C_b, C_c, C_b^m, C_c^m

A.4.2 Some Spaces of Continuous and Differentiable Functions

We now give some additional examples of Banach and normed spaces, and many more appear throughout this volume. Continuity is defined precisely in Section A.7.2 below.

Definition A.20. The *support* of a function $f\colon \mathbb{R} \to \mathbb{C}$ is the closure of the set of points where f is nonzero:

$$\operatorname{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}.$$

Since the support of a function is a closed set, a function on $\mathbb{R}$ has compact support if and only if it is zero outside of a finite interval.

Exercise A.21. Let $C_b(\mathbb{R})$ denote the space of continuous, bounded functions $f\colon \mathbb{R} \to \mathbb{C}$. Show that $C_b(\mathbb{R})$ is a Banach space with respect to the *uniform norm*

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|.$$

Furthermore, show that the subspace

$$C_0(\mathbb{R}) = \Bigl\{f \in C_b(\mathbb{R}) : \lim_{|t|\to\infty} f(t) = 0\Bigr\}$$

is also a Banach space with respect to the uniform norm, but the subspace

$$C_c(\mathbb{R}) = \bigl\{f \in C_b(\mathbb{R}) : \operatorname{supp}(f) \text{ is compact}\bigr\} \tag{A.3}$$

is a normed space that is not a Banach space under the uniform norm (compare Exercise A.62).

Beware, some authors use the symbols C_0 to denote the space that we refer to as C_c.

Proof:

A.21 (a) The fact that $C_b(\mathbb{R})$ is a vector space and that $\|\cdot\|_\infty$ is a norm on $C_b(\mathbb{R})$ is clear, so we only need to show completeness.

Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C_b(\mathbb{R})$ with respect to the uniform norm. Then for each x, we have

$$|f_m(x) - f_n(x)| \;\le\; \|f_m - f_n\|_\infty,$$

so $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence of scalars, and hence converges. Define $f(x) = \lim_{n\to\infty} f_n(x)$.

Now choose $\varepsilon > 0$. Then there exists an N such that $\|f_m - f_n\|_\infty < \varepsilon$ for all m, $n > N$. Fix $n > N$. For each $M > 0$ we have

$$|f(x) - f_n(x)| \;=\; \lim_{m\to\infty} |f_m(x) - f_n(x)| \;\le\; \|f_m - f_n\|_\infty \;\le\; \varepsilon,$$

so $\|f - f_n\|_\infty \le \varepsilon$ for all $n > N$. Also, $\|f\|_\infty \le \|f - f_n\|_\infty + \|f_n\|_\infty$, so f is bounded. Finally, the uniform limit of continuous functions is continuous, so $f \in C_b(\mathbb{R})$ and $f_n \to f$ uniformly. This shows that $C_b(\mathbb{R})$ is complete.

(b) The fact that $C_0(\mathbb{R})$ is a normed space is also clear. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $C_0(\mathbb{R})$. Then it is Cauchy in $C_b(\mathbb{R})$, so by part (a) there exists an $f \in C_b(\mathbb{R})$ such that $f_n \to f$ uniformly. We need only show that $f \in C_0(\mathbb{R})$.

Fix any $\varepsilon > 0$. Then there exists an $N > 0$ such that $\|f - f_n\|_\infty < \varepsilon$ for all $n \ge N$. In particular, $\|f - f_N\|_\infty < \varepsilon$. Now, since $f_N \in C_0(\mathbb{R})$, there exists an $R > 0$ such that $|f_N(x)| < \varepsilon$ for all $|x| > R$. Hence for $|x| > R$ we have

$$|f(x)| \;\le\; |f(x) - f_N(x)| + |f_N(x)| \;\le\; \varepsilon + \varepsilon \;=\; 2\varepsilon.$$

Thus $f(x) \to 0$ as $|x| \to \infty$, so $f \in C_0(\mathbb{R})$, and therefore $C_0(\mathbb{R})$ is complete.

(c) Again the fact that $C_c(\mathbb{R})$ is a normed space is clear. To show that $C_c(\mathbb{R})$ is not complete, choose a function like $g(t) = e^{-t^2}$ in $C_0(\mathbb{R})$ that is nonzero everywhere. Then define

$$g_N(x) \;=\; \begin{cases} g(x), & |x| \le N, \\ \text{linear}, & N \le |x| \le N+1, \\ 0, & |x| > N+1. \end{cases}$$

Each g_N belongs to $C_c(\mathbb{R})$, and $g_N \to g$ uniformly. Hence $\{g_N\}_{N\in\mathbb{N}}$ is Cauchy with respect to the uniform norm. However, g_N does not converge to a function in $C_c(\mathbb{R})$, so $C_c(\mathbb{R})$ is not complete.

Exercise A.22. Let $C_b^m(\mathbb{R})$ be the space of all m-times differentiable functions on $\mathbb{R}$ each of whose derivatives is bounded and continuous, i.e.,

$$C_b^m(\mathbb{R}) \;=\; \bigl\{f \in C_b(\mathbb{R}) : f, f', \dots, f^{(m)} \in C_b(\mathbb{R})\bigr\}.$$

Show that $C_b^m(\mathbb{R})$ is a Banach space with respect to the norm

$$\|f\|_{C_b^m} \;=\; \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^{(m)}\|_\infty.$$

Proof:

A.22 Let us show that $C_b^1(\mathbb{R})$ is complete. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C_b^1(\mathbb{R})$. Then $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $C_b(\mathbb{R})$, so there exists an $f \in C_b(\mathbb{R})$ such that $f_n \to f$ uniformly. Additionally, by definition of $C_b^1(\mathbb{R})$, we know that

$$\|f'_m - f'_n\|_\infty \;\le\; \|f_m - f_n\|_\infty + \|f'_m - f'_n\|_\infty \;=\; \|f_m - f_n\|_{C_b^1},$$

so $\{f'_n\}_{n\in\mathbb{N}}$ is Cauchy with respect to the uniform norm. That is, $\{f'_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C_b(\mathbb{R})$. Since $C_b(\mathbb{R})$ is complete, there exists a $g \in C_b(\mathbb{R})$ such that $f'_n \to g$ uniformly. So, the remaining point is to show that $g = f'$, for then we will have that $f_n \to f$ in the norm of $C_b^1(\mathbb{R})$. This is a standard undergraduate real analysis theorem.

To see it, fix any $x \in \mathbb{R}$ and any $\varepsilon > 0$. Then there exists an $N > 0$ such that $\|f'_m - f'_n\|_\infty < \varepsilon$ whenever m, $n > N$. Consider any $y \in \mathbb{R}$ and m, $n > N$. Applying the Mean-Value Theorem to the function $f_m - f_n$, there exists a point c (depending on m, n, x, and y) between x and y such that

$$(f_m - f_n)(y) - (f_m(x) - f_n)(x) \;=\; (y-x)\,(f'_m - f'_n)(c).$$

Consequently,

$$\left|\frac{f_m(y) - f_m(x)}{y-x} - \frac{f_n(y) - f_n(x)}{y-x}\right| \;=\; |f'_m(c) - f'_n(c)| \;\le\; \|f'_m - f'_n\|_\infty \;<\; \varepsilon.$$

Letting $m \to \infty$, we conclude that

$$\left|\frac{f(y) - f(x)}{y-x} - \frac{f_n(y) - f_n(x)}{y-x}\right| \;\le\; \varepsilon.$$

This is valid for any x and y as long as $n > N$.

Now, since f_n is differentiable, there exists a $\delta > 0$ such that

$$|x-y| < \delta \implies \left|f'_n(x) - \frac{f_n(y) - f_n(x)}{y-x}\right| \;<\; \varepsilon.$$

Further, since $f'_n \to g$ uniformly, there exists an M such that $\|f'_n - g\|_\infty < \varepsilon$ whenever $n > M$. Hence for $n > M, N$ and $|x-y| < \delta$ we have

$$\begin{aligned}\left|g(x) - \frac{f(y)-f(x)}{y-x}\right| \;&<\; |g(x) - f'_n(x)| + \left|f'_n(x) - \frac{f_n(y)-f_n(x)}{y-x}\right| \\ &\quad + \left|\frac{f_n(y)-f_n(x)}{y-x} - \frac{f(y)-f(x)}{y-x}\right| \\ &<\; \varepsilon + \varepsilon + \varepsilon \;=\; 3\varepsilon.\end{aligned}$$

Hence

$$g(x) \;=\; \lim_{y\to x} \frac{f(y)-f(x)}{y-x},$$

so f is differentiable at x, and $f'(x) = g(x)$. Therefore we conclude that $f_n \to f$ in the norm of $C_b^1(\mathbb{R})$, and hence this space is complete.

Repeating this argument yields a proof by induction that $C_b^m(\mathbb{R})$ is complete for each m. ∎

Problem

The above work shows that $C_b^1(\mathbb{R})$ is complete with respect to the norm

$$\|f\|_{C_b^1} = \|f\|_\infty + \|f'\|_\infty.$$

On the other hand, $C_b^1(\mathbb{R}) \subseteq C_b(\mathbb{R})$, so it is also a normed space w.r.t. the norm on $C_b(\mathbb{R})$, which is $\|f\|_\infty$. Show that $C_b^1(\mathbb{R})$ is not complete w.r.t. $\|\cdot\|_\infty$. Specifically, show $\exists f_n \in C_b^1(\mathbb{R})$ and $f \in C_b(\mathbb{R}) \setminus C_b^1(\mathbb{R})$ such that

$\|f - f_n\|_\infty \longrightarrow 0.$

Although they are not normed spaces, it is sometimes important to consider the space of functions that are continuous or m-times differentiable but not bounded. We denote these by:

$$\begin{aligned} C(\mathbb{R}) &= \{f : \mathbb{R} \to \mathbb{C} \ : \ f \text{ is continuous}\}, \\ C^m(\mathbb{R}) &= \{f \in C(\mathbb{R}) \ : \ f, f', \dots, f^{(m)} \in C(\mathbb{R})\}. \end{aligned}$$

Additionally, we sometimes need to consider spaces of infinitely differentiable functions, including the following:

$$\begin{aligned} C^\infty(\mathbb{R}) &= \{f \in C(\mathbb{R}) \ : \ f, f', \dots \in C(\mathbb{R})\}, \\ C_b^\infty(\mathbb{R}) &= \{f \in C_b(\mathbb{R}) \ : \ f, f', \dots \in C_b(\mathbb{R})\}. \end{aligned}$$

The corresponding subspaces containing only those functions that decay at infinitely or are compactly supported are denoted by

$$\begin{aligned} C_0^m(\mathbb{R}) &= \{f \in C_0(\mathbb{R}) \ : \ f, f', \dots, f^{(m)} \in C_0(\mathbb{R})\}. \\ C_c^m(\mathbb{R}) &= \{f \in C_c(\mathbb{R}) \ : \ f, f', \dots, f^{(m)} \in C_c(\mathbb{R})\}. \\ C_0^\infty(\mathbb{R}) &= \{f \in C_0(\mathbb{R}) \ : \ f, f', \dots \in C_0(\mathbb{R})\}. \\ C_c^\infty(\mathbb{R}) &= \{f \in C_c(\mathbb{R}) \ : \ f, f', \dots \in C_c(\mathbb{R})\}. \end{aligned}$$

The space $C_c^\infty(\mathbb{R})$ will be especially important to us in Chapter 3 and in Appendix E. Although not a normed space, it is topological vector space (see Appendix E), and is often denoted by the symbols

$$\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R}).$$

Its dual space $\mathcal{D}'(\mathbb{R})$ is the *space of distributions* (see Chapter 3).