## C.6 Adjoints for Operators on a Hilbert Space

If $A$ is an $m \times n$ complex matrix and $x \cdot y$ is the ordinary dot product on $\mathbb{C}^d$, then

$$Ax \cdot y \;=\; x \cdot A^* y, \qquad x \in \mathbb{C}^m,\; y \in \mathbb{C}^n,$$

where $A^* = \overline{A^{\mathrm{T}}}$ is the *Hermitian*, or conjugate transpose, of $A$.

As an application of the Riesz Representation Theorem, we will show that there is an analogue of the Hermitian matrix for linear operators on Hilbert spaces. In Section **??** we will see that this extends further to operators on Banach spaces, but in that setting we need to appeal to the Hahn–Banach Theorem in order to construct the adjoint.

Throughout this section, we will let $H$ and $K$ denote Hilbert spaces.

### C.6.1 Definition and Basic Properties

**Exercise C.43.** Let $L\colon H \to K$ be a bounded linear operator.

(a) For each $g \in K$, define a functional $\mu_g\colon H \to \mathbb{C}$ by

$$\langle f, \mu_g \rangle \;=\; \langle Lf, g \rangle, \qquad f \in H.$$

Note that, following Notation C.40, $\langle f, \mu_g \rangle$ denotes the action of the functional $\mu_g$ on the vector $f$, while $\langle Lf, g \rangle$ denotes the inner product of the vectors $Lf,\, g \in K$. Show that $\mu_g \in H^*$ and conclude that there exists a unique element $g^* \in H$ such that

$$\langle f, \mu_g \rangle \;=\; \langle f, g^* \rangle, \qquad f \in H.$$

(b) Define $L^*\colon K \to H$ by $L^*g = g^*$. Show that $L^*$ is a bounded linear map, that $(L^*)^* = L$, and that $\|L^*\| = \|L\|$.

We formalize this as a definition.

**Definition C.44 (Adjoint).** The *adjoint* of $L \in \mathcal{B}(H,K)$ is the unique operator $L^*\colon K \to H$ that satisfies

$$\langle Lf, g \rangle \;=\; \langle f, L^*g \rangle, \qquad f \in H,\; g \in K.$$

When $H = K$, we have the following additional terminology.

**Definition C.45 (Self-Adjoint and Normal Operators).** Let $A \in \mathcal{B}(H)$ be given.

(a) We say that $L$ is *self-adjoint* or *Hermitian* if $L = L^*$. Equivalently, $L$ is self-adjoint if

$$\forall\, f, g \in H, \quad \langle Lf, g \rangle \;=\; \langle f, Lg \rangle.$$

(b) We say that $L$ is *normal* if $L$ commutes with its adjoint, i.e., if $LL^* = L^*L$.

All self-adjoint operators are normal, but not all normal operators are self-adjoint, compare Problem C.21.

*Remark C.46.* Each complex $m \times n$ matrix $A$ determines a linear map of $\mathbb{C}^n$ to $\mathbb{C}^m$. The adjoint of this map corresponds to the conjugate transpose of $A$:

$$A^* = \overline{A^{\mathbb{T}}},$$

which is called the *Hermitian of* $A$ (sometimes denoted by $A^{\mathrm{H}}$).

Thus, in the language used so far, a matrix $A$ is self-adjoint if $A = A^*$. However, for matrices it is customary to instead say that $A$ *is Hermitian* if $A = A^*$.

Note that if $L \in \mathcal{B}(H, K)$, then $L^*$ is a linear map of $H$ into $K$, but the mapping of $\mathcal{B}(H, K)$ to $\mathcal{B}(K, H)$ given by $L \mapsto L^*$ is antilinear:

$$(\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*.$$

Exercise C.43 also shows that this map is an isometry.

We will need some facts about the relationship between invariant subspaces and adjoints.

**Definition C.47.** Let $A \in \mathcal{B}(H)$ be given. Then a closed subspace $M \subseteq H$ is *invariant under* $A$ if

$$A(M) \subseteq M,$$

where $A(M) = \{Ax : x \in M\}$.

Note that it we do not require that $A(M)$ be equal to $M$.

The simplest example of an invariant subspace is $M = \operatorname{span}\{f\}$, where $f$ is an eigenvector of $A$.

**Exercise C.48.** Show that if a closed subspace $M \subseteq H$ is invariant under $A \in \mathcal{B}(H)$, then $M^\perp$ is invariant under $A^*$.

### C.6.2 Adjoints of Unbounded Operators

Adjoints can also be defined for unbounded operators, although now we must be careful with domains. For example, consider the differentiation operator $Df = f'$. This operator is not defined on all of $L^2(\mathbb{R})$, but instead is densely defined in the sense of Notation C.3. For example, $D$ maps the dense subspace

$$S = \bigl\{f \in C_0^1(\mathbb{R}) : f, f' \in L^2(\mathbb{R})\bigr\}$$

into $L^2(\mathbb{R})$. Although $D\colon S \to L^2(\mathbb{R})$ is unbounded, if $f$, $g \in S$ then we have $fg \in C_0(\mathbb{R})$, so integration by parts yields

$$\langle Df, g\rangle = \int f'(x)\, g(x)\, dx = -\int f(x)\, g'(x)\, dx = -\langle f, Dg\rangle.$$

Hence $D^* = -D$ as an operator mapping $S$ into $L^2(\mathbb{R})$.

The important point in the preceding example is that if $g \in S$ is fixed, then $f \mapsto \langle Df, g\rangle$ is actually a bounded linear functional on $S$, even though $D$ is an unbounded operator. The following exercise extends this to general operators.

**Exercise C.49.** Suppose that $S$ is a dense subspace of $H$, and $L\colon S \to K$ is a linear, but not necessarily bounded, operator. Let

$$S^* \;=\; \{g \in K : f \mapsto \langle Lf, g\rangle \text{ is a bounded linear functional on } S\}.$$

Show that there is an operator $L^*\colon S^* \to H$ such that

$$\langle Lf, g\rangle \;=\; \langle f, L^*g\rangle, \qquad f \in S,\; g \in S^*.$$

If necessary, we can always restrict a densely defined operator to a smaller but still dense domain. Given an operator $L$ mapping some dense subspace of $H$ into $H$, if we can find some dense subspace $S$ on which $L$ is defined and such that

$$\langle Lf, g\rangle \;=\; \langle f, Lg\rangle, \qquad f, g \in S,$$

then we say that $L$ is *self-adjoint.*

### C.6.3 Bounded Self-Adjoint Operators on Hilbert Spaces

We now focus in more detail on bounded self-adjoint operators on Hilbert spaces, which have many useful properties and appear often throughout this volume.

In this volume, unless specifically stated otherwise we always assume that a given vector space is to be taken over the complex field. Thus, unless otherwise stated, every Hilbert space in this volume is a complex Hilbert space. For complex Hilbert spaces, we have the following characterization of self-adjoint operators (this does not not hold for real Hilbert spaces).

**Theorem C.50.** *Let $H$ be a Hilbert space (implicitly complex), and let $A \in \mathcal{B}(H)$ be given. Then:*

$$A \text{ is self-adjoint} \quad\Longleftrightarrow\quad \langle Af, f\rangle \in \mathbb{R} \quad \forall\, f \in H.$$

*Proof.* $\Rightarrow$. Assume $A = A^*$. Then for any $f \in H$ we have

$$\overline{\langle Af, f\rangle} \;=\; \langle f, Af\rangle \;=\; \langle A^*f, f\rangle \;=\; \langle Af, f\rangle,$$

so $\langle Af, f\rangle$ is real.

$\Leftarrow$. Assume that $\langle Af, f\rangle$ is real for all $f$. Choose any $f$, $g \in H$. Then

$$\langle A(f+g), f+g\rangle \;=\; \langle Af, f\rangle + \langle Af, g\rangle + \langle Ag, f\rangle + \langle Ag, g\rangle.$$

Since $\langle A(f+g), f+g\rangle$, $\langle Af, f\rangle$, and $\langle Ag, g\rangle$ are all real, we conclude that $\langle Af, g\rangle + \langle Ag, f\rangle$ is real. Hence it equals its own complex conjugate:

$$\langle Af, g\rangle + \langle Ag, f\rangle \;=\; \overline{\langle Af, g\rangle + \langle Ag, f\rangle} \;=\; \langle g, Af\rangle + \langle f, Ag\rangle. \tag{C.11}$$

Similarly, after examining the equation

$$\langle A(f+ig), f+ig\rangle \;=\; \langle Af, f\rangle - i\langle Af, g\rangle + i\langle Ag, f\rangle + \langle Ag, g\rangle,$$

we conclude that

$$\langle Af, g\rangle - \langle Ag, f\rangle \;=\; -\langle g, Af\rangle + \langle f, Ag\rangle. \tag{C.12}$$

Adding (C.11) and (C.12) together, we obtain

$$2\langle Af, g\rangle \;=\; 2\langle f, Ag\rangle \;=\; 2\langle A^*f, g\rangle.$$

Since this is true for every $f$ and $g$, we conclude that $A = A^*$. □

The next result gives us an alternative formula for the operator norm of a self-adjoint operator.

**Theorem C.51.** *If $A \in \mathcal{B}(H)$ is self-adjoint, then*

$$\|A\| \;=\; \sup_{\|f\|=1} |\langle Af, f\rangle|.$$

*Proof.* Set $M = \sup_{\|f\|=1} |\langle Af, f\rangle|$. By Cauchy–Bunyakowski–Schwarz and the definition of operator norm, it follows that $M \le \|A\|$.

Choose any unit vectors $f, g \in H$. Then, by expanding the inner products, canceling terms, and using the fact that $A = A^*$, we see that

$$\begin{aligned}\big\langle A(f+g),\, f+g\big\rangle - \big\langle A(f-g),\, f-g\big\rangle \;&=\; 2\,\langle Af, g\rangle + 2\,\langle Ag, f\rangle \\ &=\; 2\,\langle Af, g\rangle + 2\,\langle g, Af\rangle \\ &=\; 4\,\mathrm{Re}\,\langle Af, g\rangle.\end{aligned}$$

Therefore, applying the definition of $M$ and using the Parallelogram Law, we have

$$\begin{aligned}4\,\mathrm{Re}\,\langle Af, g\rangle \;&\le\; |\langle A(f+g), f+g\rangle| + |\langle A(f-g), f-g\rangle| \\ &\le\; M\,\|f+g\|^2 + M\,\|f-g\|^2 \\ &=\; 2M\left(\|f\|^2 + \|g\|^2\right) \;=\; 4M.\end{aligned}$$

That is, $\mathrm{Re}\,\langle Af, g\rangle \le M$ for every choice of unit vectors $f$ and $g$. Write $|\langle Af, g\rangle| = \alpha\,\langle Af, g\rangle$ where $\alpha \in \mathbb{C}$ satisfies $|\alpha| = 1$. Then $\bar{\alpha}g$ is another unit vector, so

$$|\langle Af, g\rangle| \;=\; \alpha\langle Af, g\rangle \;=\; \langle Af, \bar{\alpha}g\rangle \;\le\; M.$$

Hence

$$\|Af\| \;=\; \sup_{\|g\|=1} |\langle Af, g\rangle| \;\le\; M,$$

and therefore $\|A\| \le M$. □

As a corollary, we obtain a useful fact for an operator on a complex Hilbert space (for an operator on a real Hilbert space, the assumption that $A = A^*$ must be added).

**Exercise C.52.** If $A \in \mathcal{B}(H)$ and $\langle Af, f\rangle = 0$ for every $f$, then $A = 0$.

### C.6.4 Positive and Positive Definite Operators on Hilbert Spaces

Among the self-adjoint operators, we distinguish the special class of positive operators.

**Definition C.53 (Positive and Positive Definite Operators).** Let $A \in \mathcal{B}(H)$ be given.

(a) We say that $A$ is *positive* or *nonnegative*, denoted $A \geq 0$, if $\langle Af, f\rangle \geq 0$ for every $f \in H$.

(b) We say that $L$ is *positive definite* or *strictly positive*, denoted $A > 0$, if $\langle Af, f\rangle > 0$ for every nonzero vector $f \in H$.

The term *nonnegative operator* is perhaps more correct than *positive operator*, but we will use the latter terminology.

By Theorem C.50, since we are dealing with complex Hilbert spaces, all positive and positive definite operators are self-adjoint. When dealing with real Hilbert spaces, the assumption of self-adjointness should be added in the definition of positive and positive definite operators.

The next exercise gives an important example of positive operators.

**Exercise C.54.** Show that if $A \in \mathcal{B}(H,K)$, then $A^*A \in \mathcal{B}(H)$ and $AA^* \in \mathcal{B}(K)$ are both positive operators. Determine conditions on $A$ that imply that $A^*A$ or $AA^*$ is positive definite.

### Additional Problems

**C.14.** Let $H_1$, $H_2$, $H_3$ be Hilbert spaces. Show that if $A \in \mathcal{B}(H_1, H_2)$ and $B \in \mathcal{B}(H_2, H_3)$, then $(A^*)^* = A$ and $(BA)^* = A^*B^*$.

**C.15.** Show that $A \mapsto A^*$ defines an involution on $\mathcal{B}(H)$ in the sense of Definition C.36.

**C.16.** Show that if $A \in \mathcal{B}(H,K)$ is a topological isomorphism then $A^* \in \mathcal{B}(K,H)$ is also a topological isomorphism, and $(A^{-1})^* = (A^*)^{-1}$.

**C.17.** Show that if $A \in \mathcal{B}(H,K)$, then $\|A\| = \|A^*\| = \|A^*A\|^{1/2} = \|AA^*\|^{1/2}$.

Remark: In the language of operator theory, the fact that $\|A^*A\| = \|A\|^2$ means that $\mathcal{B}(H)$ is an example of a $C^*$-*algebra*.

**C.18.** Given $A \in \mathcal{B}(H)$, show that $A$ is normal if and only if $\|Af\| = \|A^*f\|$ for every $f \in H$.

**C.19.** Show that if $A \in \mathcal{B}(H, K)$ then the following statements hold.

(a) $\ker(A) = \mathrm{range}(A^*)^\perp$.

(b) $\ker(A)^\perp = \overline{\mathrm{range}(A^*)}$.

(c) $A$ is injective if and only if $\mathrm{range}(A^*)$ is dense in $H$.

**C.20.** Given $A \in \mathcal{B}(H)$, show that

$$\ker(A) = \ker(A^*A) \qquad \text{and} \qquad \overline{\mathrm{range}(A^*A)} = \overline{\mathrm{range}(A^*)}.$$

**C.21.** Fix $\lambda \in \ell^\infty$, and let $M_\lambda$ be the multiplication operator defined in Exercise C.14. Find $M_\lambda^*$, and show that $M_\lambda$ is normal. Determine when $M_\lambda$ is self-adjoint, positive, or positive definite.

**C.22.** Fix $\phi \in L^\infty(\mathbb{R})$, and let $M_\phi$ be the multiplication operator defined in Exercise C.15. Find $M_\phi^*$, and show that $M_\phi$ is normal. Determine when $M_\phi$ is self-adjoint, positive, or positive definite.

**C.23.** Given $k \in L^2(\mathbb{R}^2)$, let $L_k$ be the corresponding integral operator. Show that the adjoint operator $(L_k)^*$ is the integral operator $L_{k^*}$ whose kernel is $k^*(x,y) = \overline{k(y,x)}$.

**C.24.** Let $L$ and $R$ be the left- and right-shift operators from Problem C.4. Show that $R = L^*$, and conclude that $L$ and $R$ are not normal.

**C.25.** Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for a separable Hilbert space $H$. Let $T\colon H \to \ell^2(\mathbb{N})$ be the *analysis operator* $Tf = \{\langle f, e_n\rangle\}_{n\in\mathbb{N}}$. Find a formula for the *synthesis operator* $T^*\colon \ell^2(\mathbb{N}) \to H$.

**C.26.** Show that if $U\colon H \to H$ is unitary, then $U^* = U^{-1}$, $U$ is normal, and any eigenvalue $\lambda$ of $U$ satisfies $|\lambda| = 1$.

**C.27.** Let $M$ be a closed subspace of $H$, and let $P \in \mathcal{B}(H)$ be given. Show that $P$ is the orthogonal projection of $H$ onto $M$ if and only if $P^2 = P$, $P^* = P$, and $\mathrm{range}(P) = M$.

**C.28.** Let $L\colon H \to H$ be a self-adjoint operator on $H$, either bounded or unbounded and densely defined.

(a) Show that all eigenvalues of $L$ are real.

(b) Show that eigenvectors of $L$ corresponding to distinct eigenvalues are orthogonal.

**C.29.** Let $L \in \mathcal{B}(H)$ be normal.

(a) Show that if $\lambda$ is an eigenvalue of $L$, then $\bar{\lambda}$ is an eigenvalue of $L^*$, with the same eigenvector as $L$.

(b) Show that eigenvectors of $L$ corresponding to distinct eigenvalues are orthogonal.

**C.30.** Show that if $A \in \mathcal{B}(H)$, then $A + A^*$, $AA^*$, $A^*A$, and $AA^* - A^*A$ are all self-adjoint (in fact, $AA^*$ and $A^*A$ are positive).

**C.31.** Show that if $A$, $B \in \mathcal{B}(H)$ are self-adjoint, then $ABA$, and $BAB$ are self-adjoint. Show that $AB$ is self-adjoint if and only if $AB = BA$, and exhibit self-adjoint operators $A$, $B$ that do not commute.

**C.32.** Let $A \in \mathcal{B}(H)$ be given.

(a) Show that if $A \in \mathcal{B}(H)$ is a positive operator, then all eigenvalues of $A$ are real and nonnegative.

(b) Show that if $A \in \mathcal{B}(H)$ is a positive definite operator, then all eigenvalues of $A$ are real and strictly positive.