

B.8 Repeated Integration

Let $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ be measurable. If f is a measurable function on $E \times F$ then there are three natural integrals of f on $E \times F$. First, there is the integral of f on $E \times F$ as a subset of \mathbb{R}^{m+n} , which we write as the *double integral*

$$\iint_{E \times F} f = \iint_{E \times F} f(x, y) (dx dy).$$

Second, for each fixed y we can integrate $f(x, y)$ as a function of x ; and then integrate the result in y , obtaining the *iterated integral*

$$\int_F \left(\int_E f(x, y) dx \right) dy.$$

Third, we also have the iterated integral

$$\int_E \left(\int_F f(x, y) dy \right) dx.$$

In general these three integrals need not be equal, even if they all exist.

In this section we review Fubini's and Tonelli's Theorems, which give sufficient conditions under which we can exchange the order of integration. We begin with Tonelli's Theorem, which states that interchange is allowed if f is nonnegative. In particular, this suggests that a counterexample to equality of the integrals must be related to the indeterminateness of $\infty - \infty$ (see Problem B.19).

Theorem B.65 (Tonelli's Theorem). *Let E be a measurable subset of \mathbb{R}^m and F a measurable subset of \mathbb{R}^n . If $f: E \times F \rightarrow [0, \infty]$ is measurable, then the following statements hold.*

- (a) $f_x(y) = f(x, y)$ is measurable on F for $x \in E$.
- (b) $f^y(x) = f(x, y)$ is measurable on E for $y \in F$.
- (c) $g(x) = \int_F f_x(y) dy$ is a measurable function on E .
- (d) $h(y) = \int_E f^y(x) dx$ is a measurable function on F .
- (e) As extended real numbers, we have

$$\begin{aligned} \iint_{E \times F} f(x, y) (dx dy) &= \int_F \left(\int_E f(x, y) dx \right) dy \\ &= \int_E \left(\int_F f(x, y) dy \right) dx. \end{aligned} \quad (\text{B.5})$$

As a corollary, we obtain the useful fact that to test whether a given function belongs to $L^1(E \times F)$ we can simply show that any one of three possible integrals is finite.

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Corollary B.66. Let E be a measurable subset of \mathbb{R}^m and F a measurable subset of \mathbb{R}^n . If f is a measurable function on $E \times F$, then (as extended real numbers):

$$\iint_{E \times F} |f(x, y)| (dx dy) = \int_F \left(\int_E |f(x, y)| dx \right) dy = \int_E \left(\int_F |f(x, y)| dy \right) dx.$$

Consequently, if any one of these three integrals is finite, then $f \in L^1(E \times F)$.

Fubini's Theorem allows the interchange of integrals if f is integrable (thereby again avoiding the ambiguity that is $\infty - \infty$).

Theorem B.67 (Fubini's Theorem). Let E be a measurable subset of \mathbb{R}^m and F a measurable subset of \mathbb{R}^n . If $f \in L^1(E \times F)$, then the following statements hold.

- (a) $f_x(y) = f(x, y)$ is measurable and integrable on F for almost every $x \in E$.
 (b) $f^y(x) = f(x, y)$ is measurable and integrable on E for almost every $y \in F$.
 (c) $g(x) = \int_F f_x(y) dy$ is a measurable and integrable function on E .
 (d) $h(y) = \int_E f^y(x) dx$ is a measurable and integrable function on F .
 (e) We have

$$\iint_{E \times F} f(x, y) (dx dy) = \int_F \left(\int_E f(x, y) dx \right) dy = \int_E \left(\int_F f(x, y) dy \right) dx.$$

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Additional Problems

B.19. Show that the following iterated integral has the value

$$\int_1^\infty \left(\int_1^\infty \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \right) dx = -\frac{\pi}{4},$$

while

$$\int_1^\infty \left(\int_1^\infty \frac{x^2 - y^2}{(x^2 + y^2)^2} dx \right) dy = \frac{\pi}{4}.$$

Conclude that equality need not hold in equation (B.5) if the hypotheses of Fubini's Theorem are not fulfilled.

B.20. Let $f(x, y)$ be a measurable function on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$, and fix $1 \leq p < \infty$. Prove *Minkowski's Integral Inequality*:

$$\left(\int \left(\int |f(x, y)| dy \right)^p dx \right)^{1/p} \leq \int \left(\int |f(x, y)|^p dx \right)^{1/p} dy. \quad (\text{B.6})$$

This equation is perhaps more revealing if we rewrite it as follows. Set $f_x(y) = |f(x, y)|$. Then equation (B.6) is equivalent to

See also
Section 6.3
in Folland

$$\left\| \int f_x(\cdot) dx \right\|_p \leq \int \|f_x\|_p dx.$$

Thus, Minkowski's Integral Inequality is an integral version of the Triangle Inequality (also known as Minkowski's Inequality) on $L^p(\mathbb{R}^n)$.

B.21. If $x > 0$, then $\int_0^\infty e^{-tx} dt = \frac{1}{x}$. Combine this with Fubini's Theorem to evaluate the integral $\int_0^a \frac{\sin x}{x} dx$. Then apply the Lebesgue Dominated Convergence Theorem to show that

$$\lim_{a \rightarrow \infty} \int_0^a \frac{\sin x}{x} dx = \frac{\pi}{2},$$

Thus, even though $\frac{\sin x}{x}$ is not integrable, the improper Riemann integral $\int_0^\infty \frac{\sin x}{x} dx$ does exist and equals $\frac{\pi}{2}$. The *principal value* of $\int \frac{\sin x}{x} dx$ is defined to be

$$\text{pv} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{\sin x}{x} dx = \pi. \quad (\text{B.7})$$

(This integral can also be evaluated by used contour integration.)

B.22. This problem will establish a version of *Hardy's Inequalities*.

(a) Given $1 \leq p < \infty$ and $\alpha < -1$, show there exists a constant $C(\alpha, p)$ such that for any measurable $f: (0, \infty) \rightarrow [0, \infty]$,

$$\int_0^\infty \left(\int_0^x f(t) dt \right)^p x^\alpha dx \leq C(\alpha, p) \int_0^\infty f(t)^p t^{\alpha+p} dt.$$

If $\alpha > -1$, the inequality is

$$\int_0^\infty \left(\int_x^\infty f(t) dt \right)^p x^\alpha dx \leq C(\alpha, p) \int_0^\infty f(t)^p t^{\alpha+p} dt.$$

(b) For the case $\alpha = -p < -1$, show that the optimal constant is

$$C(-p, p) = (p')^p = \left(\frac{p}{p-1} \right)^p.$$

(c) Suppose $f \in L^p(\mathbb{R})$ where $1 < p < \infty$. Define $F(x) = \frac{1}{x} \int_0^x |f(t)| dt$ for $x \in \mathbb{R}$, and show that

$$\|F\|_p \leq p' \|f\|_p, \quad (\text{B.8})$$

with p' being the best possible constant. Also show that equality holds in equation (B.8) if and only if $f = 0$ a.e.

Problem Hints

B.19 Hint: Use the fact that

$$\frac{d}{dy} \frac{y}{x^2 + y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

to show that for each $x > 1$ the function $f_x(y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$ is integrable on $(1, \infty)$, and to find the value of $\int_1^\infty \frac{x^2 - y^2}{(x^2 + y^2)^2} dy$.

B.20 Hint: For $1 < p < \infty$, use a technique similar to the one given in the hint to Exercise A.18, which proves the Triangle Inequality on ℓ^p when $1 < p < \infty$.

B.22 Hints: (a) For the case $\alpha < -1$, fix $p + \alpha < \eta < p - 1$, and use Hölder's Inequality to show that

$$\left(\int_0^x f(t) dt \right)^p \leq C \int_0^x f(t)^p t^\eta dt x^{p-1-\eta}.$$

(b) To show $C(-p, p) \leq (p')^p$, set $\alpha = -p$, and minimize over η . To show that $(p')^p$ is best constant, consider $f(t) = t^{-1/p} \chi_{[0, R]}(t)$.

(c) To show that equality holds only for the zero function, note that there is only one step where inequality can occur in the proof of part (a), namely, when Hölder's Inequality is applied.