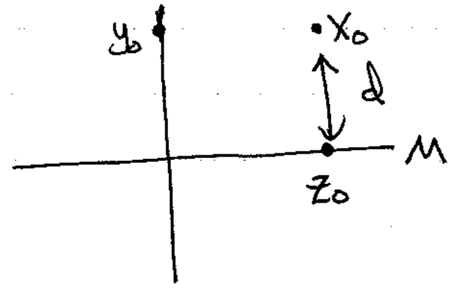
$$\|\mu\| = \frac{1}{d}.$$

Remark
Again, the proof for the case of a Hilbert space $H$ is easy by making use of orthogonal complements.
Suppose $x_0 \notin M$. Then we can write $x_0$ uniquely as

$$x_0 = y_0 + z_0, \quad y_0 \in M^\perp, \; z_0 \in M$$

Note that

$$\|y_0\| = \|x_0 - z_0\| = \text{dist}(x_0, M) = d.$$




Now consider a different orthogonal decomposition of $H$. The space $S = \text{span}\{y_0\}$ is 1-dimensional, hence is closed. Therefore

$$H = S \oplus S^{\perp}.$$

Hence every $x \in H$ can be written uniquely as

$$x = c_x y_0 + z_x, \quad c_x \in \mathbb{C}, \; z_x \in S^{\perp}.$$

Define $\mu : H \to \mathbb{C}$ by

$$\langle x, \mu \rangle = c_x$$

Then $\mu$ is linear, & $\langle x_0, \mu \rangle = 1$. Given $x \in H$, we have by the Pythagorean Theorem that

$$\|x\|^2 = \| c_x y_0 + z_x \|^2$$

$$= |c_x|^2 \, \|y_0\|^2 + \|z_x\|^2$$

$$\geq |\langle x, \mu \rangle|^2 \cdot d^2 + 0$$

Rearranging, we obtain $|\langle x, \mu \rangle| \leq \frac{1}{d} \|x\| \quad \forall x \in H$, so $\|\mu\| \leq \frac{1}{d}$. Also, $y_0/d$ is a unit vector and

$$\left| \left\langle \frac{y_0}{d}, \mu \right\rangle \right| = \frac{1}{d}$$

so $\|\mu\| \geq \frac{1}{d}$. Thus $\mu$ is our desired element of $H^*$.

<u>Proof of the Corollary</u>

Let $\pi : X \longrightarrow X/M$ be the canonical projection.

Then, by definition,

$$\|x_0 + M\|_{X/M} = \mathrm{dist}(x_0, M) = d.$$

Further, by a previous Hahn-Banach corollary,

$$d = \|x_0 + M\|_{X/M} = \sup_{\|\lambda\|_{(X/M)^*} = 1} |\langle x_0 + M, \lambda \rangle|$$

and the supremum is achieved. Hence,

$\exists \lambda \in (X/M)^*$ with $\|\lambda\|_{(X/M)^*} = 1$ and

$$|\langle x_0 + M, \lambda \rangle| = d.$$

Write

$$\langle x_0 + M, \lambda \rangle = c\,|\langle x_0 + M, \lambda\rangle|$$

where $|c| = 1$, & replace $\lambda$ by $\bar{c}\lambda$ to get

$$\langle x_0 + M, \lambda \rangle = d.$$

We now have the following diagram:

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, "\pi"'] & \mathbb{F} \\
X/M \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

Define $\mu = \frac{1}{d}\lambda \circ \pi : X \longrightarrow \mathbb{F}$.

Since $\lambda$ & $\pi$ are both linear & continuous, $\mu$ is also, so $\mu \in X^*$. Further,

$$\begin{aligned}
\langle x_0, \mu \rangle &= \frac{1}{d}\langle \pi(x_0), \lambda \rangle \\
&= \frac{1}{d}\langle x_0 + M, \lambda \rangle \\
&= \underline{1}.
\end{aligned}$$

Additionally, $M = \ker(\pi) \subseteq \ker(\mu)$, so $\mu|_M = 0$.

Finally, the Isomorphism Theorem for normed spaces tells us that, since $\mu = \frac{1}{d}(\lambda \circ \pi)$,

$$\|\mu\| = \frac{1}{d}\|\lambda\| = \frac{1}{d}. \blacksquare$$

The next result is an application of the preceding corollary of Hahn-Banach.

**Theorem**

Let $X$ be a normed linear space. Then

$$X^* \text{ separable} \implies X \text{ separable.}$$

Exercise: The converse fails (e.g., consider $\ell^1$).

**Proof:**

Suppose $X^*$ is separable. Then $X^*$ has a countable dense subset, say $S = \{\mu_n\}_{n \in \mathbb{N}}$.

Setting $\lambda_n = \dfrac{\mu_n}{\|\mu_n\|}$, it follows that $\{\lambda_n\}_{n \in \mathbb{N}}$ is a countable dense subset of the unit sphere

$$D = \{\mu \in X^* : \|\mu\|_{X^*} = 1\}.$$

Since

$$1 = \|\mu_n\|_{X^*} = \sup_{\|x\|=1} |\langle x, \mu_n\rangle|,$$

there must exist an $x_n \in X$ with $\|x_n\| = 1$ s.t.

$$|\langle x_n, \mu_n\rangle| \geq \frac{1}{2}.$$

Define

$$M = \overline{\text{span}}\, \{x_n\}_{n\in\mathbb{N}} \subseteq X.$$

This is a closed subspace of $X$.

Exercise:

$$\left\{ \sum_{n=1}^{N} c_n x_n \ : \ N>0,\ c_n \text{ rational} \right\}$$

is a countable dense subset of $M$, so $M$ is separable.

Suppose $M \neq X$. Then $\exists\, x_0 \in X \setminus M$,

and by rescaling $x_0$ we may assume that

$d = \text{dist}(x_0, M) = 1$. By Hahn-Banach,

$\exists\, \mu \in X^*$ s.t.

$$\mu|_M = 0, \qquad \langle x_0, \mu\rangle = 1, \qquad \|\mu\|_{X^*} = \frac{1}{d} = 1.$$

Hence,

$$\|\mu-\mu_n\|_{X^*} = \sup_{\|x\|=1} |\langle x, \mu-\mu_n\rangle|$$

$$\geq \sup_{n\in\mathbb{N}} |\langle x_n,\mu\rangle - \langle x_n,\mu_n\rangle|$$

$$= \sup_{n\in\mathbb{N}} |0 - \langle x_n,\mu_n\rangle| \qquad \text{since } x_n\in M$$

$$\geq \frac{1}{2}.$$

But this contradicts the fact that $\{\mu_n\}_{n\in\mathbb{N}}$ is dense in $D$ (note that $\mu\in D$). ∎

*Exercise*

Show that if $X$ is a reflexive Banach space, then

$$X \text{ is separable} \iff X^* \text{ is separable.}$$

*Example*

$\ell^1$ is separable but can't be reflexive because $\ell^\infty$ isn't separable.

## Orthogonal Complements in normed spaces

Recall that the orthogonal complement of a subset $S$ of a Hilbert space $H$ is

$$S^{\perp} = \{ g \in H : \langle f, g \rangle = 0 \ \forall f \in S \}.$$

By the Riesz Representation Theorem, each $g \in H$ is uniquely identified with an element $\mu_g \in H^*$. Therefore, we can write the orthogonal complement of $S$ as

$$S^{\perp} = \{ \mu \in H^* : \langle f, \mu \rangle = 0 \ \forall f \in S \},$$

where the equalitity is in the sense of the identification of $H$ with $H^*$.

This suggests the following definition of the orthogonal complement of a subset of an arbitrary normed space. Note, however, that when $X$ is not a Hilbert space, a set and its orthogonal complement belong to <u>different</u> spaces: $S \subseteq X$ while $S^{\perp} \subseteq X^*$. Consequently, although we can define an orthogonal complement, it does not imply a decomposition of $X$ analogous to the decomposition $H = M \oplus M^{\perp}$ for $M$ a closed subspace of a Hilbert space.

However, the orthogonal complement will still be a useful tool for certain results.

Definition

If $S$ is a subset of a normed space $X$, then the orthogonal complement of $S$ is the following subset of $X^*$:

$$S^\perp = \{\mu \in X^* : \forall x \in X, \ \langle x, \mu \rangle = 0\}.$$

Exercise: $S^\perp$ is a closed subspace of $X^*$.

Corollary

Let $M$ be a subspace of a normed space $X$. Then

$$M \text{ is dense in } X \iff M^\perp = \{0\}.$$

Proof:

$\Leftarrow$ Suppose that $M$ was not dense in $X$. Then $\bar{M}$ is a proper subspace of $X$, so we can find some vector $x_0 \in X \setminus \bar{M}$. Then by Hahn-Banach, $\exists \mu \in X^*$ s.t. $\mu|_{\bar{M}} = 0$ & $\langle x_0, \mu \rangle = 1$. Therefore $\mu \in M^\perp$ but $\mu \neq 0$, so $M^\perp \neq \{0\}$.

$\Rightarrow$ Exercise. ∎

<u>Exercise</u>

Even if $S \subseteq X$ is not a subspace, the condition $S^{\perp} = \{0\}$ corresponds to an important property, namely that $S$ is complete, i.e., its finite linear span is dense. Prove that

$$S \text{ is complete in } X \iff S^{\perp} = \{0\}.$$

# Comparisons

| Hilbert Space | Banach space |
|---|---|
| $x \in H$ | $x \in X$ |
| inner product $\langle x,y\rangle \quad x,y \in H$ | linear functionals $f(x), \quad x \in X,\ f \in X^*$ $= \langle x, f\rangle$ |
| Cauchy-Schwarz $\lvert\langle x,y\rangle\rvert \le \lVert x\rVert\,\lVert y\rVert$ | operator norm $\lvert f(x)\rvert \le \lVert f\rVert_{X^*} \lVert x\rVert_X$ |
| $\lVert y\rVert = \sup\limits_{\lVert x\rVert=1} \lvert\langle x,y\rangle\rvert$ | $\lVert f\rVert_{X^*} = \sup\limits_{\lVert x\rVert=1} \lvert f(x)\rvert$ |
| $\lVert x\rVert = \sup\limits_{\lVert y\rVert=1} \lvert\langle x,y\rangle\rvert$ | $\lVert x\rVert_X = \sup\limits_{\lVert f\rVert=1} \lvert f(x)\rvert$ |
| <u>Subspaces</u> M closed: | <u>Subspaces</u> M closed: |
| $M^\perp = \{y \in M : \langle x,y\rangle = 0\ \forall x \in M\}$ $\subseteq H$ | $M^\perp = \{f \in X^* : f(x) = 0\ \forall x \in M\}$ $\subseteq X^*$ |
| $M^\perp \cong H/M$ | (next section) |
| $H = M \oplus M^\perp$ | No analogue |

# EXERCISES

*Exercise* (See Conway, p. 92-93)

Let $M$ be a closed subspace of a Banach space $X$.

Let $\rho_X : X \to X^{**}$ & $\rho_M : M \to M^{**}$ be the natural maps.

Let $i : M \to X$ be the inclusion map, $i(x) = x \ \forall x \in M$.

Show $\exists$ isometry $\phi : M^{**} \to X^{**}$ s.t.

$$\rho_X \circ i = \phi \circ \rho_M .$$

$$\begin{tikzcd} X \arrow[r, "\rho_X"] & X^{**} \\ M \arrow[u, "i"] \arrow[r, "\rho_M"'] & M^{**} \arrow[u, dashed, "\exists \phi"'] \end{tikzcd}$$

Prove further that $\phi(M^{**}) = (M^{\perp})_{\perp}$.

(Do this exercise after we have covered Hahn-Banach.)

*Exercise*

Use the preceding exercise to show that if $X$ is reflexive, then any closed subspace of $X$ is reflexive.

Exercise

Let $X$ be a Banach space, & let $x_1, \dots, x_n$ be finitely many linearly independent vectors in $X$. Given $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, show $\exists \mu \in X^*$ s.t. $\langle x_k, \mu \rangle = \alpha_k$ for $k = 1, \dots, n$.

*Exercise*

Let $M$ be a subspace of a normed space $X$. Show that

$$\overline{M} = \bigcap \{ \ker(\mu) : \mu \in X^* \ \& \ \mu|_M = 0 \}$$

Note that $\mu|_M = 0$ implies $M \subseteq \ker(\mu)$.

*Exercise*

Suppose that $\mu \in C[0,1]^*$ (i.e., $\mu$ is a complex Radon measure on $[0,1]$). Show that

$$\langle x^n, \mu \rangle = 0 \ \forall n \geq 0 \implies \mu = 0.$$

Hint: Weierstrass Approximation Theorem.

*Exercise*

Let $X$ be a normed space. Show that

$$\langle x, \mu \rangle = 0 \ \forall \mu \in X^* \iff x = 0.$$

Definition

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a Banach space $X$.

a. $\{x_n\}_{n\in\mathbb{N}}$ is minimal if no $x_m$ lies in the closed span of the other $x_n$, i.e.,

$$\forall m\in\mathbb{N}, \quad x_m \notin \overline{\text{span}}\{x_n\}_{n\neq m}.$$

b. A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is biorthogonal to $\{x_n\}_{n\in\mathbb{N}}$ if

$$\langle x_m, \mu_n\rangle = \delta_{mn} = \begin{cases} 1, & m=n, \\ 0, & m\neq n. \end{cases}$$

Exercise

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a Banach space $X$.
Prove TFAE:

a. $\{x_n\}_{n\in\mathbb{N}}$ is minimal.

b. $\exists\ \{\mu_n\}_{n\in\mathbb{N}}$ biorthogonal to $\{x_n\}_{n\in\mathbb{N}}$.

Also prove that TFAE:

a'. $\{x_n\}_{n\in\mathbb{N}}$ is minimal & complete.

b'. $\exists$ unique sequence $\{\mu_n\}_{n\in\mathbb{N}}$ biorthogonal to $\{x_n\}_{n\in\mathbb{N}}$.

Exercise

Let $x_1, \dots, x_n \in \mathbb{C}^d$ be given. Prove TFAE:

a. $x_1, \dots, x_n$ are linearly independent.

b. $x_1, \dots, x_n$ are minimal.

Exercise

Let $\{x_1, \dots, x_d\}$ be a basis for $\mathbb{C}^d$. Show

$\exists$ unique sequence $\{y_1, \dots, y_d\}$ biorthogonal to $\{x_1, \dots, x_d\}$,

this sequence is itself a basis for $\mathbb{C}^d$, &

$$\forall x \in \mathbb{C}^d, \qquad x = \sum_{k=1}^{d} (x \cdot y_k)\, x_k = \sum_{k=1}^{d} (x \cdot x_k)\, y_k.$$

Remark

These simple equivalences do not extend to infinite-dimensional spaces! In general,

$$\text{minimal} \underset{\not\Leftarrow}{\Rightarrow} \text{finitely independent}$$

$$\text{minimal + complete} \underset{\Leftarrow}{\not\Rightarrow} \text{Schauder basis}$$

Exercise: Fix $1 < p < \infty$.
Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in a Banach space $X$.
Suppose that

$$T : X^* \longrightarrow \ell^p$$
$$\mu \longmapsto \{\langle x_n, \mu \rangle\}_{n \in \mathbb{N}}$$

is a bounded map of $X^*$ into $\ell^p$. Show that

$$\forall\, (c_n)_{n \in \mathbb{N}} \in \ell^{p'}, \quad \sum_{n=1}^{\infty} c_n x_n \text{ converges unconditionally in } X.$$

Q. What if $p = 1$ or $p = \infty$, is the result still valid?

Remark

In fact, by using the Uniform Boundedness Principle or the Closed Graph Theorem, the hypothesis that $T(\mu) = \{\langle x_n, \mu \rangle\}_{n \in \mathbb{N}}$ maps $X^*$ into $\ell^p$ implies that $T$ must be bounded, so this is actually not needed as a hypothesis in the exercise.

## Remark

The special case where $X$ is a Hilbert space and $p=2$ occurs often and has a special name.

## Definition

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space $H$ is a <u>Bessel sequence</u> if

$$\forall x \in H, \quad \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 < \infty. \qquad (*)$$

Again, by using either the Uniform Boundedness Principle or the Closed Graph Theorem, hypothesis $(*)$ is equivalent to the hypothesis that $\exists\, B>0$ s.t.

$$\forall x \in H, \quad \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2.$$

A subclass of the Bessel sequences is especially useful.

## Definition

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space $H$ is a <u>frame</u> for $H$ if $\exists\, A, B>0$ s.t.

$$\forall x \in H, \quad A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2.$$