A.10 Complete Sequences

In this section we define complete sequences of vectors in normed spaces. In finite dimensions, these are simply spanning sets. However, in infinite dimensions there is a subtle, but important, distinction between a complete set and a spanning set.

Definition A.71 (Span). Let A be a subset of a normed linear space X. The *finite linear span* of A, denoted $\mathrm{span}(A)$, is the set of all finite linear combinations of elements of A:

$$\mathrm{span}(A) \;=\; \left\{\sum_{k=1}^{n} \alpha_k f_k : n > 0,\, f_k \in A,\, \alpha_k \in \mathbb{C}\right\}.$$

We also refer to the finite linear span of A as the *finite span*, the *linear span*, or simply the *span* of A.

In particular, if A is a countable sequence, say $A = \{f_k\}_{k\in\mathbf{N}}$, then

$$\mathrm{span}(\{f_k\}_{k\in\mathbf{N}}) \;=\; \left\{\sum_{k=1}^{n} \alpha_k f_k : n > 0,\, \alpha_k \in \mathbb{C}\right\}.$$

Example A.72. Let $e_n = (\delta_{mn})_{m\in\mathbf{N}} = (0,\dots,0,1,0,0,\dots)$ denote the sequence which has a 1 in the nth component and zeros elsewhere. Then

$$\mathrm{span}(\mathcal{E}) \;=\; \mathrm{span}(\{e_n\}_{n\in\mathbf{N}}) \;=\; c_{00}.$$

In particular, the finite span of $\{e_n\}_{n\in\mathbf{N}}$ is not ℓ^p for any p.

Definition A.73 (Closed Span). Let A be a subset of a normed linear space X. The *closed finite span* of A, denoted $\overline{\mathrm{span}}(A)$, is the closure of the set of all finite linear combinations of elements of A:

$$\overline{\mathrm{span}}(A) \;=\; \overline{\mathrm{span}(A)} \;=\; \{z \in X : \exists\, y_n \in \mathrm{span}(A) \text{ such that } y_n \to z\}.$$

We also refer to the closed finite span of A as the *closed linear span* or the *closed span* of A.

Beware: The definition of the closed span does NOT imply that

$$\overline{\mathrm{span}}(A) \;=\; \left\{\sum_{k=1}^{\infty} \alpha_k f_k : f_k \in A, \alpha_k \in \mathbb{C}\right\} \qquad \leftarrow \text{This need not hold!}$$

In particular it is NOT true that an arbitrary element of $\overline{\mathrm{span}}(A)$ can be written $f = \sum_{k=1}^{\infty} \alpha_k f_k$ for some $f_k \in A$, $\alpha_k \in \mathbb{C}$ (see Exercise A.78 for a counterexample). Instead, to illustrate the meaning of the closed span, consider the case of a countable set $A = \{f_k\}_{k\in\mathbf{N}}$. Here we have

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$$\overline{\text{span}}(\{f_k\}_{k\in\mathbb{N}}) = \Big\{f \in X : \exists\, \alpha_{k,n} \in \mathbb{C} \text{ such that } \sum_{k=1}^{n} \alpha_{k,n} f_k \to f \text{ as } n \to \infty\Big\}.$$

That is, an element f lies in the closed span if there exist $\alpha_{k,n} \in \mathbb{C}$ such that

$$\sum_{k=1}^{n} \alpha_{k,n} f_k \to f \quad \text{as } n \to \infty.$$

In contrast, to say that $f = \sum_{k=1}^{\infty} \alpha_k f_k$ means that

$$\sum_{k=1}^{n} \alpha_k f_k \to f \quad \text{as } n \to \infty. \tag{A.5}$$

In particular, in order for equation (A.5) to hold, the scalars α_k must be *independent* of n. This condition is related to the definition of Schauder bases, which we will consider in more detail in Section ??, after development of the Hahn–Banach Theorem.

A complete subset is one whose closed span is the entire space.

Definition A.74 (Complete Sequence). If X is a Banach space then a subset $A \subseteq X$ is *complete* in X if $\text{span}(A)$ is dense in X, i.e., if $\overline{\text{span}}(A) = X$.

There are many other terminologies in use for complete sequences, e.g., they are also called *total* or *fundamental* sequences.

Schauder bases are examples of complete sequences.

Definition A.75 (Schauder Basis). A sequence $\mathcal{F} = \{f_k\}_{k\in\mathbb{N}}$ in a Banach space X is a *Schauder basis* for X if we can write every $f \in X$ as

$$f = \sum_{k=1}^{\infty} \alpha_k(f)\, f_k \tag{A.6}$$

for a unique choice of scalars $\alpha_k(f)$, where the series converges in the norm of X.

In particular, $g_N = \sum_{k=1}^{N} \alpha_k(f)\, f_k$ belongs to $\text{span}(\{f_k\}_{k\in\mathbb{N}})$ for each N, and $g_N \to f$ as $N \to \infty$, so every f belongs to the closed span of $\mathcal{F} = \{f_k\}_{k\in\mathbb{N}}$. Thus every Schauder basis is a complete sequence.

Exercise A.76. Let $e_n = (\delta_{mn})_{m\in\mathbb{N}} = (0, \dots, 0, 1, 0, 0, \dots)$ be as in Example A.72. Show that if $1 \le p < \infty$, then $\{e_n\}_{n\in\mathbb{N}}$ is a Schauder basis for ℓ^p, and hence is complete in ℓ^p.

Show that $\{e_n\}_{n\in\mathbb{N}}$ is not a Schauder basis for ℓ^∞, but instead is a Schauder basis for the proper closed subspace c_0.

We refer to $\mathcal{E} = \{e_n\}_{n\in\mathbb{N}}$ as the *standard basis* for ℓ^p (p finite) or c_0 ($p = \infty$).

While every Schauder basis is complete, the converse fails in general. We will next give an example of a complete sequence $\{f_k\}_{k\in\mathbb{N}}$ which does *not* have the property that every vector f can be written in the form given in equation (A.6). For this example we will need the following very useful theorem on approximation by polynomials (which we prove in Chapter 1, see Theorem 1.86). We define the space

$$C[a,b] \;=\; \{f\colon [a,b] \to \mathbb{C} \;:\; f \text{ is continuous}\}, \tag{A.7}$$

which is a Banach space under the uniform norm.

Theorem A.77 (Weierstrass Approximation Theorem). *If $f \in C[a,b]$ and $\varepsilon > 0$, then there exists a polynomial p such that*

$$\|f - p\|_\infty \;=\; \sup_{x\in[a,b]} |f(x) - p(x)| \;<\; \varepsilon.$$

Exercise A.78. Use the Weierstrass Approximation Theorem to show that the set of monomials $\{x^k\}_{k=0}^\infty$ is complete in $C[a,b]$. However, show that there exist functions $f \in C[a,b]$ which cannot be written as $f(x) = \sum_{k=0}^\infty \alpha_k x^k$ with convergence of the series in the uniform norm.

We will explore the distinctions between bases and complete sets in more detail in Section ??.

Additional Problems

A.22. Let X be a Banach space. Show that if there exists a countable subset $\{f_n\}_{n\in\mathbb{N}}$ in X that is complete, then X is separable.