Orthogonal Projections & Multiplication Operators

C.2 Some Useful Operators

In this section we give some examples of operators that will be useful to us.

C.2.1 Orthogonal Projections

We begin with orthogonal projections in Hilbert spaces.

Definition C.12 (Orthogonal Projection). Let M be a closed subspace of a Hilbert space H. Define $P\colon H \to H$ by $Ph = p$, where p is the orthogonal projection of h onto M, (see Definition A.92). The operator P is the *orthogonal projection of H onto M*.

Exercise C.13 (Properties of Orthogonal Projections). Let $M \neq \{0\}$ be a closed subspace of a Hilbert space H, and let P be the orthogonal projection of H onto M. Show that the following statements hold.

(a) If $h \in H$ then Ph is the unique vector in M such that $h - Ph \in M^{\perp}$.

(b) $\|h - Ph\| = \operatorname{dist}(h, M)$ for every $h \in H$.

(c) P is linear, $\|Ph\| \leq \|h\|$ for every $h \in H$, and $\|P\| = 1$.

(d) P is *idempotent*, i.e., $P^2 = P$.

(e) $\ker(P) = M^{\perp}$ and $\operatorname{range}(P) = M$.

(f) $I - P$ is the orthogonal projection of H onto $M^{\perp}$.

A characterization of orthogonal projections is given in Problem C.27.

C.2.2 Multiplication Operators

Next we consider two types of multiplication operators. The first type multiplies each term in an orthonormal basis expansion by a fixed scalar.

Exercise C.14. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for a separable Hilbert space H. Then by Exercise A.97, we know that every $f \in H$ can be written

$$f = \sum_{n=1}^{\infty} \langle f, e_n \rangle \, e_n.$$

Fix any sequence of scalars $\lambda = (\lambda_n)_{n\in\mathbb{N}}$. For those $f \in H$ for which the following series converges, define

$$M_\lambda f = \sum_{n=1}^{\infty} \lambda_n \, \langle f, e_n \rangle \, e_n. \tag{C.2}$$

Prove the following facts.

(a) The series defining $M_\lambda f$ in (C.2) converges for every $f \in H$ if and only if $\lambda \in \ell^\infty$. In this case M_λ is a bounded linear mapping of H into itself, and $\|M_\lambda\| = \|\lambda\|_\infty$.

(b) If $\lambda \notin \ell^\infty$, then M_λ defines an unbounded linear mapping from the domain

$$\operatorname{domain}(M_\lambda) \;=\; \Big\{f \in H : \sum_{n=1}^{\infty} |\lambda_n \langle f, e_n\rangle|^2 < \infty\Big\} \tag{C.3}$$

into H. Note that domain(M_λ) contains the finite span of $\{e_n\}_{n\in\mathbb{N}}$, and hence is dense in H.

If $H = \ell^2$ and $\{e_n\}_{n\in\mathbb{N}}$ is the standard basis for ℓ^2, then the multiplication operator M_λ defined in equation (C.2) is simply the "componentwise multiplication" given by

$$M_\lambda x \;=\; \lambda x \;=\; (\lambda_1 x_1, \lambda_2 x_2, \dots), \qquad x = (x_1, x_2, \dots) \in \ell^2.$$

This is a discrete version of the multiplication operator defined in the next exercise.

Exercise C.15. Let $\phi\colon \mathbb{R} \to \mathbb{C}$ and $1 \le p \le \infty$ be given.

(a) Show that if $\phi \in L^\infty(\mathbb{R})$, then $M_\phi f = f\phi$ is a bounded mapping of $L^p(\mathbb{R})$ itself, and $\|M_\phi\| = \|\phi\|_\infty$.

(b) Conversely, show that if $f\phi \in L^p(\mathbb{R})$ for every $f \in L^p(\mathbb{R})$, then we must have $\phi \in L^\infty(\mathbb{R})$.

Hints

Exercises from Appendix C

C.14 Hint: (b) If $\lambda \notin \ell^\infty$, then there exists a subsequence $(\lambda_{n_k})_{k\in\mathbb{N}}$ such that $|\lambda_{n_k}| \geq k$ for each k. Let $c_{n_k} = 1/k$ and define all other c_n to be zero. Show that $f = \sum c_n e_n$ converges but $Lf = \sum \lambda_n c_n e_n$ does not converge.

C.15 Hints: (a) For the case $p < \infty$, to show the inequality $\|M_\phi\| \leq \|\phi\|_\infty$, choose any $\varepsilon > 0$. Then $E = \{|\phi| > \|\phi\|_\infty - \varepsilon\}$ has positive measure, so some set $E_k = E \cap [k, k+1]$ must have positive (and finite) measure. Consider $f = |E_k|^{-1/p}\chi_{E_k}$.

(b) Suppose that $p < \infty$, and assume $\phi \notin L^\infty(\mathbb{R})$. The sets

$$E_k = \{k \leq |\phi| < k+1\}.$$

are measurable and disjoint, and since ϕ is not in $L^\infty(X)$ there must be infinitely many E_k with positive measure, say E_{n_k} for $k \in \mathbb{N}$. Choose $F_k \subseteq E_{n_k}$ with $0 < |F_k| < \infty$, and consider $f(x) = n_k\,|F_k|^{-1/p}$ for $x \in F_k$, and $f(x) = 0$ otherwise.

Additional Problems

C.10. Choose $\lambda \in \ell^\infty$, and set $\delta = \inf_n |\lambda_n|$. Define M_λ as in Exercise C.14, and prove the following.

(a) Each λ_n is an eigenvalue for M_λ with corresponding eigenvector e_n.

(b) M_λ is injective if and only if $\lambda_n \neq 0$ for every n.

(c) M_λ is surjective if and only if $\delta > 0$.

(d) If $\delta = 0$ but $\lambda_n \neq 0$ for every n then range(M_λ) is a dense but proper subspace of H.

(e) M_λ is unitary if and only if $|\lambda_n| = 1$ for every n.

C.11. Let $\phi \in L^\infty(\mathbb{R})$ be fixed, and let M_ϕ be defined as in Exercise C.15. Fix $1 \leq p \leq \infty$.

(a) (a) Determine a necessary and sufficient condition on ϕ that implies that $M_\phi\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is injective.

(b) Determine a necessary and sufficient condition on ϕ that implies that $M_\phi\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is surjective.

(c) Show directly that if M_ϕ is injective but not surjective then the inverse mapping $M_\phi^{-1}\colon \text{range}(M_\phi) \to L^p(\mathbb{R})$ is unbounded.